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DECOMPOSITION OF A COMPLETE EQUIPARTITE GRAPH INTO ISOMORPHIC SUBGRAPHS

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A complete equipartite graph $K_n(k)$, where n and k are positive integers, is a graph whose vertex set is the union of pairwise disjoint sets $P_1, \ldots, P_n$ (called parts of this graph), each of which has k elements, and in which two vertices x, y are adjacent if and only if $x \in P_i$, $y \in P_j$, $i \neq j$.

A graph G is said to be divisible by a positive integer t (denoted by $t|G$) if there exists a decomposition of G into t pairwise isomorphic and edge-disjoint subgraphs.

In [1] F. Harary, R. W. Robinson and N. C. Wormald expressed a conjecture that every complete equipartite graph is divisible by every positive integer which divides its number of edges (the number of edges of $K_n(k)$ is evidently equal to $\frac{1}{2}n(n-1)k^2$). We shall present some partial results in this direction. In particular we shall study decompositions of complete equipartite graphs into isomorphic subgraphs which are complete graphs.

We shall prove some theorems.

Theorem 1. *Let n, k, t be positive integers with the property that t divides $\frac{1}{2}n(n-1)k$. Then $t|K_n(k)$.*

Proof. Let r be the greatest common divisor of t and $\frac{1}{2}n(n-1)$, let $s = t/r$. As r divides $\frac{1}{2}n(n-1)$, the complete graph K_n can be decomposed into r pairwise isomorphic and edge-disjoint subgraphs $H_1, \ldots, H_r$ (this was proved in [1]). Let $v_1, \ldots, v_n$ be the vertices of K_n; we may suppose that each of the graphs $H_1, \ldots, H_r$ contains all of them (some of them may be isolated). For $i = 1, \ldots, r$ let H_i^* be the graph on the vertex set $V = \bigcup_{j=1}^{n} P_j$ such that two vertices x, y of this set are adjacent if and only if $x \in P_l$, $y \in P_m$ and the vertices v_l, v_m are adjacent in H_i. Evidently the graphs $H_1^*, \ldots, H_r^*$ form a decomposition of $K_n(k)$ into r pairwise isomorphic and edge-disjoint subgraphs. As r is the greatest common divisor of $\frac{1}{2}n(n-1)$ and t and the number t divides $\frac{1}{2}n(n-1)k$, the number s divides k. Therefore each P_i for $i = 1, \ldots, n$ can be decomposed into s pairwise disjoint sets $Q_{i1}, \ldots, Q_{is}$ of the same cardinality. For each $j = 1, \ldots, s$ we construct the graph G_j in the following way.

The vertex set of G_j is V and the edge set consists exactly of all edges xy of H_1^*, where $x \in Q_{lj}$, $y \in P_m$, where $l < m$. Evidently the graphs $G_1, \ldots, G_s$ form a decomposition of H_1^* into s pairwise isomorphic and edge-disjoint subgraphs. As $H_1^*, \ldots, H_r^*$ are pairwise isomorphic, such a decomposition exists for each of them. If we decompose each of the graphs $H_1^*, \ldots, H_r^*$ into pairwise isomorphic and edge-disjoint subgraphs which are all isomorphic to $G_1, \ldots, G_s$, we obtain a required decomposition of $K_n(k)$ into $rs = t$ subgraphs.

Theorem 2. *Let k, n be two positive integers such that k is divisible by no integer p such that $2 \leqq p \leqq n-1$. Then there exists a decomposition of $K_n(k)$ into k^2 pairwise edge-disjoint subgraphs which are all isomorphic to K_n.*

Proof. Let $P_1, \ldots, P_n$ be the parts of $K_n(k)$; the vertices of P_i for $i = 1, \ldots, n$ will be denoted by $[i, j]$ for $j = 1, \ldots, k$. Let a, b be integers, $1 \leqq a \leqq k$, $1 \leqq b \leqq k$. By the symbol $f_i(a, b)$, where $1 \leqq i \leqq n$, we denote the integer z such that $1 \leqq z \leqq k$, $z \equiv ai + b \pmod{k}$. By $G(a, b)$ we shall denote the subgraph of $K_n(k)$ induced by the vertex set $\{[i, f_i(a, b)] | 1 \leqq i \leqq n\}$. The graph $G(a, b)$ contains exactly one vertex from each P_i, therefore it is isomorphic to K_n. Consider two graphs $G(a_1, b_1)$, $G(a_2, b_2)$ and suppose that they have a common edge; let the end vertices of this edge be $[i, j]$ and $[l, m]$. Evidently $i \neq l$. As the vertex $[i, j]$ is in both $G(a_1, b_1)$ and $G(a_2, b_2)$, we have simultaneously

$$j = f_i(a_1, b_1) \equiv a_1 i + b_1 \pmod{k},$$
$$j = f_i(a_2, b_2) \equiv a_2 i + b_2 \pmod{k},$$

hence

$$a_1 i + b_1 \equiv a_2 i + b_2 \pmod{k},$$

which implies

$$(a_1 - a_2) i \equiv b_2 - b_1 \pmod{k}.$$

Analogously we obtain

$$(a_1 - a_2) l \equiv b_2 - b_1 \pmod{k}.$$

Hence

$$(a_1 - a_2)(i - l) \equiv 0 \pmod{k}.$$

As $i \neq l$, we have $1 \leqq |i - l| \leqq n - 1$. As k is divisible by no integer p such that $2 \leqq p \leqq n - 1$, the numbers $i - l$ and k are relatively prime and we have

$$a_1 - a_2 \equiv 0 \pmod{k}.$$

As both a_1, a_2 are between 1 and k, we have

$$a_1 = a_2.$$

This implies also

$$b_1 = b_2.$$

We have proved that the graphs $G(a_1, b_1)$, $G(a_2, b_2)$ have a common edge if and only if they coincide. Hence the graphs $G(a, b)$ for all ordered pairs $[a, b]$ with $1 \leqq a \leqq k$, $1 \leqq b \leqq k$ form the required decomposition.

Theorem 3. *Let k, n be two positive integers such that $n \leqq k+1$ and k is a power of a prime number. Then there exists a decomposition of $K_n(k)$ into k^2 pairwise edge-disjoint subgraphs which are all isomorphic to K_n.*

Proof. First suppose $n = k+1$. As k is a power of a prime number, there exists a finite projective geometry with the property that on each line there are exactly $k+1$ points and for each point there are exactly $k+1$ lines going through it. Let c be a point of this geometry. There are $k+1$ lines $p_1, \ldots, p_{k+1}$ going through it. Let P_i be the set of all points on p_i except c for $i = 1, \ldots, k+1$. Consider the sets P_i as parts of a complete equipartite graph $K_{k+1}(k)$. There are k^2 lines which do not contain c. Each of them contains exactly one vertex from each P_i and any two of them have exactly one common vertex. To each line r which does not contain c we assign a subgraph G_r of $K_{k+1}(k)$ induced by the set of vertices which correspond to the points of r; each G_r is a complete graph on $k+1$ vertices. The graphs G_{r_1}, G_{r_2} for $r_1 \neq r_2$ are edge-disjoint, because otherwise they would have at least two common vertices and the lines r_1, r_2 would have at least two common points, which is impossible. Hence these graphs form the required decomposition. If $n < k+1$, then the graph $K_n(k)$ can be considered as a subgraph of $K_{k+1}(k)$ induced by the set $\bigcup_{i=1}^{n} P_i$. We construct the required decomposition for $K_{k+1}(k)$ and from each graph of this decomposition we delete all vertices belonging to $\bigcup_{i=n+1}^{k+1} P_i$.

Corollary 1. *Let k, n be two positive integers. Let k_0 be a divisor of k which is divisible by no integer p such that $2 \leqq p \leqq n-1$. Then there exists a decomposition of $K_n(k)$ into k_0^2 pairwise edge-disjoint subgraphs which are all isomorphic to $K_n(k/k_0)$.*

Corollary 2. *Let k, n be two positive integers. Let k_0 be a divisor of k such that $n \leqq k_0 + 1$ and k_0 is a power of a prime number. Then there exists a decomposition of $K_n(k)$ into k_0^2 pairwise edge-disjoint subgraphs which are all isomorphic to $K_n(k/k_0)$.*

Corollary 3. *Let G be a graph with the vertices $v_1, \ldots, v_n$. Let k be an integer which fulfils the assumptions of Theorem 2 or of Theorem 3 with respect to n. Let G^* be the graph whose vertex set is the union of pairwise disjoint sets $V_1, \ldots, V_n$ of the cardinality k and in which two vertices x, y are adjacent if and only if $x \in V_i$, $y \in V_j$ and i, j are such numbers that v_i, v_j are adjacent in G. Then there exists a decomposition of G^* into k^2 pairwise disjoint subgraphs which are all isomorphic to G.*

The proofs can be easily done by the reader.

Theorem 4. *Let $k \geqq 2$, n be positive integers, $n \geqq k+2$. Then the graph $K_n(k)$ cannot be decomposed into k^2 pairwise edge-disjoint subgraphs which would be all isomorphic to K_n.*

Proof. Suppose that the required decomposition exists. Let $P_1, \ldots, P_n$ be the parts of $K_n(k)$. Let $u \in P_n$; the vertex u is evidently contained in exactly k graphs $G_1, \ldots, G_k$ of the decomposition. Any two of them have at most one vertex in common, otherwise they would have a common edge. This vertex is u, hence the graphs $G_1, \ldots, G_k$ cover the vertex set $\bigcup_{i=1}^{n-1} P_i$. Let G be a graph of the decomposition which does not contain u. All vertices of G except one are in $\bigcup_{i=1}^{n-1} P_i$, hence $n-1$ vertices of G are common vertices of G with the graphs $G_1, \ldots, G_k$. As $n-1>k$, by the Pigeon Hole Principle there exists at least one integer i such that $1 \leqq i \leqq k$ and G has at least two common vertices with G_i. Hence G and G_i have a common edge, which is a contradiction.

Theorem 5. *Let k, n, t be positive integers such that $t \mid \frac{1}{2}n(n-1)k^2$. Let r be the greatest common divisor of t and $\frac{1}{2}n(n-1)$, let $s=t/r$. If $k \mid s$ and either s/k is divisible by no integer p such that $2 \leqq p \leqq n-1$, or s/k is a power of a prime number and $n \leqq s/k+1$, then $t \mid K_n(k)$.*

Proof. Evidently $s \mid k^2$, hence s/k divides k. According to Corollary 1 or Corollary 2 there exists a decomposition of $K_n(k)$ into s^2/k^2 pairwise edge-disjoint subgraphs which are all isomorphic to $K_n(k^2/s)$. The graph $K_n(k^2/s)$ has $\frac{1}{2}n(n-1)(k^2/s)^2$ edges and the number k^2r/s divides $\frac{1}{2}n(n-1)k^2/s$, hence according to Theorem 1 there exists a decomposition of $K_n(k^2/s)$ into k^2r/s pairwise isomorphic and edge-disjoint subgraphs. If each of the mentioned s^2/k^2 graphs isomorphic to $K_n(k^2/s)$ is decomposed in this way, we obtain a decomposition of $K_n(k)$ into t pairwise isomorphic and edge-disjoint subgraphs, which implies $t \mid K_n(k)$.

Corollary 4. *Let k, n, t be positive integers, let $t \mid \frac{1}{2}n(n-1)k^2$ and let k be a prime number greater than $n-2$. Then $t \mid K_n(k)$.*

Proof. The only possibilities for s are: $s=1$, k or k^2. If $s=1$, k then use Theorem 1, if $s=k^2$ then use Theorem 5.

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РАЗБИЕНИЕ ПОЛЬНОГО ЭКВИПАРТИТНОГО ГРАФА НА ИЗОМОРФНЫЕ ПОДГРАФЫ

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Резюме

Граф называется делимый на положительное число т, если он может быть разбит на т между собой изоморфных и дизъюнктных подграфов. Харари, Робинсон и Вормалд высказали гипотезу, что каждый граф делится на каждое т, которое делит число его ребер. В работе показываются некоторые специальные результаты в этом направлении.