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ORIENTABILITY OF TOTAL SPACES OF FIBRE BUNDLES OVER RP^n

MILOŠ BOŽEK

1. Introduction

There are two well-known results on orientability of topological manifolds:

Theorem A. *Any open submanifold of orientable manifold is orientable.*

Theorem B. *The product-manifold is orientable if and only if both factors are orientable.*

Theorem A can be reformulated in the following way:

Theorem A'. *Every manifold containing an open non-orientable submanifold is non-orientable.*

The part "if" of Theorem B fails for total spaces of fibrations as the Klein bottle shows regarded as a total space of the standart fibration over S^1 with the fibre S^1. On the other hand, the part "only if" of Theorem B remains valid for a large class of fibrations[(1)].

Theorem 1. *The total space E of a locally trivial fibration $\xi=(E, p, B)$ with a non-orientable fibre F is non-orientable.*

Proof. By Theorem B, every manifold $U\times F$, $U\subset B$ open, is non-orientable. This means that E contains a non-orientable open submanifold, thus by Theorem A' E is non-orientable.

Let RP^{n-1} be a hyperplane in the n-dimensional real projective space RP^n. The main result of this paper is the following

Theorem 2. *Let $\xi=(E, p, RP^n)$, $n\geqq 2$ be a fibre bundle with a compact connected and orientable fibre F. Then the total space E of ξ is orientable if and only if the manifold $E'=p^{-1}(RP^{n-1})$ is non-orientable.*

For every $k=0, 1, \ldots, n$ we define the k^{th} *derivative of the fibre bundle* $\xi=(E, p, rp^n)$ as the manifold $E^{(k)} = p^{-1}(RP^{n-k})$. Clearly $E^{(0)}=E$ and $E^{(n)}$ is

[(1)] In this paper all fibrations belong to the category of topological manifolds and continuous maps. Under a fibre bundle we mean a fibration associated with a locally trivial principal fibration [2, Chap. 4].

homeomorphic to F. For every manifold M put $\omega(M)=1$ or 0 if M is orientable or non-orientable, respectively. The next Theorem is an easy consequence of Theorem 2.

Theorem 3. *Under the assumptions of Theorem 2 we have*

$$\omega(E)\equiv\omega(E^{(k)})+k \pmod 2$$

for all $k=1, \ldots, n-1$.

Theorem 2 will be proved in Section 3. An application of Theorems 1 and 2 will be given in Section 4.

2. Very strong deformation retracts

In the proof of Theorem 2 we shall make use of some special kind of deformation retracts.

A very strong deformation retraction of a topological space X to a subspace A is a retraction $r\colon x\to A$ for which there exists a homotopy $h_t\colon X\to X$, $t\in I=[0, 1]$ with the following properties:

(i) $h_0=1_X$,

(ii) $h_1=i\circ r$, where $i\colon A\to X$ is the inclusion map,

(iii) $h_t|A=1_A$,

(iv) $r\circ h_t=r$

for all $t\in I$.

A subspace A of X is called a *very strong deformation retract* of X if there exists a very strong deformation retraction of X to A.

Clearly every very strong deformation retraction (retract) is a strong deformation retraction (retract) in the usual sense cf. [4, p. 30].

Example 1. Let there be given a topological space X consisting of all points (x, y) of $\boldsymbol{R}^2$ such that $0\leqq x, y\leqq 1$ and $x=1$ or $y=0$ or $y=1$ and let A be a subspace of X given by $y=0$ (see Fig. 1). Then the map $r\colon X\to A$ defined by $r(x, y) = (x, 0)$ is a strong deformation retraction of X to A but it is not a very strong deformation retraction. However, A is a very strong deformation retract of X under another retraction $r'\colon X\to A$ defined by $r'(x, y)=(1, 0)$ if $y>0$ and $r'(x, y) = (x, y)$ otherwise.

Problem. Is every strong deformation retract a very strong deformation retract?

Example 2. Let $(x_0, x_1, \ldots, x_n)$ be homogeneous coordinates in RP^n. Let us consider the following five subspaces of RP^n:

$$RP^0\colon x_1=\ldots x_n=0;\quad RP^{n-1}\colon x_0=0;$$

$$S^{n-1}\colon x_0^2-x_1^2-\ldots-x_n^2=0;\quad X_1=RP^n-RP^{n-1};$$

$$X_2=RP^n-RP^0.$$

Then RP^0, RP^{n-1} and S^{n-1} are very strong deformation retracts of X_1, X_2 and $X_3 = X_1 \cap X_2$, respectively. The corresponding homotopies h_t^i, $i = 1, 2, 3$, $t \in I$, are defined by

$$h_t^1(x_0, x_1, \ldots, x_n) = (x_0, (1-t)x_1, \ldots, (1-t)x_n),$$
$$h_t^2(x_0, x_1, \ldots, x_n) = ((1-t)x_0, x_1, \ldots, x_n),$$
$$h_t^3(x_0, x_1, \ldots, x_n) = (cx_0, (t + c(1-t))x_1, \ldots, (t + c(1-t))x_n),$$

$$\text{where} \quad c = \sqrt{\frac{x_1^2 + \ldots + x_n^2}{x_0^2}}.$$

The next Proposition will explain the reason of introducing the notion "very strong deformation retract".

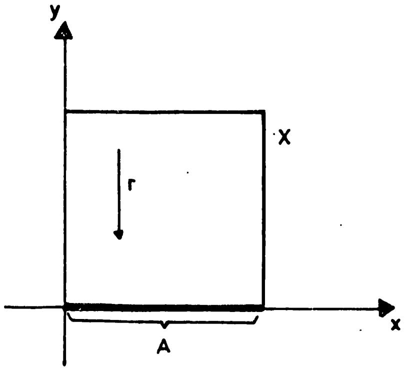

Fig. 1

Proposition 1. *Let $\xi = (E, p, B)$ be a fibre bundle and let $\tilde{B}$ be a very strong deformation retract of B. Then $\tilde{E} = p^{-1}(\tilde{B})$ is a strong deformation retract of E.*

Proof. Let i: $\tilde{B} \to B$ and i': $\tilde{E} \to E$ be inclusion maps and let r: $B \to \tilde{B}$ be a very strong deformation retraction and h_t, $t \in I$ its corresponding homotopy. Finally, let $\tilde{\xi}$ be the restriction of the fibre bundle ξ to $\tilde{B}$. It is known that there exists a canonical isomorphism $r^*\tilde{\xi} \cong (i \circ r)^*\xi$ in the category Bun_B of all fibrations over B. As $i \circ r$ is homotopic to the identity map 1_B it is $(i \circ r)^*\xi \cong \xi$. Hence there exists an isomorphism u: $r^*\tilde{\xi} \cong \xi$. It is easy to show that $u^{-1}(\tilde{E}) = \{(b, x) \in r^*\tilde{E} \mid b \in \tilde{B}\}$ and the map $\bar{r}$: $r^*\tilde{E} \to u^{-1}(\tilde{E})$ given by $\bar{r}(b, x) = (r(b), x)$ for all $(b, x) \in r^*\tilde{E}$ is a well-defined retraction. The equality $r \circ h_t = r$ implies that there is a homotopy $\bar{h}_t$: $r^*\tilde{E} \to r^*\tilde{E}$ defined by $\bar{h}_t(b, x) = (h_t(b), x)$ for all $(b, x) \in r^*\tilde{E}$, $t \in I$. The properties (i), (ii), (iii) of h_t yield the corresponding properties for $\bar{h}_t$. It means that $u^{-1}(\tilde{E})$ is a strong deformation retract of $r^*\tilde{E}$ and, going back to ξ via the isomorphism u: $r^*\tilde{\xi} \cong \xi$, we see that $\tilde{E}$ is a strong deformation retract of E.

Remark. In fact we have proved that $\tilde{E}$ is a very strong deformation retract of E.

3. Proof of Theorem 2

Throughout this paragraph the symbols $\xi=(E, p, B)$, n, F, RP^n and RP^{n-1} are assumed to satisfy the assumptions of Theorem 2. In addition the homogeneous coordinates $(x_0, x_1, \ldots, x_n)$ in RP^n are arranged in such a way that the hyperplane RP^{n-1} is given by the equation $x_0=0$. Finally, let RP^0, S^{n-1}, X_1, X_2 be subspaces of RP^n as in Example 2.

Proposition 2. *There is a long exact sequence*

$$\ldots\to\tilde{H}_q(F)\oplus\tilde{H}_q(E')\to\tilde{H}_q(E)\to\tilde{H}_{q-n}(F)\to \\ \to\tilde{H}_{q-1}(F)\oplus\tilde{H}_{q-1}(E')\to\ldots \tag{1}$$

for all $q\geqq n$.

Proof. Using the results of Example 2 and Proposition 1 we get the following homotopy equivalences

$$p^{-1}(X_1)\sim p^{-1}(RP^0)=F, \tag{2}$$

$$p^{-1}(X_2)\sim p^{-1}(RP^{n-1})=E', \tag{3}$$

$$p^{-1}(X_1\cap X_2)\sim p^{-1}(S^{n-1}). \tag{4}$$

Recall that the base X_1 of the restricted fibre bundle $\xi|X_1$ is contractible. By [1, Theorem 4.9.9] the fibre bundle $\xi|X_1$ is trivial, therefore the subbundle $\xi|S^{n-1}$ of $\xi|X_1$ is trivial as well, hence there is a homeomorphism $\alpha\colon p^{-1}(S^{n-1})\approx S^{n-1}\times F$. Now, the sequence (1) follows from the Mayer—Vietoris sequence of the excisive triad $(E; P^{-1}(X_1), p^{-1}(X_2))$ and from the natural isomorphism $\beta\colon \tilde{H}_{q-1}(S^{n-1}\times F)\cong\tilde{H}_{q-n}(F)$.

Let us denote $m=\dim F$. Then $\dim E=n+m$ and $\dim E'=n+m-1$. Further, $\dim F<\dim E-1$ because of $n\geqq 2$. Putting $q=n+m$ in (1) we obtain the first assertion of the following

Proposition 3. (a) *There is an exact sequence*

$$0\to\tilde{H}_{n+m}(E)\to\tilde{H}_m(F)\xrightarrow{\varphi}\tilde{H}_{n+m-1}(E'). \tag{5}$$

(b) *If the manifold E' is orientable, then φ is injective.*

Proof. Let $r\colon X_2\to RP^{n-1}$ be the retraction h_1^2 from Example 2, i.e. $r(x_0, x_1, \ldots, x_n)=(0, x_1, \ldots, x_n)$ and let $\bar{r}\colon p^{-1}(X_2)\to E'$ be the "lift" of r given by Proposition 1. Finally let $\tilde{r}=\bar{r}|p^{-1}(S^{n-1})$.

First we prove

(6) $$\ker \varphi \cong \ker \bar{r}_{*, n+m-1}.$$

From the construction of the sequence (1) we have

$$\varphi = (\bar{r} \circ j \circ i \circ \alpha)_{*, n+m-1} \circ \beta^{-1}$$

where $i: p^{-1}(S^{n-1}) \to p^{-1}(X_1 \cap X_2)$ and $j: p^{-1}(X_1 \cap X_2) \to p^{-1}(X_2)$ are inclusion maps. Obviously $\bar{r} \circ j \circ i = \bar{r}$, therefore $\varphi = \bar{r}_{*, n+m-1} \circ \alpha_{*, n+m-1} \circ \beta^{-1}$, which implies (6).

Now we are going to prove that

(7) $\bar{r}: p^{-1}(S^{n-1}) \to E'$ *is a double covering.*

As usually $r^*E' = \{(b, x) \in X_2 \times E' \mid r(b) = p(x)\}$. The retraction $r: X_2 \to RP^{n-1}$ is a homotopy equivalence, therefore there is a homeomorphism $u: p^{-1}(X_2) \to r^*E'$ such that $p \circ u^{-1}(b, x) = b$ and $\bar{r} \circ u^{-1}(b, x) = u^{-1}(r(b), x)$ for all $(b, x) \in r^*E'$. Hence

$$u(p^{-1}(S^{n-1})) = \{(b, x) \in S^{n-1} \times E' \mid r(b) = p(x)\}$$

and $\bar{r} \circ u^{-1}(b, x) = (r(b), x)$ for all $(b, x) \in u(p^{-1}(S^{n-1}))$. Clearly, the map $r \mid S^{n-1}: S^{n-1} \to RP^{n-1}$ is the standart double covering and (7) follows.

Let us return to the proof of the part (b) of Proposition 3. If E' is orientable, then (7) yields that $p^{-1}(S^{n-1})$ is orientable, too, and that $\bar{r}_{*, n+m-1}$ is injective. The assertion (6) implies injectivity of φ, which concludes the proof of Proposition 3.

We can now easily prove Theorem 2. By our assumptions regarding F we have $\tilde{H}_m(F) \cong \mathbf{Z}$. Further $\tilde{H}_{n+m-1}(E') \cong \mathbf{Z}$ or 0 if E' is orientable or non-orientable, respectively. The second statement of Proposition 3 says that $\ker \varphi = 0$ or $\tilde{H}_m(F)$ in the corresponding cases. Theorem 2 follows then from the exact sequence (5).

4. Orientability of the incidence manifold of RP^n

In paper [3] E. Ružický studied the submanifold $F(n)$ of the product-manifold $RP^n \times G_1(RP^n)$[2] consisting of all couples (x, y) for which $x \in y$. He has proved that for all n odd $F(n)$ is non-orientable. This result can be strengthened in the following way.

Theorem 4. *The manifold $F(n)$ is orientable if and only if n is even for all $n \geqq 2$.*

Proof. Let us consider the fibre bundle $\xi = (F(n), p, RP^n)$ where $p(x, y) = x$ for all $(x, y) \in F(n)$. The fibre F of ξ is homeomorphic to RP^{n-1}, thus F is non-orientable for n odd. In virtue of Theorem 1 $F(n)$ is non-orientable for n odd.

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[2] $G_1(RP^n)$ or $G_1(E^n)$ is the first Grassmannian of the projective space RP^n or the euclidean space E^n, respectively.

From now on let us assume that n is even, which implies that the fibre F is orientable. With respect to Theorem 2 we have to prove that the manifold $F(n)' = p^{-1}(RP^{n-1})$ is non-orientable. According to Theorem A' to prove this it is sufficient to show that the open submanifold $M(n)$ of $F(n)'$ consisting of all the elements (x, y) of $F(n)$ for which $x \in RP^{n-1}$ and $y \not\subset RP^{n-1}$ is non-orientable. Since $y \cap RP^{n-1} = \{x\}$ for all $(x, y) \in M(n)$, $M(n)$ is homeomorphic to the Grassmannian $G_1(E^n)$. The rest of the proof of Theorem 4 is a consequence of the following

Lemma. *If n is even, then $G_1(E^n)$ is non-orientable.*

Proof. If $n=2$, then $G_1(E^2) = G_1(RP^2) - \{RP^1\} \approx RP^2 - RP^0$, therefore $G_1(E^2)$ is homeomorphic to the (open) Möbius band, and so $G_1(E^2)$ is non-orientable.

If $n>2$, choose a point o of E^n and denote by $\tilde{G}_1(E^n)$ the open submanifold of $G_1(E^n)$ consisting of all lines in E^n not passing through o. Consider the fibre bundle $\xi = (\tilde{G}_1(E^n), \tilde{p}, E^n - \{o\})$ where $\tilde{p}(y)$ is the orthogonal projection of the point o into the line y for all $y \in \tilde{G}_1(E^n)$. The fibre $\tilde{F}$ of ξ is homeomorphic to RP^{n-2}, thus $\tilde{F}$ is non-orientable. A direct application of Theorems 1 and A' concludes the proof of Lemma.

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ОРИЕНТИРУЕМОСТЬ ТОТАЛЬНЫХ ПРОСТРАНСТВ РАССЛОЕННЫХ ПРОСТРАНСТВ НАД RP^n

Милош Божек

Резюме

Основными результатами работы являются: 1) тотальное пространство локально тривиального расслоения с неориентируемым слоем является неориентируемым многообразием; 2) тотальное пространство расслоенного пространства $\xi = (E, p, RP^n)$, $n \geqq 2$, компактным связным ориентируемым слоем F ориентируемо тогда и только тогда, когда многообразие $E' = p^{-1}(RP^{n-1})$ неориентируемо. В качестве приложения решен вопрос об ориентируемости многообразия $F(n)$, точками которого являются все пары $(x, y) \in RP^n \times G_1(RP^n)$, для которых $x \in y$.