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ENUMERATION OF GRAPHS MAXIMAL WITH RESPECT TO CONNECTIVITY

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The notions and symbols not defined here will be used in the sense of [4].

One of the most important goals of the theory of k-vertex (edge) connected graphs is to construct all k-vertex (edge) connected graphs. This task has been accomplished for vertex connectivity by Dirac [2] and Plummer [7] for $k=2$, by Tutte [9] for $k=3$, by Slater [8] for $k=4$ and by Chaty and Chein for 2-edge connected graphs [1]. For $k \geqslant 5$ this problem seems be very difficult. Therefore minimal k-vertex (edge) connected graphs are investigated. The dual question of maximal k-connected graphs has been studied by Gliviak [3].

The aim of this paper is to determine the number of maximal k-vertex (edge) connected graphs.

Let us denote, as usual, by $\varkappa(G)$ the vertex connectivity, by $\lambda(G)$ the edge connectivity of a graph G. Let G be a non-complete graph. Then following [3] G is called *$\varkappa_n$-maximal* if $\varkappa(G)=n$ and $\varkappa(G+x)>\varkappa(G)$ holds for every edge $x \in E(\bar{G})$. Analogically G is called *λ_n-maximal* if $\lambda(G)=n$ and $\lambda(G+x) > \lambda(G)$, for every edge $x \in E(\bar{G})$. Further, let the symbol $A(G; n)$ denote the class of graphs that arose from graph G by adding n new edges.

Let graphs G_1 and G_2 have disjoint vertex sets V_1 and V_2 and disjoint edge sets E_1 and E_2, respectively. Their union is the graph $G=G_1 \cup G_2$, which has the vertex set $V=V_1 \cup V_2$ and the edge set $E=E_1 \cup E_2$. Their join G_1+G_2 consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2.

Theorem 1. ([3]) *Let G be a graph and n, r, s be natural numbers. Then G is*

a) *$\varkappa_0$-maximal iff $G \simeq K_r \cup K_s$;*
b) *$\varkappa_n$-maximal iff $G \simeq K_n+(K_r \cup K_s)$;*
c) *λ_0-maximal iff $G \simeq K_r \cup K_s$;*
d) *λ_n-maximal iff $G \simeq A(K_r \cup K_s; n)$, where either $r=1$, $s \geqslant n+1$, or s, $r \geqslant n+2$.*

From Theorem 1 it is easy to see:

Theorem 2. *Let p, n be natural numbers, $p \geqslant n+2$. Then the number of $\varkappa_n$-maximal graphs with p vertices is equal to* $\left[\frac{p-n}{2}\right]$.

The number of λ_n-maximal graphs will be determined by applying Polya's Enumeration Theorem.

Let G be a graph. Then the symbol $\Gamma(G)$ denotes the vertex-group of G and $\Gamma_1(G)$ denotes the edge-group of G. Following [5] we shall say that $G_1, G_2 \in A(G; n)$ are similar if there is $\varphi \in \Gamma(G)$ such that $\varphi: G_1 \to G_2$ is an isomorphism. If G_1, G_2 are not similar, they are called dissimilar.

Theorem 3. *Suppose that $G_1, G_2 \in A(K_s \cup K_r; n)$, where $s, r \geqslant n+2$. Then G_1 and G_2 are similar iff $G_1 \simeq G_2$.*

Proof. Let $G_1, G_2 \in A(K_s \cup K_r; n)$, where $s, r \geqslant n+2$. As $G_1, G_2 \in A(K_s \cup K_r; n)$ we have $V(G_1) = V(G_2) = V(K_s \cup K_r) = V(K_s) \cup V(K_r)$. To express it more clearly let us denote $V(G_i) \cap V(K_s) = A_i$, $V(G_i) \cap V(K_r) = B_i$, for $i = 1, 2$. Obviously $A_1 = A_2$, $B_1 = B_2$. It is well known that $\varphi \in \Gamma(K_s \cup K_r)$ iff the components of graph $K_s \cup K_r$ are invariable with respect to φ, for $s \neq r$, and $\varphi \in \Gamma(K_s \cup K_s)$ iff either the components of this graph are invariable with respect to φ or φ maps any vertex of one component onto a vertex from the other component.

The necessity of the condition is straightforward. Let now $G_1 \simeq G_2$. We shall show that G_1 and G_2 are similar. We shall prove it indirectly.

Let us consider the case $s \neq r$. Let $\varphi: G_1 \to G_2$ be an isomorphism and let $\varphi \notin \Gamma(K_s \cup K_r)$. Thus there is a vertex $u \in A_1$ such that $\varphi(u) \in B_2$. Put

$$A = \{u\,; u \in A_1, \varphi(u) \in B_2\},$$
$$B = \{u\,; u \in B_1, \varphi(u) \in A_2\}.$$

Obviously $|A| = |B|$. Let $|A| = p$ (as $\varphi \notin \Gamma(K_s \cup K_r)$, in the case of $s = r$ there must be $p < s$). Let $u, v \in A_1$ and $\varphi(u) \in A_2$, $\varphi(v) \in B_2$. Since $uv \in E(G_1)$, and φ is an isomorphism, $\varphi(u)\varphi(v) \in E(G_2)$. As $\varphi(u) \in A_2$, $\varphi(v) \in B_2$, the edge $\varphi(u)\varphi(v)$ does not belong to $E(K_s \cup K_r)$. Analogously for $z, w \in B_1$, $\varphi(z) \in A_2$ and $\varphi(w) \in B_2$ we have $\varphi(z)\varphi(w) \in E(G_2)$ but $\varphi(z)\varphi(w) \notin E(K_s \cup K_r)$. Therefore

$$n \geqq (r-p)p + (s-p)p\,.$$

On the other hand, for $r > s \geqslant n+2$, $1 \leqslant p \leqslant s$ (is the case of $s = r$, $s \geqslant n+2$, $1 \leqslant p \leqslant s-1$) there holds

$$(r-p)p + (s-p)p > n$$

and we have a contradiction. Thus $\varphi \in \Gamma(K_s \cup K_r)$. Q.E.D.

Let us denote, as usual, by $Z(H)$ the cycle index of the permutation group H. The polynomial $Z(H, 1+x)$ is determined by substituting $1 + x^k$ for each variable s_k in $Z(H)$.

Theorem 4. *Let $p, n > 0$ be natural numbers, $p \geq n+2$. Then the number of λ_n-maximal graphs with p vertices is*

a) $$1 \quad \text{for} \quad n+2 \leq p \leq 2n+3,$$

b) *the coefficient of x^n in*

$$x^n + \sum_{i=n+2}^{[p/2]} Z(\Gamma_1(K_{i,\,p-i}, 1+x) \quad \text{for} \quad 2n+4 \leq p,$$

where $K_{s,r}$ denotes the complete bipartite graph.

Remark 1. The cycle index of the edge group of $K_{s,r}$ can be found in [6].

Remark 2. As the graph G is λ_0-maximal iff G is $\varkappa_0$-maximal, the number of λ_0-maximal graphs is given in Theorem 2.

Proof. Let G be a graph with p vertices. Suppose that $n+2 \leq p \leq 2n+3$. From Theorem 1 it follows that G is λ_n-maximal iff $G \in A(K_1 \cup K_{p-1}; n)$. As $|A(K_1 \cup K_{p-1}; n)| = 1$, there is only one λ_n-maximal graph. Now, let $2n+4 \leq p$. According to Theorem 1 G is λ_n-maximal iff either $G \in A(K_1 \cup K_{p-1}; n)$ or $G \in A(K_s \cup K_{p-s}; n)$, where $s, p-s \geq n+2$. Let $G_1 \in A(K_s \cup K_{p-s}; n)$, $G_2 \in A(K_r \cup K_{p-r}; n)$, where $r \neq s \neq p-r$. Then G_1 cannot be isomorphic to G_2 because $|E(G_1)| \neq |E(G_2)|$. Thus the number of λ_n-maximal graphs is equal to

$$1 + \sum_{i=n+2}^{[p/2]} |A(K_i \cup K_{p-i}; n)|.$$

From [5] it follows that the number of dissimilar graphs in $A(G; n)$ is the coefficient of x^n in $Z(\Gamma_1(\bar{G}), 1+x)$. By Theorem 3 we have that the number of dissimilar graphs in $A(K_s \cup K_r; n)$ is equal to the number of nonisomorphic graphs in this class, for $s, r \geq n+2$. Thus the number of λ_n-maximal graphs is the coefficient of x^n in

$$x^n + \sum_{i=n+2}^{[p/2]} Z(\Gamma_1(\overline{K_i \cup K_{p-i}}, 1+x).$$

However, $\overline{K_p \cup K_q} = K_{p,q}$, and the proof is complete.

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ЧИСЛО ГРАФОВ, МАКСИМАЛЬНЫХ ОТНОСИТЕЛЬНО СВЯЗНОСТИ

Петер Горак

Резюме

В работе определено число графов, максималшных отностиельно вершинной (реберной) связности.