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HYPERINVARIANT SUBSPACE LATTICE OF SOME C_0-CONTRACTIONS

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1. Introduction

Let $\mathfrak{H}$ be a separable Hilbert space. Denote by $\mathscr{S}(\mathfrak{H})$ the set of all subspaces of $\mathfrak{H}$ (As usual, a subspace means a closed linear manifold). If $\mathfrak{M}_1$, $\mathfrak{M}_2$ are from $\mathscr{S}(\mathfrak{H})$, then by $\mathfrak{M}_1 \vee \mathfrak{M}_2$ we mean the smallest subspace of $\mathfrak{H}$ containing $\mathfrak{M}_1$ and $\mathfrak{M}_2$. $\mathfrak{M}_1 \cap \mathfrak{M}_2$ denote their intersection. Together with these operations $\mathscr{S}(\mathfrak{H})$ forms a complete lattice. Let T be a bounded linear operator on $\mathfrak{H}$. We denote by hyperlat (T) the lattice of all $\mathfrak{M} \in \mathscr{S}(\mathfrak{H})$ that are invariant under each operator that commutes with T. In [5] it was proved:

Theorem 1.1. *For a linear transformation T on a finite-dimensional complex vector space V* hyperlat (T) *is the smallest sublattice of $\mathscr{S}(V)$ which contains all subspaces that are either the kernel or the range of a polynomial in T.*

The purpose of the presented paper is to show some generalizations of this result for some contractions on the separable Hilbert space.

We shall use the functional calculus for Hilbert space contractions developed by Foias and Sz.-Nagy [6, chap. III.]. H^∞ and H^2 will denote the corresponding Hardy classes. A contraction T on $\mathfrak{H}$ is of class C_0 if there exists a function $m \in H^\infty$ such that $m(T) = 0$. By m_T we denote the minimal function of T, m_T is always an inner function and it can be factored into a Blaschke product and a singular function (For details see [6, chap. III.].

In what follows the range of an operator T will be denoted by rng T, the closure of rng T by $\overline{\text{rng}}\, T$ and the kernel of T by ker T. The basic lattice-theoretic terminology and results may be found in [4].

Definition 1.2. *Let T be a completely non-unitary contraction. We say that T has the property* (L) *if and only if* hyperlat (T) *is the smallest complete sublattice of $\mathscr{S}(\mathfrak{H})$ which contains all subspaces that are of the form* ker $u(T)$ *or* $\overline{\text{rng}}\, v(T)$ *for u and v from H^∞.*

Pei Yuan Wu [10] has proved that every operator $T \in C_0$ and of finite defect indices has the property (L). In section II we shall use methods similar to the method of [5] to prove that some contractions of class C_0 with not necessarily finite defect indices have the property (L) too.

In section III we shall prove that the property (P) introduced by Hari Bercovici [2] implies (L). Consequently every weak contraction of class C_0 has the property (L).

II.

In this section we assume that $T \in C_0$ and that m_T is a Blaschke product.

1. For every complex number a: $0<|a|<1$

set

$$b_a(z)=\frac{\bar{a}}{a}\frac{a-z}{1-\bar{a}z}$$

and

$$b_0(z)=z$$

Let

$$m_T=\prod_{i=1}^{\infty} b_{a(i)}^{n(i)} \tag{2.1}$$

$$(|a(i)|<1,\ \Sigma n(i)\,(1-|a(i)|)<\infty),\ [6, (III.1.12)].$$

The natural number $n(i)$ is the multiplicity of $a(i)$ as a zero of m_T.

Setting for every natural nimber i

$$\mathfrak{H}_i=\ker b_{a(i)}^{n(i)}(T) \tag{2.2}$$

we have [6, propositions III.7.1, III.7.2]

$$\mathfrak{H}=\mathfrak{H}_i \dot{+} \bigvee_{j\neq i}\mathfrak{H}_j \tag{2.3}$$

($\dot{+}$ denotes the direct, non-necessarily orthogonal, sum.).

Theorem 2.1. *Let T be a contraction of class C_0, let m_T satisfy* (2.1) *and* $T_i=T|\mathfrak{H}_i$ $(i=1, 2, \ldots)$. *Then*

(i) *For every* $\mathfrak{M}\in$ hyperlat (T) *there holds*

$$\mathfrak{M}\cap\mathfrak{H}_i\in \text{hyperlat}\,(T_i)\quad (i=1, 2, \ldots)$$

and

$$\mathfrak{M}=\bigvee_{i=1}^{\infty}\mathfrak{M}\cap\mathfrak{H}_i),$$

(ii) *each $\mathfrak{H}_i$ is the range of $\varphi_i(T)$ where φ_i is a suitable function from H^∞.*

Proof. Let $m_{a(i)}=m_T/b_{a(i)}^{n(i)}$. According to [6, sec. III.7.1] there exist u_i, v_i from H^∞ such that

$$b_{a(i)}^{n(i)}u_i+m_{a(i)}v_i=1$$

and it follows that if P_j is the projection onto $\mathfrak{H}_j$ along $\bigvee_{i\neq j}\mathfrak{H}_i$ and if $m_{a(j)}v_j=\varphi_j$, then $P_j=\varphi_j(T)$, thus (ii) is proved.

Let $\mathfrak{M}\in$ hyperlat (T) and let S_i be a bounded linear operator on $\mathfrak{H}_i$ commuting with T_i. Clearly the operator $S=S_iP_i$ commutes with T. $\mathfrak{M}\cap\mathfrak{H}_i\in$ hyperlat (T) implies

$$S_i(\mathfrak{M}\cap\mathfrak{H}_i)=S(\mathfrak{M}\cap\mathfrak{H}_i)\subset\mathfrak{M}\cap\mathfrak{H}_i,$$

thus $\mathfrak{M}\cap\mathfrak{H}_i\in$ hyperlat (T_i).

$T|\mathfrak{M}$ is a C_0-contraction whose minimal function is a Blaschke product. According to [6, proposition III.7.2] $\mathfrak{M}$ is generated by characteristic vectors of T that belong to $\mathfrak{M}$. On the other hand the characteristic vectors of T associated with the characteristic value $a(i)$ belong to $\mathfrak{H}_i$, thus $\mathfrak{M}\subset\bigvee_{i=1}^{\infty}(\mathfrak{M}\cap\mathfrak{H}_i)$. The other inclusion is obvious, and so (i) is proved.

Theorem 2.2. *Let T be a C_0-contraction with finite defect indices and let*

$$m_T=b_a^n\quad(|a|<1).\tag{2.4}$$

Then T has the property (L).

Proof: T is quasisimilar [3] to an operator

$$S=S(b_a^{i_1})\oplus\ldots\oplus S(b_a^{i_m}),$$

where $n=i_1\geqq i_2\geqq\ldots\geqq i_m>1$.

For any inner function m $\mathfrak{H}(m)$ denotes the orthogonal complement in the Hardy space H^2 of the subspace mH^2 and $S(m)$ is the projection of the unilateral shift onto $\mathfrak{H}(m)$ (see [6, p. 369]). Here and in the rest of this paper $\oplus$ denotes the orthogonal sum.

For all k $S(b_a^k)$ is k-dimensional [6, p. 369 and proposition III. 7.3]. Hence both S and T are finitedimensional and according to theorem 1.1 they have the property (L)

Theorem 2.3. *Let T be a contraction of class C_0 and let m_T be a Blaschke product* (2.1). *Let T_i (see theorem 2.1) have finite defect indices. Then T has the property* (L).

Proof: From theorems 2.1 and 2.2 it follows that hyperlat (T) is the smallest complete sublattice of $\mathcal{S}(\mathfrak{H})$ which contains all subspaces that are of the form ker $u(T_i)$ or rng $u(T_i)$ for some u from H^∞ and $T_i=T|$rng $\varphi_i(T)$, where φ_i is an H^∞ function. We have for every $u\in H^\infty$ and $i=1, 2, 3, \ldots$

$$\ker u(T_i) = \ker u(T) \cap \operatorname{rng} \varphi_i(T)$$

and

$$\operatorname{rng} u(T_i) = u(T_i)\varphi_i(T)\mathfrak{H} = u(T)\varphi_i(T)\mathfrak{H} = \operatorname{rng} (u\varphi_i)(T).$$

This concludes the proof of our theorem.

Example 2.4. Let $m = \prod_{i=1}^{\infty} b_{a(i)}$ be a Blaschke product (each zero of multiplicity one). Denote by $\mathfrak{H}(i)$ the orthogonal sum of i copies of $\mathfrak{H}(b_{a(i)})$ and by $S(i)$ the orthogonal sum of i copies of $S(b_{a(i)})$.

Set

$$\mathfrak{H} = \bigoplus_{i=1}^{\infty} \mathfrak{H}(i) \quad \text{and} \quad T = \bigoplus_{i=1}^{\infty} S(i).$$

It is obvious that $m_T = m$. We use [7, theorem 1] to find the Jordan model of T. Using the same notation as in [7] we have

$$\Theta = \operatorname{diag} (b_{a(1)};\, b_{a(2)}, b_{a(2)};\, b_{a(3)}, b_{a(3)}, b_{a(3)};\, \ldots)$$

$$\Omega = \operatorname{diag} (B_1;\, B_2, B_2;\, B_3, B_3, B_3;\, \ldots)$$

where $B_i = m/b_{a(i)} \cdot \psi = m$.

Then it is easy to compute that

$$E_r(\Omega) = b_{a(1)} b_{a(2)} \ldots b_{a(r-1)})$$

and the Jordan model of T is

$$\bigoplus_{r=1}^{\infty} S\left(\prod_{i=r}^{\infty} b_{a(i)}\right),$$

and so T has not finite defect indices. On the other hand T obviously satisfies the assumptions of theorem 2.3.

III.

We shall consider bounded operator-valued analytic functions as matrices over H^∞ of the type $n \times n$ $(1 \leqq n \leqq \infty)$. Let $\{m_i\}_{i=1}^{n}$ be a (finite or infinite) sequence of inner functions such that m_{i+1} divides m_i for all i: $1 \leqq i < n$. Then the matrix $\operatorname{diag}(m_1, m_2, \ldots)$ is called normal. An operator is called a Jordan operator if it is of the form $S(\mathbf{M})$ with a normal matrix $\mathbf{M}$ (see [7], [8]). $H_n^2 = H^2(E_n)$ will denote the Hardy—Hilbert space of E_n-vector valued analytic functions in the unit disc, E_n

means the n-dimensional Euclidean space. The following theorem was proved in the case $n < \infty$ in [10], the same proof will do for $n = \infty$.

Theorem 3.1. *Every Jordan operator has the property* (L).

Proof: We assume $n = \infty$. Let $\mathbf{M} = \operatorname{diag}(m_1, m_2, \ldots)$ be a normal matrix. A subspace $\mathfrak{L}$ of $H_n^2 \ominus \mathbf{M} H_n^2 = \bigoplus_{i=1}^{\infty} H^2 \ominus m_i H^2$ is hyperinvariant for $S = S(\mathbf{M})$ if and only if there exist normal matrices

$$\Theta = \operatorname{diag}(\vartheta_1, \vartheta_2, \ldots) \quad \text{and} \quad \Phi = \operatorname{diag}(\varphi_1, \varphi_2, \ldots)$$

such that $\mathbf{M} = \Theta\Phi$ and

$$\mathfrak{L} = \Theta(H_n^2 \ominus \Phi H_n^2) = \bigoplus_{i=1}^{\infty} \vartheta_i (H^2 \ominus \varphi_i H^2).$$

This was proved in [9, theorem 3] for $n < \infty$, but the same proof works for $n = \infty$. We claim that

$$\mathfrak{L} = \bigvee_{i=1}^{\infty} \ker \varphi_i(S) \cap \overline{\operatorname{rng}} \, \vartheta_i(S)$$

Denote $S_i = S(m_i)$; then $S = \bigoplus_{i=1}^{\infty} S_i$

$$\mathfrak{L}_i = \vartheta_i(H^2 \ominus \varphi_i H^2) = \ker \varphi_i(S_i) = \operatorname{rng} \vartheta_i(S_i).$$

Setting $\mathfrak{L}_{ij} = \{0\}$ for $i \neq j$ and $\mathfrak{L}_{ii} = \mathfrak{L}_i$

and

$$\mathfrak{M}_i = \bigoplus_{j=1}^{\infty} \mathfrak{L}_{ij}$$

we have

$$\mathfrak{M}_i \subset \ker \varphi_i(S) \cap \overline{\operatorname{rng}} \, \vartheta_i(S),$$

hence

$$\mathfrak{L} \subset \bigvee_{i=1}^{\infty} \ker \varphi_i(S) \cap \overline{\operatorname{rng}} \, \vartheta_i(S).$$

To prove the other inclusion fix j and let $\mathbf{x} = \bigoplus_{i=1}^{\infty} \mathbf{x}_i$ be in $\ker \varphi_j(S) \cap \overline{\operatorname{rng}} \, \vartheta_j(S)$. Let $\{\mathbf{y}_n = \bigoplus_{i=1}^{\infty} \mathbf{y}_{in}\}_{n=1}^{\infty}$ be a sequence of vectors (in $H_n^2 \ominus \mathbf{M} H_n^2$) such that

$$\lim_{n \to \infty} \vartheta_j(S) \mathbf{y}_n = \mathbf{x}$$

(in norm topology). Then for all i

$$\lim_{n\to\infty} \vartheta_j(S_i)\,\mathbf{y}_{in} = \mathbf{x}_i \tag{3.1}$$

$\mathbf{x} \in \ker\ \varphi_i(S)$ ismplies $\mathbf{x}_i \in \ker\ \varphi_j(S_i)$, and so

$$\lim_{n\to\infty} \varphi_j(S_i)\vartheta_j(S_i)\mathbf{y}_{in} = 0.$$

If $i<j$, then $\varphi_j\vartheta_j$ divides $\varphi_i\vartheta_j$ and so

$$\lim_{n\to\infty} \varphi_i(S_i)\vartheta_j(S_i)\mathbf{y}_{in} = 0;$$

then (3.1) implies

$$\varphi_i(S_i)\mathbf{x}_i = 0.$$

If $i \geqq j$, $m_i = \varphi_i\vartheta_i$ divides $\varphi_i\vartheta_j$. For all n

$$m_i(S_i)\mathbf{y}_{in} = 0$$

and so

$$\varphi_i(S_i)\vartheta_j(S_i)\mathbf{y}_{in} = 0,$$

hence (3.1) implies

$$\varphi_i(S_i)\mathbf{x}_i = 0.$$

We have $\mathbf{x}_i \in \ker\ \varphi_i(S_i) = \mathfrak{L}_i$ for all i. This finishes the proof of our theorem.

Bercovici [2] has studied the operators T of class C_0 having the following property:

(P) *Any injection X commuting with T is a quasi-affinity.* He proved [2, proposition 4.8] that if T has the property (P) and S is its Jordan model, then hyperlat (T) and hyperlat (S) are isomorphic; namely he proved the following:

Theorem 3.2. *Let T and T' be two quasisimilar operators of class C_0 acting on $\mathfrak{H}$, $\mathfrak{H}'$, respectively, and having the property* (P). *Let us define*

$$\xi\colon \text{hyperlat}\,(T) \to \text{hyperlat}\,(T')$$
$$\eta\colon \text{hyperlat}\,(T') \to \text{hyperlat}\,(T)$$

by

$$\xi(\mathfrak{M}) = \bigvee_{X\in \mathrm{I}(T',\,T)} X\mathfrak{M} \tag{3.2}$$

$$\eta(\mathfrak{N}) = \bigvee_{Y\in \mathrm{I}(T,\,T')} Y\mathfrak{N} \tag{3.3}$$

(I$(T'\ T)$ *means the set of all operators $\mathfrak{H} \to \mathfrak{H}'$ satisfying $T'X = XT$*).

Then

(i) *For any quasiaffinities*

$$A \in \mathrm{I}(T', T), \quad B \in \mathrm{I}(T, T')$$

$$\xi(\mathfrak{M}) = (A\mathfrak{M})^{-} = B^{-1}\mathfrak{M}, \quad \mathfrak{M} \in \text{hyperlat}\,(T)$$

(ii) ξ *is bijective and* $\eta = \xi^{-1}$.

Now we use this theorem to prove:

Theorem 3.3. *Every operator T having the property* (P) *has the property* (L).

Proof: Let S be the Jordan model of T. According to [2, corollary 4.3] S has the property (P) and from theorem 3.2 there exist quasi-affinities $A \in \mathrm{I}(T, S)$, $B \in \mathrm{I}(S, T)$ such that for every $\mathfrak{L} \in \text{hyperlat}\,(S)$

$$\xi(\mathfrak{L}) = (A\mathfrak{L})^{-} = B^{-1}(\mathfrak{L})$$

and the mapping ξ is a bijective lattice isomorphism from hyperlat (S) onto hyperlat (T). Now for every $\varphi \in H^{\infty}$ $TA = AS$ implies $\varphi(T)A = A\varphi(S)$, this implies

$$\xi(\ker \varphi(S)) = (A \ker \varphi(S))^{-} \subset \ker \varphi(T)$$

and

$$\xi\,(\overline{\text{rng}}\,\varphi(S)) = (A(\varphi(S)\mathfrak{H})^{-})^{-} = (A\varphi(S)\mathfrak{H})^{-} = (\varphi(T)A\mathfrak{H})^{-} \subset \overline{\text{rng}}\,\varphi(T).$$

Similarly $\varphi(S)B = B\varphi(T)$, and so

$$\ker \varphi(T) \subset B^{-1} \ker \varphi(S) = \xi\,(\ker \varphi(S))$$

and

$$\overline{\text{rng}}\,\varphi(T) \subset B^{-1}\,\overline{\text{rng}}\,\varphi(S) = \xi\,(\overline{\text{rng}}\,\varphi(S)).$$

We claim that

$$\xi\,(\ker \varphi(S)) = \ker \varphi(T)$$

and

$$\xi\,(\overline{\text{rng}}\,\varphi(S)) = \overline{\text{rng}}\,\varphi(T)$$

and this together with theorem 3.1 gives that T has the property (L).

Corollary 3.4. *Every weak contraction of class* C_0 *has the property* (L).

Proof: Every weak contraction of class C_0 has the property (P) (see [1, corollary 2.8]).

The converse of theorem 3.3 is not true because every Jordan operator has the property (L) but need not have the property (P). It suffices to take the orthogonal sum of infinitely many copies of $S(m)$ with an arbitrary nonconstant inner function m. This follows easily from theorem 4.1 of [2].

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РЕШЕТКА ПОДПРОСТРАНСТВ, ГИПЕРИНВАРИАНТНЫХ ДЛЯ НЕКОТОРЫХ СЖАТИЙ КЛАССА C_0

Михал Заяц

РЕЗЮМЕ

В статье изучаются условия, при которых сжатие T класса C_0 имеет следующее свойство:

(L) Решетка подпространств, гиперивариантных для T — порождена подпространствами, являющимися нуль-пространством или замыканием области значения оператора $u(T)$, где u — любая функция из H^∞.

Показывается, что свойство (L) имеет место, если минимальная функция оператора T является произведением Бляшке (2.1) и у операторов $T_i = T|\mathfrak{H}_i$ (смотри (2.2)) конечные дефектные индекси. Показывается тоже, что любой оператор Жордана обладает свойством (L) и что свойство (P) (Если X — коммутирующий с T линейный оператор и $\ker X = 0$, то $\ker X^* = 0$) введенное Берковичем, влечет за собой (L).