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NOTE ON LINEAR ARBORICITY

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It is possible to define many variations of packing and covering invariants for graphs which involve paths or cycles, as one can see from [1], which is especially devoted to this problem.

A concept which will be of value to us in what follows is the linear arboricity. It was defined at first by Harary in [2] and independently, as a path chromatic index, by Hilton [3].

A linear forest is a graph in which each component is a path. The linear arboricity $\Xi(G)$ of a graph G is the minimum number of linear forests whose union is G. The linear arboricity of the complete graph coincides with its path number which was determined in [4]. It is not easy to determine the value of linear arboricity for general graphs. Thus it is investigated for specific families of graphs. In [1] there was expressed a conjecture (an analogous conjecture is given also in [3]):

Conjecture. *The linear arboricity of an r-regular graph G is* $\left\{\frac{r+1}{2}\right\}$.

This conjecture was proved there for $r=2,3$. For the case $r=4$ the authors mentioned that they verified this conjecture, too. But they do not know whether it holds for $r\geq 5$. Our aim is to prove it for the case of $r=6$.

Prior to prove it some facts are needed.

Theorem 1. *Let G be a graph with degree sequence* $\{4, 3, 3, \ldots, 3\}$. *Then* $\Xi(G)=2$.

Proof. Delete any edge e incident with the vertex of degree four from G. Although the resulting graph is not regular, it can be easily supplemented to a regular one. Since for $r=3$ [1] the conjecture holds, consider a decomposition of $G-e$ into two linear forests H_1 and H_2, say blue and red, respectively.

Denote the vertex of degree two in $G-e$ by x and the second vertex of the edge e by y. Let neither H_1+e nor H_2+e be linear forests.

Only two cases are possible (see Fig. 1.):

* (i) one blue edge is incident to x in $G-e$

(ii) two blue edges are incident to x in $G-e$.

* Remark: stright ine — blue, waveline — red

In case (i) the vertices x and y lie on some red path and in both cases some of the vertices p, q may coincide with some of the vertices a, b, c.

First investigate case (i): Once again two cases are possible — see Fig. 2.

In both cases after a suitable interchange of colours one may add the edge e to the red linear forest.

Now look at case (ii): A path in $G-e$ is said to be alternating if the colours of its edges alternate. First we notice that:

There exists a non alternating path of a length at least two in $G - e$ beginning in the vertex x.

Fig 1

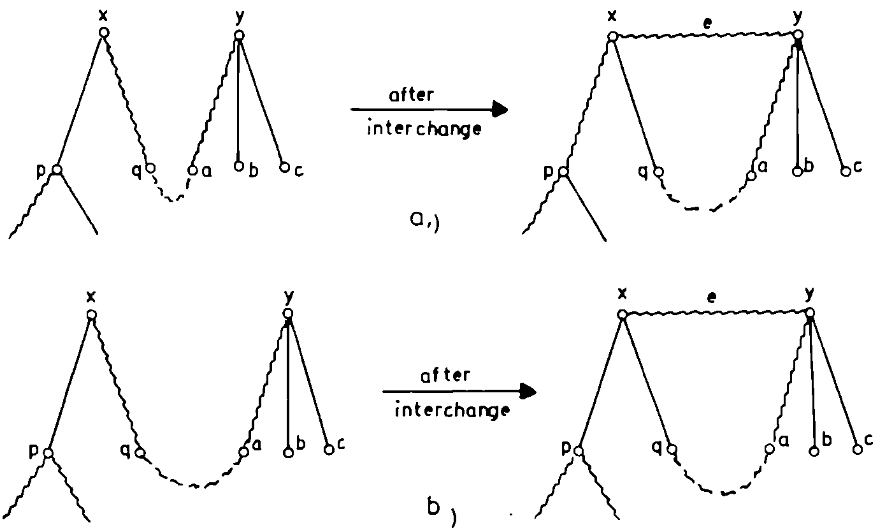

Fig 2

Let us prove it. Let, on the contrary, each path from x be alternate. Take the longest one. Such a path exists because $G-e$ is finite. Let t be the last vertex of this path. The vertex t has degree three and thus two edges e_1, e_2 not contained in this longest path are incident with t. Without loss of generality we can assume the last edge of this path to be blue. Furthermore, let e_1 be also blue (analogously for e_2).

The second end vertex of e_1 lies on this path (otherwise there is a nonalternating path in $G-e$). But in that case one can easily find a non-alternating path, too. Thus e_1 and e_2 are red and their end vertices lie on the longest path. But also in this case it is not so hard to find a non-alternating path in $G-e$ from x. This is a contradiction. Hence the result.

Let us continue in the analysis of case (ii): Assume that x and y lie on a blue path meeting q (analogously for p) — see Fig. 3.

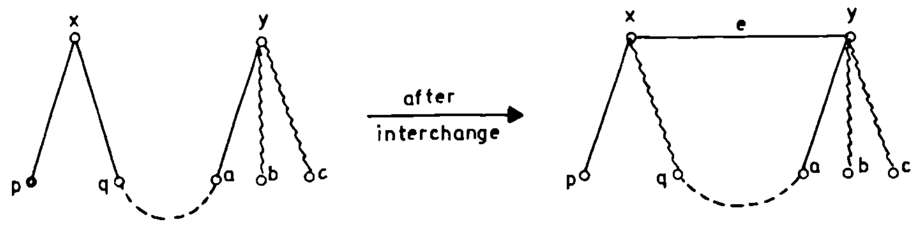

Fig. 3

After a suitable interchange of colours (see Fig. 3.) one can add the edge e to the blue linear forest.

In the next we can assume that no blue path joins x and y. As we notice above there exists in $G-e$ a non-alternating path from x. Take the shortest one. Without loss of generality we can assume the last two edges of it to be red. Excepting the last edge interchange the colours of this path. We assert that neither H_1 nor H_2 contains a cycle. In fact, if such a cycle exists, then one can easily find a shorter non-alternating path from x, which is a contradiction with the assumption. Moreover, neither H_1 nor H_2 can contain a vertex of degree three, thus they remain linear forests.

After this interchange of the colours the vertex x is incident with the blue and red edges. Since no blue path joins x and y, one may add the edge e to the blue linear forest. In the case y incident with two blue edges (after the interchange) one can proceed as in case (i). The theorem is proved.

And now we are able to prove

Theorem 2. *Let G be a finite connected 6-regular graph. Then*

$$\Xi(G)=4.$$

Proof. I. Let the number of edges of G be even. Consider an Eulerian trail in G. Colour the edges of this trail alternately blue and red. We obtain two cubic factors. Each of them has (by [1]) the linear arboricity equal to two. Hence $\Xi(G)=4$.

II. Let the number of edges of G be odd. Once again consider an Eulerian trail in G. Colour the edges of this trail alternately blue and red. We obtain a decomposition of G into two factors. The blue one has degree sequence $\{4, 3, 3, \ldots, 3\}$ and the red one $\{2, 3, \ldots, 3\}$. Applying Theorem 1 we obtain the assertion of Theorem 2.

Remark. *An interesting open problem is to determine the maximal number of $(r+1)$'s in a degree sequence (of a given length)*

$$\{r+1, r+1, \ldots, r+1, r, r, \ldots, r\}$$

of a graph G with the linear arboricity to be

$$\Xi(G) = \left\{\frac{r+1}{2}\right\} \quad \text{for odd} \quad r \geq 3.$$

Added in proof: B. Paroche (On partition of graphs into linear forests and dissections, Rapport de recherche; Centre National de la recherche scientifique) proved the Conjecture for r = 5, 6.

REFERENCES

[1] AKLYAMA J. EXOO G. HARARY F.: Covering and packing in graphs III. Cyclic and acyclic invariants, Math. Slovaca 30, 1980.

[2] HARARY F: Covering and packing in graphs I., Ann. N. Y. Acad. Sci., 175, 1970, 198—205.

[3] HILTON A. J. W.: Canonical edge-colourings of locally finite graphs, (submitted to J. Comb. Theory).

[4] STANTON R.—COWAN D. JAMES L.: Some results on path numbers, Proc. Louisiana Conf. Combinatorics, Graph Theory and Computing, (R.C. Read ed.) Academic Press, New York 1972, 285 294.

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ЗАМЕЧАНИЕ К ЛИНЕЙНОЙ ДРЕВЕСНОСТИ

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Резюме

Линейная древесность $\Xi(G)$ графа G это минимальное число линеиных лесов, соединение которых равно G. В работах [1] и [3] независимо была высказана гипотеза, что линейная древесность r-правильного графа равна

$$\left\{\frac{r+1}{2}\right\}.$$

В работе [1] она была доказана для $r-2,3$. Целью этого замечания доказать ее для r 6.