Ján Plesník

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1. Introduction

There are known many inequalities (bounds) for the chromatic number of a graph G (most of them can be found in [6, Chap. 12] and for others see [2, 3, 4, 7, 8, 11, 13, 16, 18, 19, 21, and 22]). Some of the bounds do not explicitly depend on the chromatic number of other graphs (related to G), while the other bounds do so. Our results belong to the second group. We bound $\chi(G)$ from above by the chromatic numbers of certain parts of G. A theorem is presented and then three partitioning algorithms are discussed. Finally the biparticity [7] is studied.

Our terminology is based on Harary [6]. Given a graph G, $V(G)$ and $E(G)$ denote its point set and line set, respectively. If U is a subset of $V(G)$, then $G[U]$ is the induced subgraph of G with the point set U.

This paper is related mainly to the following two assertions. The first of them is a result due to Zykov [22, Th. 2] (see also [3]).

Lemma 1. *If a graph G is decomposed into factors $F_1, \ldots, F_k$, then $\chi(G) \leqslant \Pi\chi(F_i)$.*

There is a trivial "sum" analogy:

Lemma 2. *If $G_1, \ldots, G_k$ are pairwise point-disjoint induced subgraphs of G including in union all points, then $\chi(G) \leqslant \Sigma\chi(G_i)$.*

2. A coloring theorem

The main purpose of this section is to decrease the number of summands in the inequality from Lemma 2.

Theorem 1. *Let a graph G be point-decomposed into subsets $V_1, \ldots, V_k$ and line-decomposed into the induced subgraphs $G[V_1], \ldots, G[V_k]$, and a spanning subgraph F. If $G[V_i]$ is g_i-colorable $(1 \leqslant i \leqslant k)$ with $g_1 \geqslant g_2 \geqslant \ldots \geqslant g_k$ and F is f-colorable (clearly, we can always assume that $f \leqslant k$), then the graph G is $(g_1 + \ldots + g_f)$-colorable.*

Proof. Let $b_1, \ldots, b_f$ be the colors used at some f-coloring of the spanning subgraph F and let $c_1^j, \ldots, c_{g_j}^j$ be the colors used at some g_j-coloring of the induced subgraph $G[V_j]$, $j = 1, \ldots, k$. Thus any point of G has a pair of colors. Now we form a coloring of G as follows. For every $i = 1, \ldots, f$ and $j = 1, \ldots, k$, any point with a pair (b_i, c_s^j) gets the color c_s^i if $s \leqslant g_f$ and the color c_s^j otherwise. One can easily verify that this is a coloring of G which uses only the colors $c_1^1, \ldots, c_{g_1}^1$, $c_1^2, \ldots, c_{g_2}^2, \ldots, c_1^f, \ldots, c_{g_f}^f$. This completes the proof.

Remark 1. Obviously, the union of all G_i ($G_i = G[V_i]$) has the chromatic number equal to $\chi(G_1)$. Therefore, if $\chi(G_1) = \ldots = \chi(G_f)$, then our theorem is a consequence of Lemma 1.

Problem. It would be interesting to make an analogy of our theorem in the case when the sets V_i may intersect. Vizing's theorem [20], considered as the point-coloring result for a line graph, can serve as a prototype of the desired results.

It is a custom to test every new coloring result on the famous four color conjecture (which has only recently been proved; cf. Appel and Haken [1]). There is an extensive list of equivalent conjectures (see, e.g., Ore [17]). Using our theorem, we can immediately add the next result.

Corollary. *A planar graph G has $\chi(G) \leqslant 4$ if and only if there are pairwise point-disjoint induced subgraphs $G_1, \ldots, G_k$ of G including in union all the points of G and a spanning subraph F of G consisting of the lines not in any G_i such that at least one of the following conditions is fulfilled:*

(i) $\chi(F) \leqslant 3$, $\chi(G_1) \leqslant 2$, $\chi(G_2) = \ldots = \chi(G_k) = 1$.

(ii) $\chi(F) \leqslant 2$, $\chi(G_1) \leqslant 3$, $\chi(G_2) = \ldots = \chi(G_k) = 1$.

(iii) $\chi(F) \leqslant 2$ and $\chi(G_1), \chi(G_2), \ldots, \chi(G_k) \leqslant 2$.

Note that the part (iii) can be strengthened. Namely, Zykov [22, Th. 5] proved (on the base of Lemma 1): *A planar graph G is 4-colorable if and only if G can be decomposed into two bipartite factors.*

3. Partitioning algorithms

A number of algorithms for finding a minimum coloring and thus the chromatic number of a graph are known; however, the computation time is exponential for all the methods (cf. Lawler [12]) and therefore often prohibitive. Thus faster (polynomial time) algorithms which do not always yield a minimum coloring are frequently used (see, e.g., Matula et al. [14]). However, Johnson [9] and Mitchem [15] have shown that typical algorithms of this kind all have associated classes of graphs for which the upper estimate for $\chi(G)$ can be arbitrarily great multiple of the actual $\chi(G)$. The result of Garey and Johnson [5] supports their conjecture that every polynomial time algorithms will do so. Here we suggest three partitioning algorithms for coloring and then we prove that they also give bad estimations.

The first algorithm is based on the proof of Theorem 1: first we find induced subgraphs G_i, then F, and after coloring them, we use the recoloring procedure as in the proof. Since one can easily find a (set-wise) maximal bipartite induced subgraph of a graph, we suggest to proceed as follows.

Algorithm 1. *Given a graph G find a maximal 2-colorable induced subgraph G_1 of G, then a maximal 2-colorable induced subgraph G_2 of $G - V(G_1)$, etc. In this way we partition all the points of G into induced subgraphs $G_1, \ldots, G_k$. Then find an f-coloring of the remaining spanning subgraph F of G (e.g. this algorithm can be applied again). Finally, form a coloring of G by the recoloring procedure from the proof of Theorem 1.*

The algorithm is rapid; however, there are cases when it gives a very bad bound for $\chi(G)$ as we shall show in the following example.

Example 1. Let us consider a 3-partite graph G (Johnson [9]) with parts $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$, and lines a_ib_j, a_ic_j, b_ic_j for all i, $j = 1, \ldots, n$, where $i \neq j$. Let us put $n = 2^r$ for some integer $r \geqslant 2$. According to Algorithm 1 we form $G_1 = G[\{a_1, a_2, b_1, b_2, c_1, c_2\}]$, $G_2 = [\{a_3, a_4, b_3, b_4, c_3, c_4\}]$, $\ldots$, $G_k = [\{a_{n-1}, a_n, b_{n-1}, b_n, c_{n-1}, c_n\}]$ $(k = 2^{r-1})$. The rest F is again 3-partite and we use our algorithm to color it. One sees that now we can generate induced subgraphs $G_1' = F[\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4\}]$, $\ldots$, $G_{k'}' = F[\{a_{n-3}, a_{n-2}, a_{n-1}, a_n, b_{n-3}, b_{n-2}, b_{n-1}, b_n, c_{n-3}, c_{n-2}, c_{n-1}, c_n\}]$ with $k' = 2^{r-2}$. Repeating this procedure for the new rest F', etc., we can show that the estimate for $\chi(G)$ will be $2^r = n$. This is a very bad result as G has $3n$ points and $\chi(G) = 3$.

By Remark 1 if $\chi(G_1) = \chi(G_g)$, then Theorem 1 has no advantages against Lemma 1. Therefore in some cases the following algorithm can be better since we need not find induced subgraphs.

Algorithm 2. *Decompose a given graph G into (maximal) bipartite factors G_1, $G_2, \ldots, G_k$. Then a point v gets a color $c(v) = (c_1(v), \ldots, c_k(v))$, where $c_i(v)$ is the color of v in G_i.* (There are at most $\Pi\chi(G_i)$ such k-tuples and a proof of Lemma 1 follows.)

In fact, this algorithm is a rough version of the following one, which is based on Lemma 2. We shall generate first F and then $G_1, \ldots, G_f$, where G_i is the induced subraph on the points of F colored by i. More precisely, for $f = 2$ we can proceed as follows.

Algorithm 3. *Find a maximal 2-chromatic factor F of a given graph G. Let G_i be the induced subgraph of G on the points with color i $(i = 1,2)$. By Lemma 2 if G_i is g_i-colorable $(i = 1,2)$, then G is $(g_1 + g_2)$-colorable. To estimate $\chi(G_1)$ and $\chi(G_2)$ we can apply the same algorithm as for $\chi(G)$.*

Obviously, Algorithms 2 and 3 are fast, but we shall show that Algorithm 3 (and thus also Algorithm 2) gives a bad estimate for $\chi(G)$.

Example 2. For $i=1, 2, \ldots$ let us form a graph $G(i)$ with 2^i points as follows. Put $G(1)=K_2$. If a graph $G(i-1)$ is known, consider the cycle C_{2^i} which is bipartite with parts A and B (each of 2^{i-1} points) and insert two copies of $G(i-1)$ into C_{2^i} in such a way that one copy will be placed (arbitrarily) on points of part A and the other on points of part B; the result will be denoted by $G(i)$. One can verify that any graph $G(i)$ has 2^i points, is regular of degree $2i-1$, and thus $\chi(G(i))\leq 2i$. However, Algorithm 3 allows to take $F=C_{2^i}$ and $G_1=G_2=G(i-1)$, which gives: $\chi(G(i))\leq 2\chi(G(i-1))$. Repeating this process, we obtain $\chi(G(i))\leq 2^{i-2}\chi(G(i-(i-2)))=2^i$, which is incomparable with $2i$. Note that also Algorithm 2 gives in the worst case only the bound 2^i.

Remark 2. Matula et al. [14] have tested sequential algorithms on complete k-partite graphs and shown that such graphs will always be k-colored. As for our algorithms, only the first gives always a k-coloring (as it can be easily proved) while Algorithms 2 and 3 fail even for the complete 3-partite graph with 4 points.

Remark 3. One might suggest a *"union"* of algorithms, i.e. an algorithm in which we apply the first algorithm for coloring, then the second algorithm, etc., and finally from the obtained colorings we choose a minimum coloring. However, it seems that if there is a "bad graph" for each partial algorithm, then the *disjoint union* of such graphs is a "bad graph" for the *union* of algorithms. This is certainly true if the partial algorithms are sequential [14] or (our) partitioning.

4. Biparticity and chromatic number

What happens in Algorithm 2 if we demand k to be the least possible? We shall see that in this case Algorithm 2 gives a good upper bound. Namely, this question has already been studied by Matula [13] and Harary et al. [7]. According to [7] the biparticity $\beta(G)$ of the graph G is the minimum number of bipartite subgraphs covering $E(G)$ (if G has no line, then $\beta(G)=0$). (Let $[x]$ denote the least integer greater than or equal to x.)

Lemma 3 ([13] and [7]). *For any graph G,* $\beta(G)=[\log_2 \chi(G)]$. *Consequently,* $2^{\beta-1}<\chi(G)\leq 2^{\beta}$.

Thus, having in Algorithm 2 $k=\beta(G)$, we obtain a very good bound for $\chi(G)$. However, to determine $\beta(G)$ is not easier than to find $\chi(G)$, as we have:

Theorem 2. *For any graph G,* $\chi(G)=2^{\beta(G)}-n+1$, *where n is the minimum number such that* $\beta(G+K_n)=\beta(G)+1$.

Proof. As well known, for the join (Zykov sum) we have $\chi(G+K_n)=$

$\chi(G)+n$. Hence by Lemma 3 we can write: $2^{\beta(G)}<\chi(G)+n$ and (using the minimality of n) $\chi(G)+n-1\leqslant 2^{\beta(G)}$, which gives the desired result.

Corollary. *The problem of determining for any graph G the biparticity $\beta(G)$ is NP-complete* (see e.g. [10] for the notion).

Proof. The determination of n from Theorem 2 demands only a polynomial number of steps. Thus the NP-completeness of the biparticity problem follows from Theorem 2, Lemma 3, and the NP-completeness of the chromatic number problem [10].

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Katedra numerickej matematiky
Prírodovedeckej fakulty UK
Mlynská dolina,
816 31 Bratislava

РАСКРАСКА ГРАФОВ РАЗБИЕНИЕМ

Ян Плесник

Резюме

На основании разбиения приводится простая теорема (т. 1) для раскраски вершин графа. Показаны некоторые декомпозиционные алгоритмы и примеры показывающие, что эти алгоритмы дают плохие оценки для хроматического числа. Теорема 2 связывает хроматическое число графа $\chi(G)$ и минимальное число $\beta(G)$ бихроматических факторов графа G, на которые можно G разложить [13, 7]. Из этого сразу следует, что задача нахождения числа $\beta(G)$ является NP-полной комбинаторной задачей.