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ON STATISTICALLY CONVERGENT SEQUENCES OF REAL NUMBERS

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The notion of the statistical convergence of sequences of real numbers was introduced in papers [1] and [5]. In the present paper we shall show that the set of all bounded statistically convergent sequences of real numbers is a nowhere dense subset of the linear normed space $\boldsymbol{m}$ (with the sup-norm) of all bounded sequences of real numbers and the set of all statistically convergent sequences of real numbers is a dense subset of the first Baire category in the Fréchet space s.

1. Introduction

In this part of the paper we shall introduce some definitions, notations and two auxiliary results.

If $A \subset N = \{1, 2, \ldots, n, \ldots\}$ then we put $A(n) = \sum_{a \leqq n,\, a \in A} 1$. If there exists $\lim_{n\to\infty} \frac{A(n)}{n}$, it will be called the asymptotic density of the set A and will be denoted by $\delta(A)$. Obviously we have $\delta(A) = 0$ provided that A is a finite set of positive integers.

Definition 1.1. *The sequence $x = \{\xi_k\}_{k=1}^{\infty}$ of real numbers is said to converge statistically to the real number ξ (this fact will be denoted by*

$$\lim_{k\to\infty} \operatorname{stat} \xi_k = \xi \tag{1}$$

or briefly $x \xrightarrow[\text{stat}]{} \xi$) if for each $\varepsilon > 0$ we have $\delta(A_\varepsilon) = 0$, where $A_\varepsilon = \{n \in N;\ |\xi_n - \xi| \geqq \varepsilon\}$.

In paper [5] instead of (1) the notation D-$\lim \xi_k = \xi$ is used and the statistical convergence is called the D-convergence.

It is easy to see that if (1) holds, then the number ξ is determined uniquely.

If $\lim_{k\to\infty} \xi_k = \xi$, then (1) holds, too, since the set A_ε is in this case finite for each

$\varepsilon > 0$. The converse is not true (see example 1,1). Thus the statistical convergence is a natural generalization of the usual convergence of sequences.

Let us further observe that the condition $\delta(A_\varepsilon) = 0$ is equivalent to the condition $\delta(A'_\varepsilon) = 1$, where

$$A'_\varepsilon = \{n \in N;\ |\xi_n - \xi| < \varepsilon\} (= N - A_\varepsilon).$$

The sequence which converges statistically need not be bounded. This fact can be seen from the following

Example 1,1. It is easy to see that the set

$$A = \{1^2, 2^2, \dots, n^2, \dots\}$$

has the asymptotic density 0. Since the set of all rational numbers is countable, there exists such a sequence $\{\eta_{j^2}\}_{j=1}^{\infty}$ that the set of all terms of this sequence coincides with the set of all rational numbers. Put $\eta_n = \frac{1}{n}$ for $n \neq j^2$ $(j = 1, 2, \dots)$.

Then $\lim\limits_{k\to\infty} \operatorname{stat} \eta_k = 0$ and simultaneously each real number is a limit point of the sequence $\{\eta_k\}_{k=1}^{\infty}$.

The analysis of the structure of the sequence $\{\eta_k\}_{k=1}^{\infty}$ from the previous example suggests the conjecture that the structure of each statistically convergent sequence is analogous to the structure of this sequence, i.e. if (1) holds, then there exists such a set $K \subset N$ that $\delta(K) = 1$ and $\lim\limits_{\substack{k\to\infty \\ k\in K}} \xi_k = \xi$ $\Big(\lim\limits_{\substack{k\to\infty \\ k\in K}} \xi_k = \xi$ means that for each $\varepsilon > 0$ there exists such a k_0 that for each $k > k_0$, $k \in K$ we have $|\xi_k - \xi| < \varepsilon\Big)$.

The following lemma is an affirmative solution of the mentioned conjecture.

Lemma 1,1. *Statement (1) holds if and only if there exists such a set*

$$K = \{k_1 < k_2 < \dots < k_n < \dots\} \subset N$$

that $\delta(K) = 1$ *and* $\lim\limits_{n\to\infty} \xi_{k_n} = \xi$.

Proof. 1. If there exists a set with the mentioned properties and ε is an arbitrary given positive number, then we can choose such a number $n_0 \in N$ that for each $n > n_0$ we have

$$|\xi_{k_n} - \xi| < \varepsilon. \tag{2}$$

Put $A_\varepsilon = \{n \in N;\ |\xi_n - \xi| \geqq \varepsilon\}$. Then from (2) we get

$$A_\varepsilon \subset N - \{k_{n_0+1}, k_{n_0+2}, \dots\}$$

and on the right-hand side there is a set the asymptotic density of which is 0. Therefore $\delta(A_\varepsilon) = 0$, hence (1) holds.

2. Let (1) hold. Put

$$K_j = \{n \in N;\ |\xi_n - \xi| < \tfrac{1}{j}\} \quad (j = 1, 2, \ldots).$$

Then according to definition 1,1 we have $\delta(K_j) = 1$ $(j = 1, 2, \ldots)$.

It is evident from the definition of K_j $(j = 1, 2, \ldots)$ that

$$K_1 \supset K_2 \supset \ldots \supset K_j \supset K_{j+1} \supset \ldots, \tag{3}$$

$$\delta(K_j) = 1 \quad (j = 1, 2, \ldots). \tag{3'}$$

Let us choose an arbitrary number $v_1 \in K_1$. According to (3′) there exists such a $v_2 > v_1$, $v_2 \in K_2$ that for each $n \geqq v_2$ we have $\frac{K_2(n)}{n} > \frac{1}{2}$. Further, according to (3′) there exists such a $v_3 > v_2$, $v_3 \in K_3$ that for each $n \geqq v_3$ we have $\frac{K_3(n)}{n} > \frac{2}{3}$ a.s.o. Thus we can construct by induction such a sequence

$$v_1 < v_2 < \ldots < v_j < \ldots$$

of positive integers that $v_j \in K_j$ $(j = 1, 2, \ldots)$ and

$$\frac{K_j(n)}{n} > \frac{j-1}{j} \tag{4}$$

for each $n \geqq v_j$ $(j = 1, 2, \ldots)$.

Let us construct the set K as follows: Each natural number of the interval $\langle 1, v_1 \rangle$ belongs to K, further, any natural number of the interval $\langle v_j, v_{j+1} \rangle$ belongs to K if and only if it belongs to K_j $(j = 1, 2, \ldots)$.

According to (3), (4) for each n, $v_j \leqq n < v_{j+1}$ we get

$$\frac{K(n)}{n} \geqq \frac{K_j(n)}{n} > \frac{j-1}{j}.$$

From this it is obvious that $\delta(K) = 1$.

Let $\varepsilon > 0$. Choose a j such that $\frac{1}{j} < \varepsilon$. Let $n \geqq v_j$, $n \in K$. Then there exists such a number $l \geqq j$ that $v_l \leqq n < v_{l+1}$. But then on the basis of the definition of K, $n \in K_l$, hence

$$|\xi_n - \xi| < \frac{1}{l} \leqq \frac{1}{j} < \varepsilon.$$

Thus $|\xi_n - \xi| < \varepsilon$ for each $n \in K$, $n \geqq v_j$, i.e. $\lim\limits_{\substack{k \to \infty \\ k \in K}} \xi_k = \xi$.

The following result can be obtained (cf. [1], [5]) directly from the definition 1,1.

Lemma 1,2. *If*

$$\lim_{k\to\infty} \text{stat}\ \xi_k = a, \quad \lim_{k\to\infty} \text{stat}\ \eta_k = b$$

and c is a real number, then

(i) $\lim_{k\to\infty} \text{stat}\ (\xi_k + \eta_k) = a + b$,

(ii) $\lim_{k\to\infty} \text{stat}\ (c\cdot\xi_k) = ca$.

It follows from lemma 1,2 that the set of all bounded statistically convergent sequences of real numbers is a linear subspace of the linear normed space $\boldsymbol{m}$ of all bounded sequences of real numbers (with the norm $\|x\| = \sup_{k=1,2,\dots} |\xi_k|$, $x = \{\xi_k\}_{k=1}^{\infty} \in \boldsymbol{m}$).

2. Bounded statistically convergent sequences of real numbers

We denote by $\boldsymbol{m}_0$ the set of all bounded statistically convergent sequences of real numbers.

Theorem 2,1. *The set $\boldsymbol{m}_0$ is a closed linear subspace of the linear normed space $\boldsymbol{m}$.*

Proof. Let $x^{(n)} \in \boldsymbol{m}_0$ $(n = 1, 2, \dots)$, $x^{(n)} \to x \in \boldsymbol{m}$. We shall show that $x \in \boldsymbol{m}_0$. According to the assumption for each n there exists such a real number a_n that

$$x^{(n)} \xrightarrow[\text{stat}]{} a_n \quad (n = 1, 2, \dots)$$

i.e. if $x^{(n)} = \{\xi_k^{(n)}\}_{k=1}^{\infty}$, then

$$\lim_{k\to\infty} \text{stat}\ \xi_k^{(n)} = a_n \quad (n = 1, 2, \dots).$$

We shall prove that

a) the sequence (of real numbers) $\{a_n\}_{n=1}^{\infty}$ converges to a real number a;

b) $x \xrightarrow[\text{stat}]{} a$ (i.e. if $x = \{\xi_k\}_{k=1}^{\infty}$, then $\lim_{k\to\infty} \text{stat}\ \xi_k = a$).

From a), b) the assertion follows on account of Lemma 1,2.

Proof of a). Since $\{x^{(n)}\}_{n=1}^{\infty}$ is a convergent sequence of elements from $\boldsymbol{m}$, for each $\varepsilon > 0$ there exists such a $n_0 \in N$ that for each j, $n > n_0$ we have

$$\|x^{(j)} - x^{(n)}\| < \frac{\varepsilon}{3}. \tag{5}$$

Further, according to Lemma 1,1 there exist such sets A_j, A_n, A_j, $A_n \subset N$ that $\delta(A_j) = \delta(A_n) = 1$ and

$$\lim_{\substack{k \to \infty \\ k \in A_j}} \xi_k^{(j)} = a_j \tag{6}$$

$$\lim_{\substack{k \to \infty \\ k \in A_n}} \xi_k^{(n)} = a_n . \tag{7}$$

The set $A_j \cap A_n$ is infinite since the asymptotic density of this set is equal to 1. Hence we can choose such a $k \in A_j \cap A_n$ that we have (see (6), (7)):

$$|\xi_k^{(j)} - a_j| < \frac{\varepsilon}{3}, \quad |\xi_k^{(n)} - a_n| < \frac{\varepsilon}{3} . \tag{8}$$

According to (5) and (8) we get for each j, $n > n_0$

$$|a_j - a_n| \leqq |a_j - \xi_k^{(j)}| + |\xi_k^{(j)} - \xi_k^{(n)}| +$$

$$+ |\xi_k^{(n)} - a_n| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Since the sequence $\{a_k\}_{k=1}^{\infty}$ fulfils the Cauchy condition for convergence, it must converge to a real number a, hence

$$a = \lim_{k \to \infty} a_k . \tag{9}$$

Proof of b). Let $\eta > 0$. It suffices to prove that there exists such a set $A \subset N$ that $\delta(A) = 1$ and for each $k \in A$ the inequality $|\xi_k - a| < \eta$ holds (see Lemma 1,1).

Since $x^{(j)} \to x$, there exists such a $p \in N$ that

$$\|x^{(p)} - x\| < \frac{\eta}{3} . \tag{10}$$

The number p can be chosen in such a way that together with (10) also the inequality

$$|a_p - a| < \frac{\eta}{3} \tag{11}$$

holds (see (9)).

Since $x^{(p)} \xrightarrow[\text{stat}]{} a_p$, there exist such a set $A \subset N$ that $\delta(A) = 1$ and for each $k \in A$ we have

$$|\xi_k^{(p)} - a_p| < \frac{\eta}{3} . \tag{12}$$

Now according to (10), (11), (12) we get for each $k \in A$

$$|\xi_k - a| \leqq |\xi_k - \xi_k^{(p)}| + |\xi_k^{(p)} - a_p| +$$

$$+ |a_p - a| < \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta .$$

Hence b) follows.

Using the previous theorem we can easily prove the following result on the structure of the set $\boldsymbol{m}_0$.

Theorem 2,2. *The set $\boldsymbol{m}_0$ is a nowhere dense set in $\boldsymbol{m}$.*

Proof. It is a well-known fact that every closed linear subspace of an arbitrary linear normed space E, different from E, is a nowhere dense set in E (cf. [2], p. 37, Exercise 4; [3]). Hence on account of Theorem 2,1 it suffices to prove that $\boldsymbol{m}_0 \neq \boldsymbol{m}$. But this is evident, since the sequence $x = \{(-1)^k\}_{k=1}^{\infty} \in \boldsymbol{m}$ does not belong to $\boldsymbol{m}_0$.

3. Statistically convergent sequences of real numbers and the space s

Denote by s_0 the set of all statistically convergent sequences of real numbers. In what follows s denotes the Fréchet metric space of all real sequences with the metric ϱ,

$$\varrho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\xi_k - \eta_k|}{1 + |\xi_k - \eta_k|} ,$$

$$x = \{\xi_k\}_{k=1}^{\infty} \in s, \quad y = \{\eta_k\}_{k=1}^{\infty} \in s .$$

In this part of the paper we shall describe some fundamental properties and the structure of s_0 in the space s.

Theorem 3,1. (i) *The set s_0 is dense in the space s.*

(ii) *The set s_0 is a set of the first Baire category in the space s.*

Corollary. *The set $s - s_0$ (of all real sequences which are not statistically convergent) is a residual set (of the second Baire category) in the space s.*

For the proof of Theorem 3,1*) we shall use the following

Lemma 3,1. *Let g_k $(k = 1, 2, \ldots)$ be complex valued continuous functions on $R = (-\infty, +\infty)$. Let us suppose that there are two distinct complex numbers c_1, c_2 such that for each sufficiently large k we have $c_1, c_2 \in g_k(R)$.*

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*) The author is indebted to Professor M. Novotný for his suggestion, which led to an improvement of the original version of the proof of this theorem.

Let (a_{nk}) *be a triangular matrix with the following properties:*

(P_1) *For each fixed* k *we have* $\lim_{n\to\infty} a_{nk}=0$;

(P_2) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}=1$.

Then the set s_1 *of all such* $x=\{\xi_k\}_{k=1}^{\infty}\in s$ *for which there exists a (finite) limit* $\lim_{n\to\infty}\sum_{k=1}^{n} a_{nk}g_k(\xi_k)$ *is a set of the first Baire category in* s.

Proof. For $x=\{\xi_k\}_{k=1}^{\infty}\in s_1$ we put

$$f_n(x)=\sum_{k=1}^{n} a_{nk}g_k(\xi_k) \quad (n=1,2,\ldots), \quad f(x)=\lim_{n\to\infty} f_n(x).$$

We shall prove that

a) f_n $(n=1,2,\ldots)$ is a continuous function on s_1;

b) f is discontinuous at each point of s_1.

a) Let $a=\{\alpha_k\}_{k=1}^{\infty}\in s_1$, $x^{(j)}=\{\xi_k^{(j)}\}_{k=1}^{\infty}\in s_1$ $(j=1,2,\ldots)$, $x^{(j)}\to a$. Since from the convergence in the space s the convergence "by coordinates" follows, for each fixed k we have $\lim_{j\to\infty}\xi_k^{(j)}=\alpha_k$. But then on account of the continuity of functions g_k $(k=1,2,\ldots)$ we get

$$\lim_{j\to\infty} f_n(x^{(j)})=\lim_{j\to\infty}\sum_{k=1}^{n} a_{nk}g_k(\xi_k^{(j)})=$$

$$=\sum_{k=1}^{n} a_{nk}\lim_{j\to\infty} g_k(\xi_k^{(j)})=\sum_{k=1}^{n} a_{nk}g_k(\alpha_k)=f_n(a).$$

Thus f_n $(n=1,2,\ldots)$ is continuous on s_1.

b) Let $b=\{\beta_k\}_{k=1}^{\infty}\in s_1$. Denote by v such a number c_i $(i=1$ or $2)$ that differs from $f(b)$. It suffices to prove that in each sphere $S(b,\delta)=\{x\in s_1;\varrho(b,x)<\delta\}$ $(\delta>0)$ there exists such an element $x=\{\xi_k\}_{k=1}^{\infty}$ that $f(x)=v$.

Let $\delta>0$. Choose a natural number m such that $\sum_{k=m+1}^{\infty} 2^{-k}<\delta$. According to the assumption there exists such an m' that for each $k>m'$ there exists such $\eta_k\in R$ that $g_k(\eta_k)=v$. Put $m_0=\max\{m,m'\}$ and define the sequence $x=\{\xi_k\}_{k=1}^{\infty}$ in the following way: $\xi_k=\beta_k$ for $k\leqq m_0$ and $\xi_k=\eta_k$ for $k>m_0$. Then $x=\{\xi_k\}_{k=1}^{\infty}\in s$ and by choice of m_0 we get $\varrho(b,x)<\delta$. Further for $n>m_0$ we have

$$f_n(x)=\sum_{k=1}^{m_0} a_{nk}(g_k(\beta_k)-v))+v\sum_{k=1}^{n} a_{nk}.$$

Now from the properties (P_1), (P_2) it follows at once that $f(x)=\lim_{n\to\infty} f_n(x)=v$.

According to previous considerations the function f is in the first Baire class and is discontinuous at each point of s_1. But it is a well-known fact that the set of discontinuity points of an arbitrary function of the first Baire class is a set of the first Baire category (cf. [6], p. 185). Hence s_1 is a set of the first Baire category in s_1 and so a set of the first Baire category in s, too. The proof is finished.

Remark. Every regular triangular matrix (a_{nk}) has the properties (P_1), (P_2) from Lemma 3,1 (cf. [4], p. 8). The converse is not true. Putting e.g.

$$a_{n1}=\frac{1}{\sqrt{n}},\quad a_{n2}=-\frac{1}{\sqrt{n}},\ \ldots,\quad a_{nn-2}=\frac{1}{\sqrt{n}},\quad a_{nn-1}=-\frac{1}{\sqrt{n}},\quad a_{nn}=1$$

for n odd and

$$a_{n1}=\frac{1}{\sqrt{n}},\quad a_{n2}=-\frac{1}{\sqrt{n}},\ \ldots,\quad a_{nn-3}=\frac{1}{\sqrt{n}},\quad a_{nn-2}=-\frac{1}{\sqrt{n}},$$

$$a_{nn-1}=a_{nn}=\frac{1}{2}$$

for n even we get the triangular matrix with the properties (P_1), (P_2), for which

$$\sum_{k=1}^{n}|a_{nk}|\to+\infty(n\to\infty).$$

Hence this matrix is not regular.

Proof of Theorem 3,1. (i) If $x=\{\xi_k\}_{k=1}^{\infty}\in s_0$ and the sequence $y=\{\eta_k\}_{k=1}^{\infty}$ of real numbers differs from x only in a finite number of terms, then evidently $y\in s_0$, too. From this statement (i) follows at once on the basis of the definition of the metric in s.

(ii) For the proof of (ii) we shall use the following result from [5] (Theorem 4).

Theorem A. *Statement (1) holds if and only if for each real number t we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}e^{it\xi_k}=e^{it\xi}.$$

Denote by s_1' the set of all such $x=\{\xi_k\}_{k=1}^{\infty}\in s$ for which the finite $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}e^{i\xi_k}$ exists. Putting in Lemma 3,1

$$g_k(t)=e^{it}(k=1,2,\ldots),\quad a_{nk}=\frac{1}{n}\ (k=1,2,\ldots,n\,;\ n=1,2,\ldots)$$

we see that s_1' is a set of the first Baire category in s. According to Theorem A we have $s_0\subset s_1'$, hence s_0 is a set of the first Baire category in s, too. The proof is finished.

In what follows denote by s^* the set of all such $x=\{\xi_k\}_{k=1}^{\infty}\in s$ for which the sequence

$$\left\{\frac{1}{n}\sum_{k=1}^{n}|\xi|\right\}_{n=1}^{\infty}$$

is bounded. The set s^* will be considered as a subspace of the space s. For $\xi\in R$ denote by $s^*(\xi)$ the set of all such $x=\{\xi_k\}_{k=1}^{\infty}\in s$ for which $\lim_{k\to\infty}\operatorname{stat}\xi_k=\xi$. We shall show that $s^*(\xi)$ is a set of the second Borel class in the space s^*.

Theorem 3,2. *The set $s^*(\xi)$ is an $F_{\sigma\delta}$-set in s^*.*

For the proof of the theorem we shall use the following lemma.

Lemma 3,2. *The sequence $x=\{\xi_k\}_{k=1}^{\infty}\in s^*$ converges statistically to the real number ξ if and only if for each rational number t we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}e^{it\xi_k}=e^{it\xi}. \tag{13}$$

Proof. 1) If (1) is true, then according to Theorem A (cf. [5]) the equality (13) holds for each real t and so for each rational number t.

2) Let (13) hold for each rational number t. Let t_0 be an arbitrary real number. We shall prove that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}e^{it_0\xi_k}=e^{it_0\xi}. \tag{14}$$

From this statement (1) follows according to Theorem A.

For $t\in R$ put

$$A_n(t_0,t)=\frac{1}{n}\sum_{k=1}^{n}e^{it_0\xi_k}-\frac{1}{n}\sum_{k=1}^{n}e^{it\xi_k}.$$

Since $e^{iv\xi}=\cos v\xi+i\sin v\xi$ $(v\in R)$, we get

$$|A_n(t_0,t)|\leqq\frac{1}{n}\sum_{k=1}^{n}\sqrt{(\cos t_0\xi_k-\cos t\xi_k)^2+(\sin t_0\xi_k-\sin t\xi_k)^2}.$$

Using the mean value theorem we get

$$|A_n(t_0,t)|\leqq\frac{\sqrt{2}}{n}|t-t_0|\sum_{k=1}^{n}|\xi_k|. \tag{15}$$

By the assumption $x=\{\xi_k\}_{k=1}^{\infty}\in s^*$. Hence there exists such a $K>0$ that for each $n=1,2,\ldots$ we have

$$\frac{1}{n}\sum_{k=1}^{n}|\xi_k|\leqq K. \tag{16}$$

It follows from (15), (16) that

$$|A_n(t_0, t)| \leqq \sqrt{2}\, K |t - t_0|. \tag{17}$$

Further, we have

$$\frac{1}{n} \sum_{k=1}^{n} e^{it_0\xi_k} = \frac{1}{n} \sum_{k=1}^{n} e^{it\xi_k} + A_n(t_0, t). \tag{18}$$

From this we get

$$\left|\frac{1}{n} \sum_{k=1}^{n} e^{it_0\xi_k} - e^{it_0\xi}\right| \leqq \left|\frac{1}{n} \sum_{k=1}^{n} e^{it\xi_k} - e^{it\xi}\right| + \tag{19}$$
$$+ |e^{it\xi} - e^{it_0\xi}| + |A_n(t_0, t)|.$$

Let $\varepsilon > 0$. According to the continuity of the function $h(x) = e^{ix\xi}$ ($x \in R$) and (17) we can choose a rational number t such that

$$|e^{it\xi} - e^{it_0\xi}| < \frac{\varepsilon}{4}, \tag{20}$$

$$|A_n(t_0, t)| < \frac{\varepsilon}{4} \quad (n = 1, 2, \dots). \tag{20'}$$

By our assumption there exists such an n_0 that for each $n > n_0$ the first summand on the right-hand side of (19) is less than $\frac{\varepsilon}{2}$. Then with respect to (20) and (20′) we get from (19)

$$\left|\frac{1}{n} \sum_{k=1}^{n} e^{it_0\xi_k} - e^{it_0\xi}\right| < \varepsilon$$

for each $n > n_0$. Hence (14) is valid.

Proof of Theorem 3,2. Denote by Q the set of all rational numbers. From Lemma 3,2 we get

$$s^*(\xi) = \bigcap_{t \in Q} \bigcap_{j=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{n=p+1}^{\infty} H(n, j), \tag{21}$$

where

$$H(n, j) = \left\{x = \{\xi_k\}_{k=1}^{\infty} \in s^*;\ \left|\frac{1}{n} \sum_{k=1}^{n} e^{it\xi_k} - e^{it\xi}\right| \leqq \frac{1}{j}\right\}.$$

It can be easily checked that the set $H(n, j)$ (for each n, j) is closed in s^*. But then the assertion of the theorem follows at once from (21).

Problems. From the definition of the set s^* it follows that

$$s^* = \bigcup_{j=1}^{\infty} s_j^*, \tag{22}$$

where

$$s_j^* = \left\{ x = \{\xi_k\}_{k=1}^{\infty} \in s \, ; \, \forall_n \frac{1}{n} \sum_{k=1}^{n} |\xi_k| \leqq j \right\} \quad (j = 1, 2, \ldots).$$

It can be easily checked that s_j^* $(j = 1, 2, \ldots)$ is a closed set in s. This is a simple consequence of the fact that the convergence in the space s is equivalent to the convergence "by coordinates". Hence according to (22) the set s^* is an F_σ-set in s. Since the set $s^*(\xi)$ is an $F_{\sigma\delta}$-set in s^* and s^* is an F_σ-set in s, the set $s^*(\xi)$ is an $F_{\sigma\delta}$-set in s. In connection with the foregoing fact the question arises whether the set $s(\xi)$ of all such $x = \{\xi_k\}_{k=1}^{\infty} \in s$, for which $\lim\limits_{k\to\infty} \text{stat}\, \xi_k = \xi$ is an $F_{\sigma\delta}$-set in s, too. Further, the following question remains open: Is the set $s_0 \subset s$ a Borel set in s and if the answer is affirmative, in which Borel class is the set s_0?

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О СТАТИСТИЧЕСКИ СХОДЯЩИХСЯ ПОСЛЕДОВАТЕЛЬНОСТЯХ ДЕЙСТВИТЕЛЬНЫХ ЧИСЕЛ

Т. Шалат

Резюме

Последовательность $\{a_n\}_{n=1}^{\infty}$ действительных чисел называется статистически сходящяяся к числу a, если для каждого $\varepsilon > 0$ асимптотическая плотность множества $\{n\,;\ |a_n - a| \geqq \varepsilon\}$ равняется нулю. В работе показано, что множество всех статистически сходящихся последовательностей пространства всех ограниченных последовательностей нигде не плотно, множество всех статистически сходящихся последовательностей пространства Фреше s являетсяя множеством первой категории Бэра.