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Γ-REGULATOR AND Π'-REGULATOR OF A LATTICE ORDERED GROUP

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Let a representable l-group G be isomorphic to a subdirect sum of linearly ordered groups $\{G_x: x \in \mathfrak{R}\}$, $\bar{G}=(G_x: x \in \mathfrak{R})$. We endow the set $\mathfrak{R}$ with the so-called induced topology whose base for closed sets is given by the set $\mathfrak{F}=\{Z(f): f \in \bar{G}\}$, where $Z(f) = \{x \in \mathfrak{R}: f(x)=0\}$, [6] sec. 7, [8] I 1.5. The corresponding topological space is denoted by $(\mathfrak{R}, \bar{G})$. It is well-known that there exists a one-to-one correspondence γ associating with $\mathfrak{R}$ a realizer $(\mathfrak{R}, \bigcup)$ of G. By a realizer of G there is meant a set $\mathfrak{R}$ together with a mapping $\bigcup: x \in \mathfrak{R} \to \bigcup x$ into the set of all prime ideals of G fulfilling $\bigcap\{\bigcup x: x \in \mathfrak{R}\} = \{0\}$ ([8] II 3; [10] I 1.1). The property $f(x)=0$ means $f \in \bigcup x$. The set $\mathfrak{F}=\{Z(f): f \in \bar{G}\}$, where $Z(f) = \{x \in \mathfrak{R}: f \in \bigcup x\}$ is thus a base of closed sets for a topology on the set $\mathfrak{R}$. The corresponding topological space is denoted by $(\mathfrak{R}, G)$. The mapping γ is evidently a homeomorphism of the topological spaces $(\mathfrak{R}, \bar{G})$ and $(\mathfrak{R}, G)$.

We obtain a generalization of the notion of the topological space $(\mathfrak{R}, G)$ replacing the realizer by the so-called regulator. The notion of a regulator can be obtained from that of a realizer by replacing prime ideals by prime subgroups in the definition. Introducing the topology on a regulator in a similar way as above we extend the domain of applicability of the induced topology from the class of representable l-groups to the class of all l-groups. A number of results concerning topologies induced by representable l-groups [5—9] can be generalized to arbitrary l-groups (see [10]). As it is clear from above, topologies will be studied on indexed systems of prime subgroups restricted by the condition of zero intersection. We can meet with another approach to this problem in [1] by S. Bernau. He supposes that the prime subgroups of the system are z-subgroups and does not suppose the zero meet of the system.

1. The purpose of the present paper is to examine the Γ-regulator and the Π'-regulator of an l-group, their mutual relations and especially various degrees of amalgamation of these regulators. We may give a more detailed description of results after introducing necessary terminology and notations (see the beginning of sec. 2).

1.1 Definition. Let G be an l-group, $\mathfrak{R} \neq \emptyset$ a set and $\bigcup: x \in \mathfrak{R} \to \bigcup x \in \mathscr{P}(G)$ a mapping of $\mathfrak{R}$ into the set $\mathscr{P}(G)$ of all prime subgroups of G. The pair $(\mathfrak{R}, \bigcup)$ is called a *regulator* of G if $\bigcap\{\bigcup x: x\in\mathfrak{R}\} = \{0\}$. A regulator $(\mathfrak{R}, \bigcup)$ is said to be *standard* if $\bigcup x\neq G$ for every $x\in\mathfrak{R}$, [8] II 3. A regulator $(\mathfrak{R}, \bigcup)$ for which $\bigcup x \parallel \bigcup y$ whenever $x, y\in\mathfrak{R}$, $x\neq y$, is called *reduced.* A reduced regulator of an l-group $G\neq\{0\}$ is evidently standard. The mapping $\bigcup$ defines a partition $\bar{\mathfrak{R}}$ on $\mathfrak{R}$ and an injection $\bar{\bigcup}$ of $\bar{\mathfrak{R}}$ into $\mathscr{P}(G)$. The pair $(\bar{\mathfrak{R}}, \bar{\bigcup})$ is clearly a regulator of G; it is called the *simplification* of $(\mathfrak{R}, \bigcup)$. A regulator $(\mathfrak{R}, \bigcup)$ is said to be *completely regular* if there holds: $x\in\mathfrak{R}$, $f\in G$, $f\in\bigcup x \Rightarrow$ there exists $g\in G$ such that $f\delta g$, $g\bar{\in}\bigcup x$ (where $f\delta g$ means that $|f|\wedge|g|=0$, the disjointness of f and g). A regulator $(\mathfrak{R}, \bigcup)$ of G is called a *realizer* if $\bigcup x$ is a prime ideal of G for every $x\in\mathfrak{R}$ ([10] I 1.1).

In every l-group $G\neq\{0\}$ there exists a standard (even reduced completely regular) regulator while the existence of a realizer characterizes representable l-groups.

Instead of $(\mathfrak{R}, \bigcup)$ we often write $\mathfrak{R}$ supposing tacitly that the mapping $\bigcup$ is given.

We say that two regulators $(\mathfrak{R}_1, \bigcup_1)$ and $(\mathfrak{R}_2, \bigcup_2)$ of G are *equal* and we write $(\mathfrak{R}_1, \bigcup_1) = (\mathfrak{R}_2, \bigcup_2)$ if a bijection φ of $\mathfrak{R}_1$ onto $\mathfrak{R}_2$ exists such that $\bigcup_2\varphi x = \bigcup_1 x$ for every $x\in\mathfrak{R}_1$.

In [8] the symbol $x(\in\mathfrak{R})$ substitutes the associated subgroup $\bigcup x$ and hence by a regulator, there is meant there an indexed family of prime subgroups of G whose meet is $\{0\}$.

1.2 Definition. Let $(\mathfrak{R}, \bigcup)$ be a standard regulator of an l-group G. Define

$$\mathfrak{F}=\{Z(f): f\in G\}, \quad \text{where} \quad Z(f)=\{x\in\mathfrak{R}: f\in\bigcup x\}.$$

1.3 Theorem. *Let $(\mathfrak{R}, \bigcup)$ be a standard regulator of an l-group G. Then $\mathfrak{F}$ is a base of closed sets for a topology on* $\mathfrak{R}$, ([10] I 1.2).

This topology is called *the topology induced on $\mathfrak{R}$ by G.* The corresponding topological space is denoted by $(\mathfrak{R}, G)$.

1.4 Definition. Denote by $\Gamma(G)$ the Boolean algebra of all polars of G. By the symbol K' we mean the complement in the algebra $\Gamma(G)$ of $K\in\Gamma(G)$, by the symbol $\Pi'(G)$ the set $\{f': f\in G\}$ of all dual principal polars of G and by $\Pi(G)$ the set $\{f'': f\in G\}$ of all principal polars of G. Here $f' = \{f\}' = \{g\in G: f\delta g\}$, $f''=(f')'$ and $f\delta g \equiv |f|\wedge|g|=0$. Thus $K'=\{g\in G: f\delta g$ for every $f\in K\}$.

1.5 Definition. An *antifilter* on a lattice Ξ is a nonempty subset $x\subseteq\Xi$ fulfilling: 1. $K\in x$, $L\in\Xi$, $L\leqslant K \Rightarrow L\in x$; 2. $K, L\in x \Rightarrow K\vee L\in x$; 3. The greatest element of Ξ (provided it exists) does not belong to x. A maximal antifilter (with regard to the inclusion) is called an *ultraantifilter.* The set of all ultraantifilters on Ξ will be denoted by $\mathfrak{U}(\Xi)$.

1.6 The sets $\mathfrak{U}(\Xi)$ where Ξ is $\Gamma(G)$ or $\Pi'(G)$ or $\Pi(G)$, play a significant role in the theory. If, in this case, $x\in\mathfrak{U}(\Xi)$ holds, we define $\bigcup x = \bigcup\{K: K\in x\}$. If $\bigcup x\neq G$, we speak of a *standard ultraantifilter*, [8] II 4.10. Every $x\in\mathfrak{U}(\Pi')$ is standard, [8] II 4.11, while $x\in\mathfrak{U}(\Gamma)$ is standard iff $x\cap\Pi'\neq\emptyset$, [8] II 4.12. The set of all standard ultraantifilters on $\Gamma(G)$ is denoted by $\mathfrak{U}_s(\Gamma)$.

For $x\in\mathfrak{U}(\Xi)$, where $\Xi=\Gamma(G)$ or $\Pi'(G)$ or $\Pi(G)$, $\bigcup x$ is a prime subgroup of G; $(\mathfrak{U}_s(\Gamma), \bigcup)$ and $(\mathfrak{U}(\Pi'), \bigcup)$ (briefly $\mathfrak{R}_\Gamma(G)$ and $\mathfrak{R}_{\Pi'}(G)$ or $\mathfrak{R}_\Gamma$ and $\mathfrak{R}_{\Pi'}$ only) are standard regulators of G. The latter is reduced and completely regular, [8] II 4.15 and 4.16 (see also [10] II 1.5(1)). $\mathfrak{R}_\Gamma$ or $\mathfrak{R}_{\Pi'}$ is called the *Γ-regulator* or the *Π'-regulator*, respectively.

1.7 Definition. On the set $\mathfrak{U}$, where $\mathfrak{U}=\mathfrak{U}_s(\Gamma)$ or $\mathfrak{U}(\Xi)$ (Ξ a lattice), we define a topology whose base for open sets is given by

$$\Sigma'=\{\mathfrak{U}f': f\in G\} \text{ or } \Sigma=\{\mathfrak{U}K: K\in\Xi\}, \text{ where } \mathfrak{U}K=\{x\in\mathfrak{U}: K\in x\},$$

[5] 1.9; [7] IV 1.10. (In the case $\mathfrak{U}=\mathfrak{U}_s(\Gamma)$ in [5] the symbol $\mathfrak{V}f'$ is used instead of $\mathfrak{U}f'$, which is reserved for another notion.)

If necessary, the notation is specified by $\mathfrak{U}_\Gamma f'$ in the former case and by $\mathfrak{U}_\Xi K$ in the latter. We shall apply the definition for $\Xi=\Pi'(G)$ or $=\Pi(G)$.

1.8 Theorem. *The topological spaces* $(\mathfrak{U}_s(\Gamma), \Sigma')$ *and* $(\mathfrak{R}_\Gamma, G)$ *are homeomorphic and the topological spaces* $(\mathfrak{U}(\Pi'), \Sigma)$ *and* $(\mathfrak{R}_\Pi, G)$ *are homeomorphic.*

See [5] 2.1, [8] II 4.18. Both assertions follow easily from the following lemma.

1.9 Lemma. *For* $x\in\mathfrak{U}(\Gamma)$ *or* $x\in\mathfrak{U}(\Pi')$ *and* $f\in G$ *we have*

$$f'\in x\equiv f\bar{\in}\bigcup x.$$

[7] Lemma 1, [8] II 4.6.

1.10 Definition. Let G and $\mathfrak{R}$ be nonempty sets and $\bigcup: \mathfrak{R}\to\exp G$ a mapping. We define a binary relation (a polarity) $\varrho\subseteq G\times\mathfrak{R}$ by the rule $f\varrho x\equiv f\in\bigcup x$.

Define

$$\Psi(A)=\{f\in G: f\varrho x \text{ for every } x\in A\} \quad (\emptyset\subseteq A\subseteq\mathfrak{R}),$$

$$Z(P)=\{x\in\mathfrak{R}: f\varrho x \text{ for every } f\in P\} \quad (\emptyset\subseteq P\subseteq G).$$

As $A=\{x\}$ or $P=\{f\}$, we have $\Psi(\{x\})=\bigcup x$ or $Z(\{f\}) = Z(f)$, respectively (Definition 1.2). Instead of $\Psi(\{x\})$ we put $\Psi(x)$. Ψ and Z are dual isotone mappings between the sets $\exp\mathfrak{R}$ and $\exp G$.

A straightforward computation shows that the following lemmas are true (see [10] 2.2, 2.3 and 2.4).

1.11 Lemma. *For every* $\emptyset\subseteq A\subseteq\mathfrak{R}$ *and* $\emptyset\subseteq P\subseteq G$ *there holds*

$$\Psi(A)=\bigcap\{\bigcup x: x\in A\}=\{f\in G: Z(f)\supseteq A\},$$

$$Z(P)=\bigcap\{Z(f): f\in P\}=\{x\in\mathfrak{R}: \bigcup x\supseteq P\}.$$

1.12 Lemma. $A \subseteq B \subseteq \mathfrak{R}$ *implies* $\Psi(A) \supseteq \Psi(B)$,

$$\Psi(A) = G \equiv A \subseteq \{x \in \mathfrak{R} : \bigcup x = G\}; \quad \Psi(\mathfrak{R}) = \bigcap \{\bigcup x : x \in \mathfrak{R}\};$$

$$P \subseteq Q \subseteq G \text{ implies } Z(P) \supseteq Z(Q),$$

$$Z(P) = \mathfrak{R} \equiv P \subseteq \bigcap \{\bigcup x : x \in \mathfrak{R}\}, \quad Z(G) = \{x \in \mathfrak{R} : \bigcup x = G\};$$

$$Z\Psi(A) \supseteq A, \ \Psi Z \Psi(A) = \Psi(A) \ (A \subseteq \mathfrak{R})$$

$$\Psi Z(P) \supseteq P, \ Z\Psi Z(P) = Z(P) \ (P \subseteq G).$$

1.13 From 1.12 it follows that the mapping $Z\Psi$: $\exp \mathfrak{R} \to \exp \mathfrak{R}$ is a closure operation in $\mathfrak{R}$. The $Z\Psi$-images of the elements of $\exp \mathfrak{R}$ (i.e. the subsets of $\mathfrak{R}$ *closed* under ϱ) are exactly the closed sets of the topological space $(\mathfrak{R}, G)$. Then the system of all closed sets of the space $(\mathfrak{R}, G)$, $\mathfrak{N}(\mathfrak{R}, G)$ is given by

$$\mathfrak{N}(\mathfrak{R}, G) = \{Z\Psi(A) : A \subseteq \mathfrak{R}\} = \{Z(P) : P \subseteq G\} = \left\{\bigcap_{f \in P} Z(f) : P \subseteq G\right\},$$

[10] I 2.9.

Similarly, the mapping ΨZ: $\exp G \to \exp G$ is a closure operation in G. The system of all subsets of G *closed* under ϱ is denoted by $\Omega(\mathfrak{R}, G)$. Thus we have

$$\Omega(\mathfrak{R}, G) = \{\Psi Z(P) : P \subseteq G\} = \{\Psi(A) : A \subseteq \mathfrak{R}\} = \left\{\bigcap_{x \in A} \Psi(x) : A \subseteq \mathfrak{R}\right\},$$

[10] I 2.11.

1.14 Theorem. *The mappings Ψ and Z are (mutually inverse) dual isomorphisms of the systems $\Omega(\mathfrak{R}, G)$ and $\mathfrak{N}(\mathfrak{R}, G)$ ordered by inclusion.*
[10] I 2.12. (Indeed, by Ψ or Z there is meant the restriction of Ψ or Z on $\mathfrak{N}(\mathfrak{N}, G)$ or $\Omega(\mathfrak{R}, G)$, respectively.)

1.15 Denote by $\mathfrak{M}(\mathfrak{R}, G)$ or $\mathcal{O}(\mathfrak{R}, G)$ (briefly $\mathfrak{M}_{\mathfrak{R}}$ or $\mathcal{O}_{\mathfrak{R}}$) the lattice of all regular closed sets or all clopen sets of the space $(\mathfrak{R}, G)$, respectively and by $\Gamma(\mathfrak{R}, G)$ (briefly $\Gamma_{\mathfrak{R}}$) the lattice of all ambiguous polars of G, i. e. polars $K \in \Gamma(G)$ with the property $x \in \mathfrak{R}$, $K \subseteq \bigcup x \Rightarrow K' \nsubseteq \bigcup x$. Then there holds

$$Z(\Gamma) = \mathfrak{M}_{\mathfrak{R}}, \quad \Psi(\mathfrak{M}_{\mathfrak{R}}) = \Gamma, \quad Z(\Gamma_{\mathfrak{R}}) = \mathcal{O}_{\mathfrak{R}}, \quad \Psi(\mathcal{O}_{\mathfrak{R}}) = \Gamma_{\mathfrak{R}},$$

[10] I 2.18 and II 2.2.

2. The aim of the present paper is to eastablish relations between the Γ-regulator and the Π'-regulator of an l-group, especially various degrees of amalgation of these regulators. The results are contained in Theorems 3.1, 4.1 and 4.2. In Theorem 3.1 there are described conditions under which the simplification of the Γ-regulator is equal to the Π'-regulator and in Theorem 4.1 conditions under which the Γ-regulator itself is equal to the Π'-regulator. In both cases the conditions are classified according to the describing objects, which are: the regulators, the ultraantifilters, the topology induced on $\mathfrak{U}_s(\Gamma)$, $\mathfrak{U}(\Pi')$, $\mathfrak{U}(\Pi)$, $\mathfrak{R}_\Gamma$ or $\mathfrak{R}_{\Pi'}$, the structures in G, the relations between the structures in G and the topology induced in $\mathfrak{R}_\Gamma$. In Theorem 4.2, the same problem as in Theorem 4.1 is examined on the supposition that the l-group contains a weak unit. This supposition is of much importance in 3.1.

2.2 Lemma. *Let G be an l-group and $x \in \mathfrak{U}_s(\Gamma)$. Then the following conditions are equivalent.*

(1) $x \cap \Pi' \in \mathfrak{U}(\Pi')$.
(2) $\bigcup x = \bigcup(x \cap \Pi')$.
(3) *There exists* $y \in \mathfrak{U}(\Pi')$ *such that* $\bigcup x = \bigcup y$ *(i.e.* $\bigcup x$ *is a minimal prime subgroup of G).*
(4) $y \in \mathfrak{U}_s(\Gamma)$, $y \supseteq x \cap \Pi'$ *implies* $y \cap \Pi' = x \cap \Pi'$.

(See [5] 2.24).

Proof. $1 \Rightarrow 3$. By 1.9 there holds

$$f \in \bigcup x \Rightarrow f' \bar{\in} x \Rightarrow f' \bar{\in} x \cap \Pi' \Rightarrow f \in \bigcup(x \cap \Pi'),$$

thus $\bigcup x \subseteq \bigcup(x \cap \Pi')$. The converse inclusion holds as well. Hence we can take $x \cap \Pi'$ for y. $\bigcup x$ is a minimal prime subgroup of G by [8] III 7.2 or [2] 3.4.15.

$3 \Rightarrow 2$. Pick $y \in \mathfrak{U}(\Pi')$ such that $\bigcup x = \bigcup y$. Then $y = x \cap \Pi'$ for

$$f' \in x \Leftrightarrow f \bar{\in} \bigcup x = \bigcup y \Leftrightarrow f' \in y$$

by 1.9, and so $y = x \cap \Pi'$ and thus $\bigcup x = \bigcup y = \bigcup(x \cap \Pi')$.

$2 \Rightarrow 4$. Fix $y \in \mathfrak{U}_s(\Gamma)$ with $y \supseteq x \cap \Pi'$. Then $y \cap \Pi' \supseteq x \cap \Pi'$, hence $\bigcup y \supseteq \bigcup(y \cap \Pi') \supseteq \bigcup(x \cap \Pi') = \bigcup x$. Thus $\bigcup y \supseteq \bigcup x$ and by [8] II 4.13 $y \cap \Pi' \subseteq x \cap \Pi'$ holds. Finally $y \cap \Pi' = x \cap \Pi'$.

$4 \Rightarrow 1$. Choose $z \in \mathfrak{U}(\Pi')$ with $z \supseteq x \cap \Pi'$ and $y \in \mathfrak{U}_s(\Gamma)$ such that $y \supseteq z$. Then $y \supseteq x \cap \Pi'$, thus $z = y \cap \Pi' = x \cap \Pi'$, i.e. $x \cap \Pi' \in \mathfrak{U}(\Pi')$.

2.3 Definition. Let P be a topological space. We define an equivalence on P, R_b by the rule
$xR_by \equiv \bar{x} = \bar{y}$, where the bar indicates the closure in P. The partition on P corresponding to the equivalence will be denoted by R_b as well. We call the atoms of the system of all closed subsets of P (ordered by inclusion) *trivial closed* subsets (of P). Similarly for open and clopen sets.

2.4 Lemma. *If all blocks of a partition R on P are trivial closed sets, then* $R = R_b$.

Proof. If $T \in R$ and $x \in T$, then $\emptyset \neq \bar{x} \subseteq \bar{T} = T$. From the minimality of T it follows that $\bar{x} = T$. Thus, if x, $y \in T$ then $\bar{x} = T = \bar{y}$, i.e. xR_by. Conversely, if xR_by and $x \in T \in R$, then $\bar{x} = \bar{y}$ and, as above, $\bar{x} = T$. Hence $y \in \bar{y} = \bar{x} = T$.

3. Let G be an l-group and $(\mathfrak{R}, \bigcup)$ a standard regulator of G. Denote by $\mathcal{O}_c(\mathfrak{R}, G)$ the family of all compact clopen sets of the space $(\mathfrak{R}, G)$.

We shall consider the following conditions (*), (**) and (1a) to (4f).

(*) $0 \neq a \in G$ implies that there exists $b \in G$ with $\{0\} \neq b' \subseteq a''$.

(**) G has a weak unit.

I. (1a) The simplification of the Γ-regulator is equal to the Π'-regulator (i.e. $\{\bigcup x: x \in \mathfrak{U}_s(\Gamma)\} = \{\bigcup y: y \in \mathfrak{U}(\Pi')\}$).

(1b) The simplification of the Γ-regulator is a reduced regulator (i.e. $x, y \in \mathfrak{U}_s(\Gamma)$, $\bigcup x \neq \bigcup y$ implies $\bigcup x \parallel \bigcup y$).

(1c) The Γ-regulator is completely regular.

(2a) $x \in \mathfrak{U}_s(\Gamma)$ implies $x \cap \Pi' \in \mathfrak{U}(\Pi')$.

(2b) $x \in \mathfrak{U}_s(\Gamma)$ implies $\bigcup x = \bigcup(x \cap \Pi')$.

(2c) $x \in \mathfrak{U}_s(\Gamma)$ implies that there exists $y \in \mathfrak{U}(\Pi')$ such that $\bigcup x = \bigcup y$ (i.e. $\bigcup x$ is a minimal prime subgroup of G).

(2d) $x, y \in \mathfrak{U}_s(\Gamma)$, $y \supseteq x \cap \Pi'$ implies $y \cap \Pi' = x \cap \Pi'$.

(3a) The trivial closed subsets of the space $(\mathfrak{U}_s(\Gamma), \Sigma')$ form a partition of $\mathfrak{U}_s(\Gamma)$.

(3b) The blocks of the partition R_b on the space $(\mathfrak{U}_s(\Gamma), \Sigma')$ are (trivial) closed sets.

(3c) The space $(\mathfrak{U}_s(\Gamma), \Sigma')$ has a base for open sets formed by closed sets.

(3d) $\mathfrak{U}_\Gamma f'$ is a closed set of the space $(\mathfrak{U}_s(\Gamma), \Sigma')$ for every $f \in G$.

II. (3e) The space $(\mathfrak{U}(\Pi'), \Sigma)$ is compact.

(3f) The space $(\mathfrak{U}(\Pi), \Sigma)$ is compact and G fulfils the condition (*).

(3g) $\mathfrak{U}_s(\Gamma) \setminus \mathfrak{U}_\Gamma f' \in \Sigma'$ for every $f \in G$.

(3h) The space $(\mathfrak{R}_{\Pi'}, G)$ is compact.

(3 i) $\mathcal{O}(\mathfrak{R}_{\Pi'}, G) = \mathcal{O}_c(\mathfrak{R}_{\Pi'}, G)$.

(4a) $\Pi(G) = \Pi'(G)$.

(4b) The lattice $\Pi(G)$ is a Boolean algebra.

(4c) The lattice $\Pi'(G)$ is a Boolean algebra.

(4d) $\Psi[\mathcal{O}_c(\mathfrak{R}_{\Pi'}, G)] = \Pi(G)$.

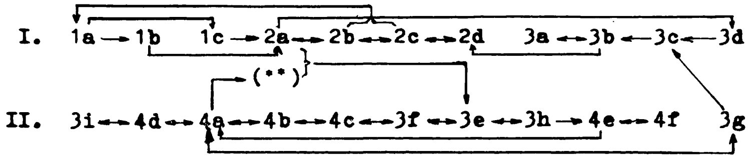
(4e) $\Gamma(\mathfrak{R}_{\Pi'}, G) = \Pi(G)$.

(4f) $\Gamma(\mathfrak{R}_{\Pi'}, G) = \Pi'(G)$.

The following theorem deals with the above mentioned conditions.

3.1 Theorem. *Let G be an l-group $\neq \{0\}$. Then the conditions of sec. I are equivalent. The conditions of sec. II are equivalent as well. Each of them implies the conditions of sec. I and the existence of a weak unit in G. If G has a weak unit, all the conditions* (1a) *to* (4f) *are equivalent.*

Scheme of the proof.



The equivalence of (2a) to (2d) follows from 2.2. Furthemore, we shall prove the following implications: (2b) $\wedge$ (2c) $\Rightarrow$ (1a) $\Rightarrow$ (1b) $\Rightarrow$ (2a), (1a) $\Rightarrow$ (1c) $\Rightarrow$ (2a) $\Rightarrow$ (3d), (3a) $\Leftrightarrow$ (3b) $\Rightarrow$ (2d), (3d) $\Rightarrow$ (3c) $\Rightarrow$ (3b). Thus the equivalence of the conditions of sec. I will be verified.

The equivalence of (4a), (4b), (4c), (3e) and (3f) follows from [10] I 1.9, (3e) $\Leftrightarrow$ (3h) by [10] I 1.7 and (4a) $\Leftrightarrow$ (4d) $\Leftrightarrow$ (3i) from [10] II 2.6 (see also 1.15 and 1.14). If we show (3g) $\Leftrightarrow$ (4a), (3h) $\Rightarrow$ (4e) $\Rightarrow$ (4a) and (4e) $\Rightarrow$ (4f), the equivalence of the conditions of sec. II will be established.

(4a) evidently implies (**). The implication (3g) $\Rightarrow$ (3c) follows directly from the definition of the base Σ', hence II $\Rightarrow$ I. It suffices to prove that (2a)$\wedge$(**) $\Rightarrow$ (3e) for the equivalence of all the conditions (1a) to (4f) to be verified, provided that G has a weak unit.

(2b)$\wedge$(2c) $\Rightarrow$ (1a). From (2c) there follows the inclusion $\subseteq$ between the compared sets in (1a) and from (2b) the converse inclusion, because for $y \in \mathfrak{U}(\Pi')$ and $x \in \mathfrak{U}_s(\Gamma)$ with $x \supseteq y$ there holds $y = x \cap \Pi'$, hence $\bigcup x = \bigcup(x \cap \Pi') = \bigcup y$.

(1a) $\Rightarrow$ (1b) follows from [8] II 4.16.

(1b) $\Rightarrow$((2a). If $x \in \mathfrak{U}_s(\Gamma)$ and the antifilter $x \cap \Pi'$ ($\neq \emptyset$ by [8] II 4.12) on Π' is not maximal, then there exists $z \in \mathfrak{U}(\Pi')$ with $z \supseteq x \cap \Pi'$ and $z \neq x \cap \Pi'$. Choose $y \in \mathfrak{U}_s(\Gamma)$ such that $y \supseteq z$. Then $y \cap \Pi' \supseteq x \cap \Pi'$, thus by [5] 2.8 or [8] II 4.6 $\bigcup y \subseteq \bigcup x$. By supposition $\bigcup y = \bigcup x$, hence by [5] 2.8 again, we have $z = y \cap \Pi' = x \cap \Pi'$, a contradiction.

(1a) $\Rightarrow$ (1c). Since the Π'-regulator is completely regular ([8] II 4.16), the Γ-regulator is completely regular.

(1c) $\Rightarrow$ (2a). Fix $x \in \mathfrak{U}_s(\Gamma)$ and for some $f \in G$, let $f' \bar{\in} x \cap \Pi'$. By 1.9 $f \in \bigcup x$ holds and by supposition, there exists $g \in G$ such that $f \delta g$ and $g \bar{\in} \bigcup x$. By 1.9 there holds $g' \in x$ and $f \delta g$ implies $f' \vee_\Gamma g' = G$. Thus $x \cap \Pi' \in \mathfrak{U}(\Pi')$.

(2a) $\Rightarrow$ (3d). [5] 2.16.

(3a) $\Rightarrow$ (3b). Lemma 2.4.

(3b) $\Rightarrow$ (3a). Since the blocks of the partition R_b are closed sets, for every $z \in \mathfrak{U}_s(\Gamma)$, $\bar{z}$ is a block of the partition R_b. Indeed, for $T \in R_b$ and $u, z \in T$ there holds $\bar{z} \subseteq \bar{T} = T$ and $u \in \bar{u} = \bar{z}$, hence $\bar{z} = T$. Every block $\bar{z}$ of the partition R_b is trivial closed, for there holds $A \subseteq \mathfrak{U}_s(\Gamma)$, $\emptyset \neq \bar{A} \subseteq \bar{z}$, $x \in \bar{A} \Rightarrow \bar{x} \subseteq \bar{A} \subseteq \bar{z} \Rightarrow \bar{x} = \bar{A} = \bar{z}$.

(3b) $\Rightarrow$ (2d). For every $z \in \mathfrak{U}_s(\Gamma)$, $\bar{z}$ is a block of the partition R_b (as in the above paragraph). Pick $x, y \in \mathfrak{U}_s(\Gamma)$ such that $y \supseteq x \cap \Pi'$. Then $y \cap \Pi' \supseteq x \cap \Pi'$, hence $x \in \bar{y}$ (see [5] 1.11). It follows that $\bar{y} = \bar{x}$ and $y \in \bar{x}$, which implies (by [5] 1.11 again) $y \cap \Pi' \subseteq x \cap \Pi'$. Hence the required equality.

(3d) $\Rightarrow$ (3c) is evident.

(3c) $\Rightarrow$ (3b). [8] IV 9.2.

(3g) $\Leftrightarrow$ (4a). [5] 2.7.

(3h) $\Rightarrow$ (4e). By [10] II 2.6 and 2.2 we have $\Pi(G) = \Psi[\mathcal{O}(\mathfrak{R}_{\Pi'}, G)] = \Gamma(\mathfrak{R}_{\Pi'}, G)$.

(4e) $\Rightarrow$ (4a). By the definition of an ambiguous polar there holds $K \in \Gamma(\mathfrak{R}_{\Pi'}, G) \Rightarrow K' \in \Gamma(\mathfrak{R}_{\Pi'}, G)$, hence from $\Pi(G) = \Gamma(\mathfrak{R}_{\Pi'}, G)$ it follows that $\Pi'(G) \subseteq \Gamma(\mathfrak{R}_{\Pi'}, G)$ and $\Pi'(G) \subseteq \Pi(G)$. Consequently $\Pi'(G) = \Pi(G)$.

(4e) $\Rightarrow$ (4f). This is an immediate consequence of the fact just proved, viz. (4e) $\Rightarrow$ (4a).

(4f) $\Rightarrow$ (4e). In a similar way as in (4e) $\Rightarrow$ (4a) we prove (4f) $\Rightarrow$ (4a) and hence (4e).

(2a)$\wedge$(**) $\Rightarrow$ (3e). $\varphi: x \in \mathfrak{U}_s(\Gamma) \to x \cap \Pi'$ is a mapping onto $\mathfrak{U}(\Pi')$, because for $y \in \mathfrak{U}(\Pi')$ and $x \in \mathfrak{U}_s(\Gamma)$ such that $x \supseteq y$ there evidently holds $x \cap \Pi' = y$. The space $(\mathfrak{U}_s(\Gamma), \Sigma')$ is compact by [5] 3.3, thus it suffices to prove that φ is continuous and for this it suffices to prove $\varphi^{-1}(\mathfrak{U}_{\Pi'} f') = \mathfrak{U}_\Gamma f'$ for every $f \in G$. Since $\varphi^{-1}(\mathfrak{U}_{\Pi'} f') = \{x \in \mathfrak{U}_s(\Gamma) : f' \in x \cap \Pi'\}$, there holds $\varphi^{-1}(\mathfrak{U}_{\Pi'} f') = \mathfrak{U}_\Gamma f'$, completing the proof of Theorem.

4. Let G be an l-group. Denote

I. (1a) The Γ-regulator of G is equal to the Π'-regulator of G.
- (1b) The Γ-regulator of G is completely regular and reduced.
- (1c) The Γ-regulator of G is reduced.
- (2a) $(\mathfrak{U}_s(\Gamma), \Sigma')$ is a T_1-space.
- (2b) $(\mathfrak{U}_s(\Gamma), \Sigma')$ is a Hausforff space.
- (2c) $(\mathfrak{U}_s(\Gamma), \Sigma')$ and $(\mathfrak{U}(\Pi'), \Sigma)$ are homeomorphic spaces.
- (3a) $x \in \mathfrak{U}_s(\Gamma) \Rightarrow x \cap \Pi' \in \mathfrak{U}(\Pi')$ and x is a unique ultraantifilter on $\Gamma(G)$ containing $x \cap \Pi'$.

II. (3b) $x \in \mathfrak{U}_s(\Gamma)$, $K \in x \Rightarrow$ there exists $f \in G$ with $f' \in x$, $f' \supseteq K$ (this is the condition (p) introduced in [5] 2.22).
- (3c) $x \in \mathfrak{U}_s(\Gamma) \Rightarrow$ there exists no $K \in \Gamma(G)$ such that $K \cup K' \subseteq \bigcup x$.
- (3d) $x \in \mathfrak{U}_s(\Gamma)$, $K \in \Gamma(G)$, $K \subseteq \bigcup x \Rightarrow K \in x$.
- (3e) $x \in \mathfrak{U}_s(\Gamma)$, $K \in \Gamma(G)$, $K \bar{\in} x \Rightarrow$ there exists $f \in G$ such that $f' \in x$, $K \vee_\Gamma f' = G$.
- (4a) The space $(\mathfrak{R}_\Gamma, G)$ is extremally disconnected (i.e. closures of open sets are open).
- (4b) The lattice $\mathfrak{M}(\mathfrak{R}_\Gamma, G)$ is a sublattice of the lattice $\mathfrak{N}(\mathfrak{R}_\Gamma, G)$.

(4c) $\mathcal{O}(\mathfrak{R}_\Gamma, G) = \mathfrak{M}(\mathfrak{R}_\Gamma, G)$.
(4d) For the complement A' in the lattice $\mathfrak{M}(\mathfrak{R}_\Gamma, G)$ of an arbitrary element $A \in \mathfrak{M}(\mathfrak{R}_\Gamma, G)$, there holds $A \cap A' = \emptyset$.
(5a) The lattice $\Gamma(G)$ is a sublattice of the lattice $\Omega(\mathfrak{R}_\Gamma, G)$.
(5b) Ψ and Z are bijections of the sets $\mathcal{O}(\mathfrak{R}_\Gamma, G)$ and $\Gamma(G)$.
(5c) $\Gamma(G) = \Gamma(\mathfrak{R}_\Gamma, G)$.
(6a) $\Gamma(G) = \Pi(G)$.
(6b) $\Gamma(G) = \Pi'(G)$.

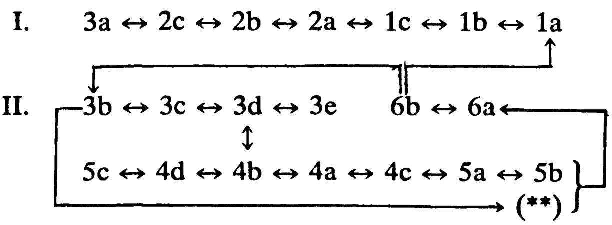
Note. We may replace the space $(\mathfrak{U}_s(\Gamma), \Sigma')$ or $(\mathfrak{U}(\Pi'), \Sigma)$ in (2a), (2b) and (2c) by the spaces $(\mathfrak{R}_\Gamma, G)$ or $(\mathfrak{R}_{\Pi'}, G)$, respectively (1.8).

4.1 Theorem. *Let G be an l-group $\neq \{0\}$. Then the conditions of sec. I are equivalent. The conditions of sec. II are equivalent as well. Each of them implies the conditions of sec. I and the existence of a weak unit in G.*

Note. It is not true, in contrast to Theorem 3.1, that all the conditions (1a) to (6b) are equivalent as soon as G has a weak unit. We give an example of an l-group G which fulfils (1c) and does not fulfil (6a). (This example is due to J. Jakubík.)

Let I be an infinite set, for each $i \in I$ let H_i be a linearly ordered group, $H_i \neq \{0\}$ and $H = \Sigma\{H_i: i \in I\}$. Let C be the additive group of integers (with the natural order) and $G = C \circ H$, where $\circ$ is the symbol of the lexicographic product. Then G has a weak unit and (6a) does not hold. We shall prove that (1c) is true in G. It suffices to verify that H fulfils (2c) because evidently $\Gamma(G)\setminus\{G\} = \Gamma(H)\setminus\{H\}$. Now let $x, y \in \mathfrak{U}_s(\Gamma)$, $K \in x$ and $K \subseteq \bigcup y$. Then a set A of elements of y covers K, thus $\bigvee A \geq K$. The lattice $\Gamma(H)$ is compactly generated (because G has a base — see [11], Satz 3) thus there exists a finite subset B of A with $\bigvee B \geq K$. However, $K = \bigvee B \in y$, and so $\bigcup x \subseteq \bigcup y$ implies $x \subseteq y$ and therefore $x = y$. Finally, $x \neq y$ implies $\bigcup x \parallel \bigcup y$.

Scheme of the proof.



The equivalence of (1c), (2a), (2b), (2c) and (3a) is proved in [5] 2.18. If we show the equivalence of (1a), (1b) and (1c), the equivalence of all the conditions of sec. I will be verified.

The equivalence of (3b), (3c), (3d) and (3e) is shown in [5] 2.23, that of (4a), (4b), (4d) and (5c) in [10] I 2.12, and that of (4a), (4c), (5a) and (5b) in [10] II 2.5. As far as we prove (4d) $\Leftrightarrow$ (3c), we have got the equivalence of the conditions (3b) to (5c). The claims (6a) $\Leftrightarrow$ (6b) and (6b) $\Rightarrow$ (3b) are evident. Furthermore, we shall show that (3b) implies (**), the existence of a weak unit in G. This enables us to prove the implication (5b) $\Rightarrow$ (6a). Thus the equivalence of the conditions of sec. II will be established. The evident implication (6b) $\Rightarrow$ (1a) means that II $\Rightarrow$ I. This completes the proof of Theorem 4.1.

(1c) $\Leftrightarrow$ (1b) $\Leftrightarrow$ (1a). The assumption (1c) implies the condition **3**(1b), hence **3**(1a), and this together with (1c) implies (1b) (for the Π'-regulator is completely regular) and (1a). The implication (1b) $\Rightarrow$ (1c) is evident, (1a) $\Rightarrow$ (1c) follows from [8] II 4.16.

(4d) $\Leftrightarrow$ (3c). If (3c) is not true, then $x \in \mathfrak{U}_s(\Gamma)$ and $K \in \Gamma$ exist such that $K \cup K' \subseteq \bigcup x$. Hence for $Z = Z_{\mathfrak{R}_\Gamma}$ we have $x \in Z(\bigcup x) \subseteq Z(K) \cap Z(K')$. Consequently the meet of the complementary elements $Z(K)$ and $Z(K')$ of the lattice $\mathfrak{M}(\mathfrak{R}_\Gamma, G)$ is nonempty, i.e. (4d) does not hold. Conversely, if (4d) does not hold, then the meet of some pair of complementary sets $A, A' \in \mathfrak{M}(\mathfrak{R}_\Gamma, G)$ is nonempty. If $x \in A \cap A'$ for some $x \in \mathfrak{U}_s(\Gamma)$, then $\bigcup x = \Psi(x) \supseteq \Psi(A) \cup \Psi(A')$, hence (3c) is not true, for $\Psi(A)$ and $\Psi(A')$ are complementary polars of G.

(3b) $\Rightarrow$ (**) (the existence of a weak unit of G). There holds $\mathfrak{U}_s(\Gamma) = \mathfrak{U}(\Gamma)$, because for $x \in \mathfrak{U}(\Gamma)$ $x \cap \Pi' \neq \emptyset$ ([8] II 4.12). The equality $\mathfrak{U}_s(\Gamma) = \mathfrak{U}(\Gamma)$ is equivalent to the existence of a weak unit in G ([8] V 12.6).

(5b)$\wedge$(**) $\Rightarrow$ (6a). By [5] 3.3, the space $(\mathfrak{U}_s(\Gamma), \Sigma')$ is compact, thus the homeomorphic space $(\mathfrak{R}_\Gamma, G)$ is compact as well (1.8). Since (3b) $\Rightarrow$ (3a) holds by [5] 2.24, there holds $\mathfrak{R}_\Gamma = \mathfrak{R}_{\Pi'}$, and so the space $(\mathfrak{R}_{\Pi'}, G)$ is compact. By [10] II 2.8, $\Psi_{\mathfrak{R}_{\Pi'}}$ maps $\mathcal{O}(\mathfrak{R}_{\Pi'}, G)$ onto $\Pi(G)$, hence $\Psi_{\mathfrak{R}_\Gamma}[\mathcal{O}(\mathfrak{R}_\Gamma, G)] = \Pi(G)$. By [10] II 2.6, from (5b) it follows that $\Psi_{\mathfrak{R}_\Gamma}$ maps $\mathcal{O}(\mathfrak{R}_\Gamma, G)$ onto $\Gamma(G)$. Consequently $\Gamma(G) = \Pi(G)$.

Recall that a subgroup A of an l-group G is said to be a *z-subgroup* if $f \in A \Rightarrow f'' \subseteq A$, [2] 3.3.8. A regulator $(\mathfrak{R}, \bigcup)$ is called a *z-regulator* if $\bigcup x$ is a z-subgroup for every $x \in \mathfrak{R}$, [10] II 2.23.

4.2 Theorem. *Let G be an l-group $\neq \{0\}$. The following conditions are equivalent.*

(a) *G has a weak unit und fulfils one of the conditions of sec. I, Theorem* 4.1.

(b) *$(\mathfrak{U}_s(\Gamma), \Sigma')$ is a Hausdorff compact space.*

(c) *$\Gamma(\mathfrak{R}_\Gamma, G) = \Pi(G)$ and $\mathfrak{R}_\Gamma$ is a z-regulator.*

(d) *$\Gamma(\mathfrak{R}_\Gamma, R) = \Pi'(G)$ and $\mathfrak{R}_\Gamma$ is a z-regulator.*

Proof. a $\Rightarrow$ b. From the condition (1a), Theorem 4.1, there follows (1a), Theorem 3.1, hence the space $(\mathfrak{U}(\Pi'), \Sigma)$ is compact by Theorem 3.1. Moreover, from (1a) ($\equiv$(2c)$\equiv$(2b)), Theorem 4.1, it follows that $(\mathfrak{U}_s(\Gamma), \Sigma')$ is compact and Hausdorff.

b $\Rightarrow$ c. If $(\mathfrak{U}_s(\Gamma), \Sigma')$ is a compact Hausdorff space, then by 4.1 ((2b) ≡ (2c)), $(\mathfrak{U}(\Pi'), \Sigma)$ is compact, hence by 3.1 ((3c) ≡ (4e)), $\Gamma(\mathfrak{R}_{\Pi'}, G) = \Pi(G)$. Again by 4.1 ((2b) ≡ (1a)), $\Gamma(\mathfrak{R}_\Gamma, G) = \Pi(G)$ and by 4.1 ((2b) ≡ (1b)), $\mathfrak{R}_\Gamma$ is completely regular, thus by [10] II 1.4 $Z(f'') = Z(f)$ for every $f \in G$ (with $Z = Z_{\mathfrak{R}_\Gamma}$). Using [10] II 2.2a, $\mathfrak{R}_\Gamma$ is a z-regulator.

c $\Leftrightarrow$ d follows immediately from the fact that evidently $K \in \Gamma(\mathfrak{R}, G) \Rightarrow K' \in \Gamma(\mathfrak{R}, G)$ for an arbitrary regulator $(\mathfrak{R}, \bigcup)$.

d $\Rightarrow$ a. c$\wedge$d implies $\Pi(G) = \Pi'(G)$, hence G has a weak unit. Write $Z_{\mathfrak{R}_\Gamma} = Z$. By [10] II 2.2a, $Z(f'') = Z(f)$ for every $f \in G$. Then $\{Z(f): f \in G\} = Z(\Pi(G)) = Z(\Gamma(\mathfrak{R}_\Gamma, G)) = \mathcal{O}(\mathfrak{R}_\Gamma, G)$ ([10] II 2.2). Thus $Z(f)$ is open for every $f \in G$. By [10] II 1.4 the Γ-regulator is completely regular, which is the condition 4.1 (1b).

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Γ-РЕГУЛЯТОР И Π'-РЕГУЛЯТОР СТРУКТУРНО УПОРЯДОЧЕННЫХ ГРУПП

Франтишек Шик

Резюме

Пара $(\mathfrak{R}, \bigcup)$ называется регулятором l-группы G, если $\mathfrak{R}$ — множество $\neq \emptyset$, $\bigcup$ — отображение $\mathfrak{R}$ в множество простых подгруппы в G и, если выполнено $\bigcap\{\bigcup x: x \in \mathfrak{R}\} = \{0\}$. Важные регуляторы представляют множества $\mathfrak{R} = \mathfrak{U}(\Xi)$ всех ультраантифильтров на структуре $\Xi = \Gamma(G)$ (всех поляр в G) или $= \Pi'(G)$ (всех дуальных главных поляр в G), если мы определяем $\bigcup x = \bigcup\{K: K \in x\}$ для всех $x \in \mathfrak{R}$. Первое из них называется Γ-регулятором, второй — Π'-регулятором. Целю работы — установить отношения между Γ-регулятором и Π'-регулятором, а именно установить различные степени слияния этих регуляторов. В теор. 3.1 найдены условия, при выполнении которых упрощение Γ-регулятора равно Π'-регулятору, а в теор. 4.1 условия, при которых сам Γ-регулятор равен Π'-регулятору. В 4.2 решается та же самая проблема при условии существования слабой единицы в G. Это предположение играет важную роль в 3.1.