Mathematica Slovaca

Anatolij Dvurečenskij

On m-joint distribution

Mathematica Slovaca, Vol. 31 (1981), No. 4, 347--353

Persistent URL: http://dml.cz/dmlcz/136275

Math. Slovaca 31, 1981, No. 4, 347–353

ON m-JOINT DISTRIBUTION

ANATOLIJ DVUREČENSKIJ

The m-joint distribution as a weak form of the compatibility of observables on a logic is studied and some results are proved. This notion allows to introduce multidimensional statistics of observables into the measurement theory of noncompatible observables.

Let L be a σ-lattice with the first and the last elements 0 and 1, respectively, with the orthocomplementation $\perp$: $a \mapsto a^\perp$ for which there hold (i) $(a^\perp)^\perp = a$ for all $a \in L$; (ii) if $a < b$, then $b^\perp < a^\perp$; (iii) $a \vee a^\perp = 1$ for all $a \in L$. Further we assume that if $a < b$, then $b = a \vee (b \wedge a^\perp)$. A σ-lattice L satisfying the above axioms will be called a logic [5].

An element $0 \neq a \in L$ is an atom of L if $b < a$ implies either $b = a$ or $b = 0$. We say that the elements $a, b \in L$, are (i) orthogonal and we write $a \perp b$ if $a < b^\perp$; (ii) compatible if there are three mutually orthogonal elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$, $b = b_1 \vee c$ and we shall write $a \leftrightarrow b$.

A state is a map m from L into $\langle 0, 1\rangle$ such that $m(1) = 1$ and $m\left(\bigvee_i a_i\right) = \sum_i m(a_i)$ if $a_i \perp a_j$, $i \neq j$. An element $a \in L$ is a carrier of a state m if $m(b) = 0$ iff $b \perp a$. If a carrier of m exists, then it is unique. A logic is full if there is a system M of states such that $a = b$ iff $m(a) = m(b)$ for all $m \in M$.

An observable is a map x: $B(R_1) \to L$ such that (i) $x(R_1) = 1$; (ii) $x(E) \perp x(F)$ if $E \cap F = \emptyset$; (iii) $x\left(\bigcup_i E_i\right) = \bigvee_i x(E_i)$ if $E_i \cap E_j = \emptyset$, $i \neq j$. We denote by $\sigma(x)$ the smallest closed set $E \subset R_1$ such that $x(E) = 1$. An observable x is purely atomic if (i) $\sigma(x) = \{\lambda_1, \lambda_2, \ldots\}$; (ii) $x(\{\lambda\})$ is an atom in L for any $\lambda \in \sigma(x)$. The range of an observable x is the set $R(x) = \{x(E): E \in B(R_1)\}$. The observables x and y are compatible and we write $x \leftrightarrow y$ if $x(E) \leftrightarrow y(F)$ for all $E, F \in B(R_1)$. If m is a state and x an observable, then m_x: $E \mapsto m(x(E))$, $E \in B(R_1)$ is a probabilty measure on $B(R_1)$.

Let m be a state on a logic L. We shall say that the observables $x_1, \ldots, x_n$ have an m-joint distribution if there exists a probability measure μ_n on $B(R_n)$ such that

$$\mu_n(E_1 \times \ldots \times E_n) = m\Big(\bigwedge_{j=1}^{n} x_j(E_j)\Big), \tag{1}$$

$$E_j \in B(R_1), \qquad j = 1, \ldots, n.$$

In this case we may study the statistical properties of the observables $x_1, \ldots, x_n$.

It is evident that if $x_1, \ldots, x_n$ are mutually compatible and m is an arbitrary state, then $x_1, \ldots, x_n$ have an m-joint distribution [5, Theorem 6.17]; in the opposite case this does not hold in general (Theorem 7).

Theorem 1. *Let L be a full logic, then $x_1, \ldots, x_n$ have an m-joint distribution for every $m \in M$ iff $x_1, \ldots, x_n$ are mutually compatible.*

Proof. Let $m \in M$. It is easy to see that for all x_i, x_j there is the m-joint distribution $m_{x_i x_j}$. Then for all E, $F \in B(R_1)$ we have

$$\begin{gathered} m(x_i(E)) = m(x_i(E) \wedge x_j(R_1)) = m_{x_i x_j}(E \times R_1) = \\ = m_{x_i x_j}(E \times F^c) + m_{x_i x_j}(E \times F) = m(x_i(E) \wedge x_j^{\perp}(F)) + m(x_i(E) \wedge x_j(F)) = \\ = m([x_i(E) \wedge x_j^{\perp}(F)] \vee [x_i(E) \wedge x_j(F)]. \end{gathered}$$

Similarly, $m(x_j(F) = m([x_j(F) \wedge x_i^{\perp}(E)] \vee [x_i(E) \wedge x_j(F)])$. From the fullness of L we have $x_i(E) \leftrightarrow x_j(F)$ for all E, $F \in B(R_1)$. Q.E.D.

We have seen that if $x_1, \ldots, x_n$ have an m-joint distribution, then any pair x_i, x_j has it. There arises a natural question regarding the converse implication. We shall show in the following that the answer in some cases is positive.

A valuation on a logic L is a map $v\colon L \to R_1$ such that

$$\begin{array}{ll} \text{(i)} & v(a \vee b) + v(a \wedge b) = v(a) + v(b),\ a, b \in L; \\ \text{(ii)} & v(a) \leq v(b) \text{ if } a < b. \end{array} \tag{2}$$

(Any state on L has the property (ii).)

Then a functional ϱ_v: $\varrho_v(a, b) = v(a \vee b) - v(a \wedge b)$ is a pseudometric on L [1, p. 230]. In this case we may define the quotient logic $\tilde{L} = L/\varrho_v$ identifying the elements $\tilde{a} = \tilde{b}$ iff $\varrho_v(a, b) = 0$, a, $b \in L$. If x is an observable on L, then $\tilde{x}(E) \equiv \widetilde{x(E)}$, $E \in B(R_1)$ is an observable on L; similarly, if m is a state on L, then $\tilde{m}$ defined by the subscription $\tilde{m}(\tilde{a}) = m(a)$ is a state on $\tilde{L}$.

Theorem 2. *Let a state m on L be a valuation; then the observables x and y have an m-joint distribution iff $\tilde{x} \leftrightarrow \tilde{y}$.*

Proof. If x, y have an m-joint distribution, then for E, $F \in B(R_1)$ we have (i) $m(x(E)) = m([x(E) \wedge y^{\perp}(F)] \vee [x(E) \wedge y(F)])$; (ii) $m(y(F)) = m([y(F) \wedge x^{\perp}(E)] \vee [x(E) \wedge y(F)])$. If $a = x(E)$ and b is the expression in parentheses on the right-hand side in (i), then $a > b$ and $\varrho_m(a, b) = 0$. Therefore

$$\begin{gathered} \tilde{x}(E) = (\tilde{x}(E) \wedge \tilde{y}^{\perp}(F)) \vee (\tilde{x}(E) \wedge \tilde{y}(F)), \\ \tilde{y}(F) = (\tilde{y}(F) \wedge \tilde{x}^{\perp}(E)) \vee (\tilde{x}(E) \wedge \tilde{y}(F)), \end{gathered}$$

that is, $\tilde{x} \leftrightarrow \tilde{y}$.

Let now $\tilde{x} \leftrightarrow \tilde{y}$; then for a state $\tilde{m}$ on $\tilde{L} = L/\varrho_m$ there is an $\tilde{m}$-joint distribution μ such that $\mu(E \times F) = \tilde{m}(\tilde{x}(E) \wedge \tilde{y}(F))$. But on the other hand $\tilde{m}(\tilde{x}(E) \wedge \tilde{y}(F)) = \tilde{m}([x(E) \wedge y(F)]^{\sim}) = m(x(E) \wedge y(F))$. Q.E.D.

Theorem 3. *If a state m is a valuation on L, then the observables $x_1, \ldots, x_n$ have an m-joint distribution iff any pair x_i, x_j, $i, j = 1, \ldots, n$ has an m-joint distribution.*

Proof. The necessity is evident. The sufficient condition follows from Theorem 2, because then we have $\tilde{x}_i \leftrightarrow \tilde{x}_j$, $i, j = 1, \ldots, n$ and there exists a probability measure μ_n on $B(R_n)$ such that $\mu_n(E_1 \times \ldots \times E_n) = \tilde{m}\left(\bigwedge_{i=1}^{n} \tilde{x}_i(E_i)\right) = m\left(\bigwedge_{i=1}^{n} x_i(E_i)\right)$. Q.E.D.

Remark. If any pair x_i, x_j from $x_1, \ldots, x_n$ has an m-joint distribution, then m on $\bigcup_{i=1}^{n} R(x_i)$ has the property (2) of the valuation. There arises the question whether m has the property (2) on the minimal sublogic L_0 generated by $R(x_i)$, $i = 1, \ldots, n$. In the case of a positive answer the proposition of Theorem 3 holds without the assumption on the valuation. The partial answer is given in the following (Corollary 6) and therefore the limitation on the valuation is not an extremely strong restriction in the study of an m-joint distribution.

Theorem 4. *Let a be the carrier of a state m. Purely atomic observables x and y with the spectra $\sigma(x) = \{\lambda_1, \lambda_2, \ldots\}$ and $\sigma(y) = \{\mu_1, \mu_2, \ldots\}$ have an m-joint distribution iff*

(i) *there are the index sets I and J such that for any $i \in I$ there is $j \in J$ with*

$$b_j \equiv y(\{\mu_j\}) = x(\{\lambda_i\}) \equiv a_i \tag{3}$$

and for any $j \in J$ there is $i \in I$ such that (3) *holds;*

(ii)
$$a < \bigvee_{i \in I} a_i = \bigvee_{j \in J} b_j .$$

Proof. Let x and y have an m-joint distribution. Then we have $1 = \sum_{i,j} m(a_i \wedge b_j) = m\left(\bigvee_{i,j} a_i \wedge b_j\right)$. Hence $a < \bigvee_{i,j} a_i \wedge b_j$. Since a_i, b_j are atoms of L, either $a_i \wedge b_j = 0$ or $a_i = b_j$. Let us denote by I the set of such indexes i for which there is j such that (3) holds. Analogically we define J. Hence (i) is satisfied.

The property (i) implies $a < \bigvee_{i,j} a_i \wedge b_j = \bigvee_{i \in I} a_i = \bigvee_{j \in J} b_j$.

Conversely, let (i) and (ii) hold. Then $\sum_{i,j} m(a_i \wedge b_j) = m\left(\bigvee_{i,j} a_i \wedge b_j\right) = m\left(\bigvee_{i \in I} a_i\right) \geqslant m(a) = 1$, which is the necessary and sufficient condition for x and y to have an m-joint distribution. Q.E.D.

Theorem 5. *Let $\{x_t, t \in T\}$ be purely atomic observables on a modular logic of the rank 3 (i.e., any set of nonzero mutually orthogonal elements of L has at most three elements). If there is $a \in L$ such that for any $t \in T$ there is $\lambda_t \in \sigma(x_t)$ with*

$$x_t(\{\lambda_t\}) = a ,$$

then the minimal sublogic generated by $L_0 = \bigcup_{t \in T} R(x_t)$ is equal to L_0.

Proof. Let x and y be two arbitrary observables of the given system of observables. If we show that for any $a, b \in R(x) \cup R(y)$ $a \vee b^{\perp}$ is an element of $R(x) \cup R(y)$, then $R(x) \cup R(y)$ is a sublogic of L and consequently L_0 is a sublogic of L, too.

There are two cases (i) $x \leftrightarrow y$, hence $R(x) = R(y)$ and the above proposition is true; (ii) $x \not\leftrightarrow y$. Let us put $a_i = x(\{\lambda_i\})$, $b_j = y(\{\mu_j\})$, $i, j = 1, 2, 3$. Let $a_1 = b_1$; then $a_2 \wedge b_2 = a_2 \wedge b_3 = a_3 \wedge b_2 = a_3 \wedge b_3 = 0$ and the following table holds for the join $\vee$

$\vee$	b_1	b_2	b_3	$b_1 \vee b_2$	$b_1 \vee b_3$	$b_2 \vee b_3$
a_1	a_1	$b_1 \vee b_2$	$b_1 \vee b_3$	$b_1 \vee b_2$	$b_1 \vee b_3$	1
a_2	$a_1 \vee a_2$	$a_2 \vee a_3$	$a_2 \vee a_3$	1	1	$a_2 \vee a_3$
a_3	$a_1 \vee a_3$	$a_2 \vee a_3$	$a_2 \vee a_3$	1	1	$a_2 \vee a_3$
$a_1 \vee a_2$	$a_1 \vee a_2$	1	1	1	1	1
$a_1 \vee a_3$	$a_2 \vee a_3$	1	1	1	1	1
$a_2 \vee a_3$	1	$a_2 \vee a_3$	$a_2 \vee a_3$	1	1	$a_2 \vee a_3$

We show, for example, $a_2 \vee b_2 = a_2 \vee a_3$. Let v be a canonical dimension function [5, p. 26]. Since $a_2 \perp a_1$, $b_2 \perp a_1$, then $a_2 \vee b_2 \perp a_1$ and $v(a_2 \vee b_2 \vee a_1) = v(a_2 \vee b_2) + v(a_1) = v(a_2) + v(b_2) + v(a_1) = 3$. Hence $a_2 \vee b_2 \vee a_1 = 1$ and $a_2 \vee b_2 = a_1^{\perp} = a_2 \vee a_3$. Similarly for other cases. Hence for any $a, b \in R(x) \cup R(y)$ $a \vee b^{\perp}$ is in $R(x) \cup R(y)$. Q.E.D.

Corollary 6. *Let $\{x_t, t \in T\}$ be a system of purely atomic observables on a modular logic of the rank 3. Let $a \in L$ be such an element that for any $t \in T$ there is $\lambda_t \in \sigma(x_t)$ with $x_t(\{\lambda_t\}) = a$. If, moreover, a is the carrier of a state m, then*

(i) *for any $s, t \in T$ x_s and x_t have an m-joint distribution;*

(ii) *m on $L_0 = L_0\left(\bigcup_{t \in T} R(x_t)\right)$ is a valuation;*

(iii) *every finite subsystem $x_{t_1}, \ldots, x_{t_n}$, $t_i \in T$, has an m-joint distribution.*

Proof. The validity of proposition (i)—(iii) is obtained by using Theorems 4, 5, 3. Q.E.D.

One of the most important examples of logics is a logic $L(H)$ of the separable Hilbert space H (real or complex), where the elements of $L(H)$ are all closed subspaces of H. The Gleason theorem [5] asserts that every state m on $L(H)$, $\dim H \geq 3$, is of the form $m(M) = \sum_i \lambda_i (M\Phi_i, \Phi_i)$, $M \in L(H)$, where $\lambda_i > 0$, $\sum_i \lambda_i = 1$, or equivalently, $m(M) = m_T(M) = \operatorname{tr}(TM)$, where $T = \sum_i \lambda_i(\cdot, \Phi_i)\Phi_i$ is a Hermitian operator of the trace class ($\{\Phi_i\}_i$ is an orthonormal system).

Gudder [3] proved the following theorem:

Theorem 7. *Let x, y be observables in $L(H)$. Then x, y have an m-joint distribution in a state $m = m_T$ iff*

$$x(E)y(F)\Phi_i = y(F)x(E)\Phi_i \tag{4}$$

for all $E, F \in B(R_1)$, $i = 1, 2, \ldots$.

We shall say that a subspace H_0 reduces an observable x on $L(H)$ if $x(E)H_0 = H_0 x(E)$ for all $E \in B(R_1)$. Then the map $x_0 : x_0(E) \equiv x(E)H_0 = x(E) \wedge H_0$ is an observable on the logic $L(H_0)$.

Theorem 8. *If x, y have an m-joint distribution in a state $m = m_T$ and if a subspace H_0 generated by $\{\Phi_i\}_i$ reduces the observables x and y, then the observables x_0 and y_0 are compatible on $L(H_0)$.*

Proof. If $M, N \in L(H)$, then $M \leftrightarrow N$ iff $MN = NM$ [5] and therefore it suffices to verify whether $x_0(E)y_0(F) = y_0(F)x_0(E)$, $E, F \in B(R_1)$. Due to Theorem 7 we have $x_0(E)y_0(F) = x(E)H_0 y(F)H_0 = x(E)y(F)H_0 = y(F)x(E)H_0 = y(F)H_0 x(E)H_0 = y_0(F)x_0(E)$. Q.E.D.

Theorem 9. *Let a subspace H_0 generated by $\{\Phi_i\}_i$ reduce the observables $x_1, \ldots, x_n$. Then $x_1, \ldots, x_n$ have an m-joint distribution in a state $m = m_T$ iff any pair x_k, x_j, $k, j = 1, \ldots, n$ has an m-joint distribution.*

Proof. Only the sufficient condition. According to Theorem 8, the observables $x_{j0}(E) = x_j(E)H_0$, $j = 1, \ldots, n$ are compatible on the logic $L(H_0)$. The state $m = m_T$ may be considered on $L(H_0)$, too. Therefore there is a probability measure μ_n on $B(R_n)$ for which we have $\mu_n(E_1 \times \ldots \times E_n) = m_T(x_{10}(E_1) \wedge \ldots \wedge x_{n0}(E_n))$. By using the methods of the Hilbert space [4] the right-hand side is equal to

$$\sum_i \lambda_i (x_{10}(E_1) \wedge \ldots \wedge x_{n0}(E_n)\Phi_i, \Phi_i) =$$

$$\sum_i \lambda_i (x_1(E_1)H_0 \wedge \ldots \wedge x_n(E_n)H_0\Phi_i, \Phi_i) =$$

$$\sum_i \lambda_i (x_1(E_1) \wedge H_0 \wedge \ldots \wedge x_n(E_n) \wedge H_0 \Phi_i, \Phi_i) =$$

$$\sum_i \lambda_i (x_1(E_1) \wedge \ldots \wedge x_n(E_n) \wedge H_0 \Phi_i, \Phi_i) =$$

$$\sum_i \lambda_i (x_1(E_1) \wedge \ldots \wedge x_n(E_n) \Phi_i, \Phi_i) = m_T\Big(\bigwedge_{j=1}^n x_j(E_j)\Big). \qquad \text{Q.E.D.}$$

Finally, an independence of observables in a state m will be investigated in this contribution. The observables $x_1, \ldots, x_n$ are independent in a state m if

$$m\Big(\bigwedge_{j=1}^n x_j(E_j)\Big) = \prod_{j=1}^n m(x_j(E_j)), \tag{5}$$

for all $E_j \in B(R_1)$, $j = 1, \ldots, n$.

If L is a sum logic [2, 3] we shall say that the summable observables $x_1, \ldots, x_n$ are strongly independent in a state m if for any finite system of bounded observables $f_1, \ldots, f_n$ there holds

$$m_{f_1 \circ x_1 + \ldots + f_n \circ x_n} = m_{f_1 \circ x_1} * \ldots * m_{f_n \circ x_n}, \tag{6}$$

where the $*$ is the convolution. (If f is a Borel function and x is an observable, then a map $f \circ x$: $E \mapsto x(f^{-1}(E))$, $E \in B(R_1)$, is an observable.)

These two notions of independence coincide in the case of compatible observables. Gudder [2] showed that the strong independence in m implies the independence in m. The equivalency was proved by him only for question observables in a pure state on a logic $L(H)$.

Theorem 10. *Let the subspace H_0 generated by $\{\Phi_i\}_i$ reduce the observables $x_1, \ldots, x_n$. Then the independence of the observables in a state $m = m_T$ implies the strong independence.*

Proof. We may easily see that $x_1, \ldots, x_n$ have an m-joint distribution in a state $m = m_T$. Therefore $x_{10}, \ldots, x_{1n}$ are compatible on $L(H_0)$ and independent in $m = m_T$. Hence they are strongly independent, too, and there holds

$$m_{T, f_1 \circ x_1 + \ldots + f_n \circ x_n} = m_{T, (f_1 \circ x_1 + \ldots + f_n \circ x_n)0} =$$

$$m_{T, f_1 \circ x_{10} + \ldots + f_n \circ x_{n0}} = m_{T, f_1 \circ x_{10}} * \ldots * m_{T, f_n \circ x_{n0}} =$$

$$m_{T, f_1 \circ x_1} * \ldots * m_{T, f_n \circ x_n}. \qquad \text{Q.E.D.}$$

REFERENCES

[1] BIRKHOFF, G.: Lattice theory. 3rd ed., 1967.

[2] GUDDER, S. P.: Hilbert space, independence, and generalized probability. J. Math. Anal. and Appl., 20, 1967, 48—67.

[3] GUDDER, S. P.: Joint distribution of observables. J. Math. and Mech., 18, 1968, 325—335.

[4] PRUGOVEČKI, E.: Quantum mechanics in Hilbert space. Acad. Press. N.Y. 1968.
[5] VARADARAJAN, V. S.: Geometry of quantum theory. Van Nostrand N.Y. 1968.

Received June 29, 1979

Ústav merania a meracej techniky SAV,
Dúbravská cesta
885 27 Bratislava

О m-СОВМЕСТНОМ РАСПРЕДЕЛЕНИИ

Анатолий Двуреченский

Резюме

Исследуется m-совместное распределение как слабая форма совместности наблюдаемых на логике и доказываются некоторые результаты. Это понятие допускает введение многомерной статистики наблюдаемых в теории несовместных наблюдаемых.