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DIRECT DECOMPOSABILITY OF CONGRUENCES IN CONGRUENCE-PERMUTABLE VARIETIES

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The set of all congruences on an algebra $\mathbf{A}$ is denoted by $Con(\mathbf{A})$. A *variety* $\mathscr{V}$ of algebras has *directly decomposable congruences* if for all $\mathbf{A}, \mathbf{B} \in \mathscr{V}$ and each $\theta \in Con(\mathbf{A} \times \mathbf{B})$ there exist $\theta_1 \in Con(\mathbf{A})$ and $\theta_2 \in Con(\mathbf{B})$ such that $\theta = \theta_1 \times \theta_2$. G. A. Fraser and A. Horn [1] gave a Mal'cev type characterization of such varieties. This condition is, however, rather impractical. It can be simplified in the case of congruence-permutable varieties by putting $n = 2$ in [1, Theorem 5] because of [1, Lemma 2]. However, we can use tolerances in the way similar as in [2] to obtain more simple Mal'cev condition which is the aim of this note.

Theorem. *Let $\mathscr{V}$ be a congruence-permuatable variety. The following conditions are equivalent:*
(1) *$\mathscr{V}$ has directly decomposable congruences*
(2) *There exist a $(2+n)$-ary polynomial p, binary polynomials $q_1, \ldots, q_n$ and ternary polynomials $r_1, \ldots, r_n$ such that*

$$\begin{aligned} x &= p(x, y, q_1(x, y), \ldots, q_n(x, y)) \\ y &= p(y, x, q_1(x, y), \ldots, q_n(x, y)) \\ z &= p(x, y, r_1(x, y, z), \ldots, r_n(x, y, z)) = \\ &= p(y, x, r_1(x, y, z), \ldots, r_n(x, y, z)). \end{aligned}$$

Let $\mathbf{A}$ be an algebra and a, b be elements of $\mathbf{A}$. Denote by $\theta(a, b)$ the least congruence an $\mathbf{A}$ containing the pair $\langle a, b \rangle$.

Lemma 1. *Let $\mathscr{V}$ be a variety of algebras. The following conditions are equivalent:*

(a) *$\mathscr{V}$ has directly decomposable congruences*
(b) *For each $\mathbf{A}, \mathbf{B} \in \mathscr{V}$ and arbitrary $a_1, a_2 \in \mathbf{A}$ and $b_1, b_2, b \in \mathbf{B}$*

$$\langle [a_1, b], [a_2, b] \rangle \in \theta([a_1, b_1], [a_2, b_2])$$

is true on $\mathbf{A} \times \mathbf{B}$.

For the proof, see [1, Theorem 4].

By a *tolerance* on an algebra **A** we mean a reflexive and symmetric binary relation T on the support of **A** which has the *Substitution Property,* i.e. T is a subalgebra of the direct product $\mathbf{A} \times \mathbf{A}$ (see e.g. [3]). Thus each congruence on **A** is a tolerance on **A** but not vice versa in a general case. For varieties, the situation is the following:

Lemma 2. *Let $\mathcal{V}$ be a variety of algebras. The following conditions are equivalent:*

(a) *Every tolerance on each $\mathbf{A} \in \mathcal{V}$ is a congruence on* **A**
(b) *$\mathcal{V}$ is congruence-permutable.*

For the proof, see [4].

The set of all tolerances on an algebra **A** forms a complete lattice with respect to the set-inclusion, [3]. Hence, for each $x, y \in \mathbf{A}$ there exists the least tolerance on **A** containing the pair $\langle x, y \rangle$. Denote it by $T(x, y)$. *Clearly* $T(x, y) \subseteq \theta(x, y)$.

Lemma 3. *Let* **A** *be an algebra and a, b, x, y its elements. The following conditions are equivalent:*
(a) $\langle a, b \rangle \in T(x, y)$
(b) *There exist a $(2+n)$-ary polynomial p and elements $c_1, \ldots, c_n$ of* **A** *such that*

$$a = p(x, y, c_1, \ldots, c_n), \quad b = p(y, x, c_1, \ldots, c_n).$$

Proof. Let R be a set of all pairs $\langle a, b \rangle$ such that $a = p(x, y, c_1, \ldots, c_n)$, $b = p(y, x, c_1, \ldots, c_n)$ for some $(2+n)$-ary polynomial p over **A** and some elements $c_1, \ldots, c_n$ of **A**. Reflexivity, symmetry and the Substitution Property of $T(x, y)$ clearly imply $R \subseteq T(x, y)$. Evidently, R is also reflexive and symmetric. The Substitution Property of R can be easily shown by induction over the rank of polynomial p, thus R is a tolerance on **A**. Since $\langle x, y \rangle \in R$, we conclude $R = T(x, y)$.

Proof of the Theorem: (1) $\Rightarrow$ (2). Let $\mathcal{V}$ have directly decomposable congruences and $\mathbf{A} = \boldsymbol{F}_2(x, y)$, $\boldsymbol{B} = \boldsymbol{F}_3(x, y, z)$ be free algebras of $\mathcal{V}$. By Lemma 1, we have

$$\langle [x, z], [y, z] \rangle \in \theta([x, x], [y, y]).$$

Since $\mathcal{V}$ is congruence-permutable, Lemma 2 implies

$$\theta([x, x], [y, y]) = T([x, x], [y, y]).$$

however, by Lemma 3,

$$\langle [x, z], [y, z] \rangle \in T([x, x], [y, y])$$

implies the existence of $(2+n)$-ary polynomial p and elements $c_1, \ldots, c_n$ of $\mathbf{A} \times \boldsymbol{B}$ with

$$[x, z]=p([x, x], [y, y], c_1, \ldots, c_n)$$
$$[y, z]=p([y, y], [x, x], c_1, \ldots, c_n).$$

Since $\mathbf{A}$, $\mathbf{B}$ are free algebras, $c_i \in \mathbf{A} \times \mathbf{B}$ implies

$$c_i=[q_i(x, y), r_i(x, y, z)]$$

for some polynomials q_i, r_i over $\mathscr{V}$ whence (2) is evident.

(2) $\Rightarrow$ (1). Let $\mathbf{A}$, $\mathbf{B} \in \mathscr{V}$, $a_1, a_2 \in \mathbf{A}$, $b_1, b_2, b \in \mathbf{B}$. Put

$$c_i=[q_i(a_1, a_2), r_i(b_1, b_2, b)].$$

By (2) and Lemma 3 we obtain

$$\begin{gathered}\langle [a_1, b], [a_2, b]\rangle = \\ =\langle [p(a_1, a_2, q_1(a_1, a_2), \ldots, q_n(a_1, a_2)), p(b_1, b_2, r_1(b_1, b_2, b), \ldots, \\ r_n(b_1, b_2, b))], [p(a_2, a_1, q_1(a_1, a_2), \ldots, q_n(a_1, a_2)), \\ p(b_2, b_1, r_1(b_1, b_2, b), \ldots, r_n(b_1, b_2, b))]\rangle = \\ =\langle p([a_1, b_1], [a_2, b_2], c_1, \ldots, c_n), p([a_2, b_2], [a_1, b_1], c_1, \ldots, c_n)\rangle \in \\ \in T([a_1, b_1], [a_2, b_2])=\theta([a_1, b_1], [a_2, b_2]).\end{gathered}$$

By Lemma 1, (1) is proved.

Exapmle. Let $\mathscr{V}$ be a variety of all rings with unit element. Thus $\mathscr{V}$ is congruence-permutable and we can put $n=2$, $p(x_0, x_1, x_2, x_3) = x_0 \cdot x_2 + x_3$ and $q_1=1$, $q_2=0=r_1$, $r_2=z$. Clearly

$$p(x, y, q_1, q_2)=x \cdot 1+0=x$$
$$p(y, x, q_1, q_2)=y \cdot 1+0=y$$
$$p(x, y, r_1, r_2) =x \cdot 0+z=z=y \cdot 0+z=p(y, x, r_1, r_2).$$

Remark. In [1, Corollary 1] it is shown that the congruence-distributivity of $\mathscr{V}$ is a sufficient condition for direct decomposability of congruences. Our Theorem implies that congruence-permutability is not sufficient for this property. Since congruence-permutability yields the congruence-modularity, also congruence-modularity is not sufficient for direct decomposability of congruences.

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ПРЯМОЕ РОЗЛОЖЕНИЕ КОНГРУЭНЦИЙ В МНОГООБРАЗИЯХ С ПЕРЕСТАНОВОЧНЫМИ КОНГРУЭНЦИЯМИ

Иван Хайда

Резюме

Дается несложное условие Мальцева для многообразие с перестановочными конгруэнциями $\mathcal{V}$ чтобы для любых алгебер $\mathbf{A}, \boldsymbol{B} \in \mathcal{V}$ и любой конгруэнции $\theta \in Con(\mathbf{A} \times \boldsymbol{B})$ существовали $\theta_1 \in Con(\mathbf{A})$ и $\theta_2 \in Con(\boldsymbol{B})$ выполняющие $\theta = \theta_1 \times \theta_2$.