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# ON THE CONFIDENCE REGION OF A VECTOR PARAMETER

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## Introduction

Throughout this paper we shall consider the following regression model:

$$E_\theta(\xi) = \mathbf{A}\theta$$

with $\xi$ an $n$-dimensional random variable. The symbol $\theta$ will denote a $k$-dimensional vector of unknown parameters and the $n \times k$ — type matrix $\mathbf{A}$ is assumed to be known. The covariance matrix of the vector $\xi$ will be denoted by $\Sigma_\xi$.

The aim of this paper is to find a confidence region of the vector parameter $\theta$ in such situations in which there would not have to exist any statistics of the form $\hat{f}(x) = \mathbf{L}'\xi$ possessing the property $E_\theta(\mathbf{L}'\xi) = \theta$ for all $\theta \in R^k$. The solutions are given under the constraints $R(\Sigma_\xi) \leq n$, $R(\mathbf{A}) \leq \min\{n, k\}$ and hence involve also singular situations.

Consequently, there is need for the use of the generalized inverses (g-inverses) as introduced in [4] both with their applications (cf. [2]).

## 1. Preliminaries

Let us consider the model of an indirectly observed vector parameter $\theta$. This model will be denoted by $(\xi, \mathbf{A}\theta, \Sigma_\xi)$ and $\xi$ will be a gaussian vector.

We shall start with a list of frequently occurring symbols and notations. Given a matrix $\mathbf{A}$, the symbol $\mathbf{A}^-$ will denote its g-inverse defined by means of the formula $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$.

Special cases:

$\mathbf{A}^-_{m(\mathbf{H})}$ — g-inverse satisfying the relations $(\mathbf{A}^-_{m(\mathbf{H})}\mathbf{A})'\mathbf{H} = \mathbf{H}\mathbf{A}^-_{m(\mathbf{H})}\mathbf{A}$ the so-called minimum $\mathbf{H}$-norm (seminorm) g-inverse of $\mathbf{A}$

$\mathbf{A}^-_{l(\mathbf{I})}$ — g-inverse satisfying the relations $(\mathbf{A}\mathbf{A}^-_{l(\mathbf{I})})' = \mathbf{A}\mathbf{A}^-_{l(\mathbf{I})}$, the so-called $\mathbf{I}$-least squares g-inverse of $\mathbf{A}$;

$\mathbf{A}^-_{l(\mathbf{I}),\, m(\Sigma_\xi)}$ — a g-inverse satisfying the relations:

$$(\mathbf{A}\mathbf{A}^-_{1(\mathbf{I}),\,m(\Sigma_\xi)})' = \mathbf{A}\mathbf{A}^-_{1(\mathbf{I}),\,m(\Sigma_\xi)}, \quad \mathbf{A}\mathbf{A}^-_{1(\mathbf{I}),\,m(\Sigma_\xi)}\mathbf{A} = \mathbf{A},$$
$$\Sigma_\xi \mathbf{A}^-_{1(\mathbf{I}),\,m(\Sigma_\xi)}\mathbf{A}\mathbf{A}^-_{1(\mathbf{I}),\,m(\Sigma_\xi)} = \Sigma_\xi \mathbf{A}^-_{1(\mathbf{I}),\,m(\Sigma_\xi)},$$
$$(\mathbf{A}^-_{1(\mathbf{I}),\,m(\Sigma_\xi)}\mathbf{A})'\Sigma_\xi = \Sigma_\xi \mathbf{A}^-_{1(\mathbf{I}),\,m(\Sigma_\xi)}\mathbf{A},$$

the so-called minimum $\Sigma_\xi$-norm (seminorm), **I**-least squares g-inverse of **A**.

Given a matrix **H** (of square type and positive semidefinite (p.s.d.)), the symbol $\|\cdot\|_{\mathbf{H}}$ will denote the norm (seminorm) in the Hilbert space with the scalar product $(x, y) = x'\mathbf{H}y$. Especially if **I** is a $k \times k$-type identity matrix, $\|\cdot\|_{\mathbf{I}}$ is the usual Euclidean norm in $R^k$.

Let us consider a consistent system $\mathbf{A}x = y$. For any matrix **H** of the above described type let $\|\cdot\|_{\mathbf{H}}$ be the norm (seminorm) on the set of all solutions of the given system. The solution $x_o$ is said to be the minimum **H**-norm (seminorm) solution provided $\|x_o\|_{\mathbf{H}} = \min_{\{x:\, \mathbf{A}x = y\}} \|x\|_{\mathbf{H}} = \min\{\|x\|_{\mathbf{H}}\colon \mathbf{A}x = y\}$. The name for $\mathbf{A}^-_{m(\mathbf{H})}$ adopted above is justified by the fact that the vector $\mathbf{A}^-_{m(\mathbf{H})}y$ is a minimum **H**-norm (seminorm) solution of the system $\mathbf{A}x = y$. The family of all minimum **H**-norm (seminorm) g-inverses of the matrix $\mathbf{A}'$ is denoted by $(\mathscr{A}')^-_{m(\mathbf{H})}$.

**Definition 1.1.** *Let $p$, $\theta$ be two column vectors with the range in $R^k$. The linear function $f(\theta) = p'\theta$ is said to be unbiassedly estimable iff there is a column vector $L \in R^n$ such that $\mathrm{E}_\theta(L'\xi) = p'\theta$ for all $\theta \in R^k$.*

**Lemma 1.1.** [2] *A function $f(\theta) = p'\theta$ is unbiassedly estimable if and only if $p \in \mathscr{M}(\mathbf{A}')$, the range of the matrix $\mathbf{A}'$.*

**Definition 1.2.** *Let $\mathrm{E}_\theta(\xi) = \mathbf{A}\theta$, where $\xi$ is an $n$-dimensional random vector. Let $\Sigma_\xi$ be its covariance matrix. Any vector $\mathbf{M}\xi$ satisfying the conditions:*

*(i)* $\forall p \in \mathscr{M}(\mathbf{A}')\ \forall \theta \in R^k\colon \mathrm{E}_\theta(p'\mathbf{M}\xi) = p'\theta$

*(ii)* $\forall p \in \mathscr{M}(\mathbf{A}')\ [\forall \mathbf{M}_1\xi \neq \mathbf{M}\xi(\mathrm{E}_\theta(p'\mathbf{M}_1\xi) = p'\theta) \Rightarrow (\sigma^2 p'\mathbf{M}\xi \leq \sigma^2 p'\mathbf{M}_1\xi)]$

*is said to be a fictitious jointly effective estimate of $\theta$. Here the symbol $\sigma^2$ is used to denote the variance, and **M** is an arbitrary matrix of type $k \times n$.*

Remark 1.1. The term fictitious should emphasize the following fact. In the case $\mathscr{M}(\mathbf{A}') = R^k$ we have $\mathrm{E}_\theta(\mathbf{M}\xi) = \theta$ for all $\theta \in R^k$. However, if $\mathscr{M}(\mathbf{A}') \subsetneqq R^k$, there is $\theta_o \in R^k$ such that $\mathrm{E}_\theta(\mathbf{M}\xi) \neq \theta_0$, hence $\mathbf{M}\xi$ is not an unbiassed estimate of $\theta_0$. On the other hand, when substituting the vector $\mathbf{M}\xi$ for the argument of any unbiassedly estimable function $f(\theta) = p'\theta$, one gets an unbiassed estimate.

Remark 1.2. Let $f(\theta) = p'\theta$ with $p \in \mathscr{M}(\mathbf{A}')$ (i.e. $f$ is estimable, cf. Lemma 1.1). An estimate $L'\xi$ is said to be unbiassed if $\mathbf{A}'L = p$. The variance of such an estimate equals $L'\Sigma_\xi L$. It is natural to conceive an unbiassed estimate $\hat{f}(\xi) = L'\xi$ to be the best if $L'\Sigma_\xi L$ is minimal, i.e. if $\|L\|_{\Sigma_\xi}$ is minimal.

**Theorem 1.1.** [2] *The best linear unbiassed estimate of an unbiassedly estimable function $f(\theta) = p'\theta$ is given by*

$$\hat{f}(\xi) = p'[(\mathbf{A}')^-_{m(\Sigma_\xi)}]'\xi.$$

Remark 1.3. Since the expression in the above theorem is given by means of g-inverses, it is not unique. One of the possible choices is the following one

$$\boldsymbol{p}'[\mathbf{A}'(\Sigma_\xi + \mathbf{AA}')^-\mathbf{A}]^-\mathbf{A}'[(\Sigma_\xi + \mathbf{AA}')^-]'\xi.$$

Its variance is given by the formula

$$\sigma^2(\hat{f}(\xi)) = \boldsymbol{p}'[(\mathbf{A}'(\Sigma_\xi + \mathbf{AA}')^-\mathbf{A})^- - \mathbf{I}]\boldsymbol{p}$$

established in [2].

**Theorem 1.2.** [2] *Let us consider the covariance matrix of the form $\sigma^2\mathbf{H}$ with $\sigma^2$ an unknown scalar and $\mathbf{H}$ a known matrix, respectively. Let $\hat{\theta}$ be a fictitious jointly effective estimate of $\theta$ (cf.* Def. 1.2.). *Then the formula*

$$\hat{\sigma}^2 = (\xi - \mathbf{A}\hat{\theta})'\mathbf{H}^-(\xi - \mathbf{A}\hat{\theta})/s$$

*gives an unbiassed estimate of $\sigma^2$. Here $s = \mathrm{R}(\mathbf{H}, \mathbf{A}) - \mathrm{R}(\mathbf{A})$ and denotes the difference of ranks of the hypermatrix* $(\mathbf{H}, \mathbf{A})$ *and of the matrix* $\mathbf{A}$.

**Lemma 1.2.** [4] *Let $\xi$ be a gaussian vector with a zero mean and the covariance matrix $\Sigma_\xi = \Sigma$. Then the quadratic form $\xi'\Sigma^-\xi$ is $\chi^2(k)$-distributed, independently of which $\Sigma^-$ was chosen, and $k = \mathrm{R}(\Sigma)$.*

## 2. Confidence Region for the Estimable Functions of a Vector Parameter

Let us consider the model $\mathrm{E}_\theta(\xi) = \mathbf{A}\theta$. We shall consider the functions $f(\theta) = \mathbf{P}'\theta$ with $\mathcal{M}(\mathbf{P}) = \mathcal{M}(\mathbf{A}')$. It is assumed that $\Sigma_\xi = \sigma^2\mathbf{H}$.

1. $\sigma^2$-known.

**Lemma 2.1.** *The quadratic form*

$$[\hat{f}(\xi) - f(\theta)]'[\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}]^-[\hat{f}(\xi) - f(\theta)]/\sigma^2$$

*is a $\chi^2(k)$ — variate with* $k = \mathrm{R}(\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\ \mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P})$.

Proof. Follows immediately from lemma 1.2.

To calculate the number $k$ in the preceding Lemma, we have to find a simpler expression for the rank of the matrix

$$\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}.$$

**Theorem 2.1.** $\mathrm{R}(\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\ \mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}) = \mathrm{R}(\mathbf{H}(\mathbf{H} + \mathbf{AA}')^-\mathbf{A})$.

Proof. Since $\mathbf{H}$ is p.s.d., there is $\mathbf{J}$ such that $\mathbf{H} = \mathbf{JJ}'$ (cf. [2]). It was pointed out above that any minimum $\mathbf{H}$-norm (seminorm) solution $\boldsymbol{x}$ of the system $\mathbf{A}'\boldsymbol{x} = \boldsymbol{y}$ has the form $\boldsymbol{x} = (\mathbf{A}')^-_{m(\mathbf{H})}\boldsymbol{y}$. Thus we get a family $\mathcal{N} = \{(\mathbf{A}')^-_{m(\mathbf{H})}\boldsymbol{y}: (\mathbf{A}')^-_{m(\mathbf{H})} \in (\mathcal{A}')^-_{m(\mathbf{H})}\}$ of solutions. If $(\mathbf{A}')^-_{m(\mathbf{H})}\boldsymbol{y}$ is a particular solution, then $\mathcal{N}=\{(\mathbf{A}')^-_{m(\mathbf{H})}\boldsymbol{y}+\boldsymbol{k}\colon \boldsymbol{k}\in \operatorname{Ker} \mathbf{A}'\cap \operatorname{Ker} \mathbf{H}\}$. Consequently $\mathcal{M}((\mathbf{A}'^-_{m(\mathbf{H})}\mathbf{A}') = \mathcal{M}((\mathbf{A}')^-_{(\mathbf{H})_o}\mathbf{A}') \vee (\operatorname{Ker} \mathbf{A}' \cap \operatorname{Ker} \mathbf{H})$, where $\mathcal{M}((\mathbf{A}')^-_{m(\mathbf{H})_o}\mathbf{A}') \vee (\operatorname{Ker} \mathbf{A}' \cap \operatorname{Ker} \mathbf{H})$ denotes the subspace of $R^n$ generated by $\mathcal{M}((\mathbf{A}')^-_{m(\mathbf{H})_o}\mathbf{A}') \cup (\operatorname{Ker} \mathbf{A}' \cap \operatorname{Ker} \mathbf{H})$. Thus the dimension of $\mathcal{M}(\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{A}')$ is invariant for any choice of the g-inverse. One possible choice is the matrix $(\mathbf{H}+\mathbf{AA}')^-\ \mathbf{A}[\mathbf{A}'(\mathbf{H}+\mathbf{AA}')^-\mathbf{A}]^-$. Since the matrix $\mathbf{H}$ and the matrix $\mathbf{AA}'$ are p.s.d., we have $\mathcal{M}(\mathbf{H}) \subset \mathcal{M}(\mathbf{H}+\mathbf{AA}')$ and $\mathcal{M}(\mathbf{A})=\mathcal{M}(\mathbf{AA}') \subset \mathcal{M}(\mathbf{H}+\mathbf{AA}')$, respectively.

It was assumed that $\mathcal{M}(\mathbf{P})=\mathcal{M}(\mathbf{A}')$. Therefore there is a nonsingular matrix $\mathbf{Q}$ with $\mathbf{P}=\mathbf{A}'\mathbf{Q}$. Substitute for $\mathbf{P}$. Then

$$\begin{aligned} R(\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}) &= R(\mathbf{Q}'\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{A}'\mathbf{Q}) = \\ &= R(\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}) = R(\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}), \end{aligned}$$

using the idempotence of the matrix $\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{H})}]'$ together with its property $\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{H})}]'\ \mathbf{H}=\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{A}'$ (cf. [4] for a more detailed discussion). Using the equivalences

$$\begin{gathered} \mathcal{M}(\mathbf{A})\subset\mathcal{M}(\mathbf{H}+\mathbf{AA}')\Leftrightarrow\exists\mathbf{C}\colon \mathbf{A}=(\mathbf{H}+\mathbf{AA}')\mathbf{C} \\ \mathcal{M}(\mathbf{H})\subset\mathcal{M}(\mathbf{H}+\mathbf{AA}')\Leftrightarrow\exists\mathbf{E}\colon \mathbf{H}=(\mathbf{H}+\mathbf{AA}')\mathbf{E}=\mathbf{E}'(\mathbf{H}+\mathbf{AA}') \end{gathered}$$

one obtains the relations

$$R(\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H})=R(\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{A}')=R(\mathbf{E}'(\mathbf{H}+\mathbf{AA}')(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{C}'(\mathbf{H}+\mathbf{AA}')).$$

In the latter relation, let us substitute for $(\mathbf{A}')^-_{m(\mathbf{H})}$ the choice $(\mathbf{H}+\mathbf{AA}')^-\times$ $\times\mathbf{A}[\mathbf{A}'(\mathbf{H}+\mathbf{AA}')^-\mathbf{A}]^-$. Then

$$R(\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{A}')=R(\mathbf{E}'(\mathbf{H}+\mathbf{AA}')\mathbf{C}(\mathbf{C}'(\mathbf{H}+\mathbf{AA}')\mathbf{C})^-\mathbf{C}'(\mathbf{H}+\mathbf{AA}')).$$

The proof is finished by establishing the relations

$$R(\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{A}')=R(\mathbf{E}'(\mathbf{H}+\mathbf{AA}')\mathbf{C})=R(\mathbf{E}'\mathbf{A})=R(\mathbf{HC})=R(\mathbf{H}(\mathbf{H}+\mathbf{AA}')^-\mathbf{A}),$$

which are an immediate consequences of the properties of g-inverses,

Q.E.D.

Remark 2.1. If $\mathcal{M}(\mathbf{A})\subset\mathcal{M}(\mathbf{H})$, then it follows easily that

$$R(\mathbf{H}(\mathbf{H}+\mathbf{AA}')^-\mathbf{A})=R(\mathbf{A}).$$

**Theorem 2.2.** *Let $\xi$ be an $n$-dimensional gaussian vector with the expectation $\mathbf{A}\theta$ and the covariance matrix $\sigma^2\mathbf{H}$, respectively. Let $\theta$ be an unknown vector parameter. Consider the function $f(\theta)=\mathbf{P}'\theta$ with $\mathcal{M}(\mathbf{P})=\mathcal{M}(\mathbf{A}')$. Then the true value of $\mathbf{P}'\theta$ lies in the set*

$$\begin{gathered} \mathcal{V}=\{\boldsymbol{y}\colon (\boldsymbol{y}-\hat{f}(\xi))'[\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}]^-(\boldsymbol{y}-\hat{f}(\xi))/\sigma^2\leq c^2\}\cap \\ \cap\{\hat{f}(\xi)+\mathcal{M}(\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P})\} \end{gathered}$$

*with the probability*

$$\text{Prob}\,\{\chi^2(r)\leqslant c^2\}.$$

*Here* $r=R(\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}$ *and* $\hat{f}(\xi)$ *is the best unbiassed estimate of* $\mathbf{P}'\theta$.

Proof. By Theorem 1.1., $\hat{f}(\xi)=\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\xi$. According to Lemma 1.2 and Theorem 2.1.

$$\begin{gathered}\text{Prob}\,\{(\mathbf{P}'\theta-\hat{f}(\xi))'[\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}]^-(\mathbf{P}'\theta-\hat{f}(\xi))/\sigma^2\leqslant c^2\}=\\ =\text{Prob}\,\{\mathbf{P}'\theta\in\{y\colon(y-\hat{f}(\xi))'[\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}]^-(y-\hat{f}(\xi))/\sigma^2\leqslant\\ \leqslant c^2\}\}=\text{Prob}\,(\chi^2(r)\leqslant c^2).\end{gathered}$$

On the other hand

$$\text{Prob}\,\{\hat{f}(\xi)\in\{\mathcal{M}(\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P})+E_\theta\hat{f}(\xi)\}\}=1$$

and hence

$$\begin{gathered}\text{Prob}\,\{\mathbf{P}'\theta\in\{y\colon(y-\hat{f}(\xi))'[\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}]^-(y-\hat{f}(\xi))/\sigma^2\leqslant\\ \leqslant c^2\}\cap\{\hat{f}(\xi)+\mathcal{M}(\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P})\}=\text{Prob}\,(\chi^2(r)\leqslant c^2).\end{gathered}$$

Q.E.D.

2. $\sigma^2$-unknown.

**Lemma 2.2.** *Let* $f(\theta)=\mathbf{P}'\theta$, $\mathcal{M}(\mathbf{P})=\mathcal{M}(\mathbf{A}')$ *within the regression model* $(\xi, \mathbf{A}\theta, \sigma^2\mathbf{H})$. *Let* $\mathbf{v}=\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{H})}]'\xi-\xi$ *(the vector* $\mathbf{v}$ *is said to be the correction vector). The random variable* $\mathbf{v}'\mathbf{H}^-\mathbf{v}$ *is a* $\sigma^2\chi^2(R(\mathbf{H}, \mathbf{A})-R(\mathbf{A}))$*-variate.*

**Lemma 2.3.** *The vectors* $\mathbf{v}$ *and* $\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\xi$ *are stochastically independent.*
In the proof cf. [1] and [2].

**Theorem 2.3.** *Let* $\xi$ *be an* $n$*-dimensional gaussian vector with* $E_\theta(\xi)=\mathbf{A}\theta$, $\Sigma_\xi=\sigma^2\mathbf{H}$. *Let* $f(\theta)=\mathbf{P}'\theta$ *be unbiassedly estimable and* $\mathcal{M}(\mathbf{P})=\mathcal{M}(\mathbf{A}')$. *Then the true value of* $\mathbf{P}'\theta$ *lies in the set*

$$\mathcal{W}=\{\mathbf{y}\colon\mathcal{A}(\mathbf{y})/B\leqslant c^2\}\cap\{\hat{f}(\xi)+\mathcal{M}(\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P})\}$$

*with the probability*

$$\text{Prob}\,(F(r, R(\mathbf{H}, \mathbf{A})-R(\mathbf{A}))\leqslant c^2).$$

*Here*

$$\mathcal{A}(y)=(y-\hat{f}(\xi))'[\mathbf{P}'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}]^-(y-\hat{f}(\xi))/r$$
$$B=\mathbf{v}'\mathbf{H}^-\mathbf{v}/(R(\mathbf{H}, \mathbf{A})-R(\mathbf{A}))$$

$$r=\begin{cases}R(\mathbf{H}(\mathbf{H}+\mathbf{A}\mathbf{A}')^-\mathbf{A}) & \text{(Theorem 2.1)}\\ R(\mathbf{A})\ \text{if}\ \mathcal{M}(\mathbf{A})\subset\mathcal{M}(\mathbf{H}) & \text{(Remark 2.1).}\end{cases}$$

To prove the Theorem let us note that the Fisher random variable is the ratio of two stochastically independent (Lemma 2.3) $\chi^2$ — variates (Lemma 1.2), each divided by the corresponding degrees of freedom (cf. Lemma 1.2, Theorem 2.1, Remark 2.1). Otherwise, the proof is similar to the proof of Theorem 2.2.

## 3. Generalization

Let us again consider the regression model $(\xi, \mathbf{A}\theta, \sigma^2\mathbf{H})$ with $\theta$ an unknown parameter. Now let us suppose that there is at least one column, say $e_i$, of the identity matrix $\mathbf{I}$ such that $e_i \notin \mathcal{M}(\mathbf{A}')$.

The columns of the identity matrix will be projected onto the manifold $\mathcal{M}(\mathbf{A}')$; the projections being considered in the Euclidean norm. The obtained vectors result in a matrix, say $\mathbf{P}_o$. Any vector $e_i$ can be decomposed in the form

$$e_i = (p_i)_o + k_i, \quad \text{where} \quad (p_i)_o \in \mathcal{M}(\mathbf{A}') \quad \text{and} \quad k_i \in \operatorname{Ker} \mathbf{A}.$$

Note that $\operatorname{Ker} \mathbf{A}$ denotes as usually the set of all solutions of the homogeneous system $\mathbf{A}x = 0$. The matrix consisting of the vectors $k_i$ will be denoted by $\mathbf{K}$. Let

$$\vartheta_1 = \mathbf{P}_0'\theta, \quad \vartheta_2 = \mathbf{K}'\theta.$$

Let us note that $\mathbf{P}_o' = \mathbf{A}'(\mathbf{A}\mathbf{A}')^-\mathbf{A}$.

**Lemma 3.1.** $\mathrm{E}_\theta(\hat{\vartheta}_2) = 0$ *for* $\hat{\vartheta}_2 = \mathbf{K}'[(\mathbf{A}')^-_{l(\mathbf{I}),\, m(\mathbf{H})}]'\xi$.

The Lemma follows if we use the definition of $\mathbf{I}$-least square g-inverse.

Remark 3.1. The only possible projection operator $P^{\mathbf{I}}_{\mathcal{M}(\mathbf{A}')}$ projecting the columns of the matrix $\mathbf{I}$ onto the manifold $\mathcal{M}(\mathbf{A}')$ is the following one:

$$P^{\mathbf{I}}_{\mathcal{M}(\mathbf{A}')} = \mathbf{A}'(\mathbf{A}\mathbf{A}')^-\mathbf{A}.$$

Thus $\vartheta_1 = \mathbf{A}'(\mathbf{A}\mathbf{A}')^-\mathbf{A}\theta$ and $\vartheta_2 = (\mathbf{I} - \mathbf{A}'(\mathbf{A}\mathbf{A}')^-\mathbf{A})\theta$.

**Lemma 3.2.** *Let* $\vartheta_1 = \mathbf{A}'(\mathbf{A}\mathbf{A}')^-\mathbf{A}\theta$. *Then the random vector* $\hat{\vartheta}_1$ *given by the relation*

$$\hat{\vartheta}_1 = \mathbf{A}'(\mathbf{A}\mathbf{A}')^-\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{H})}]'\xi$$

*has the following property*

*(i)* $\mathrm{E}(\hat{\vartheta}_1) = \vartheta_1$

*(ii) if* $\tilde{\vartheta}_1 \in \mathcal{M}(\mathbf{A}')$ *such that* $\mathrm{E}(\tilde{\vartheta}_1) = \vartheta_1$ *then* $\|\tilde{\vartheta}_1\|_{\Sigma_\xi} - \|\hat{\vartheta}_1\|_{\Sigma_\xi} \geq 0$.

**Theorem 3.1.** *Let* $\xi$ *be a gaussian vector with* $\mathrm{E}_\theta(\xi) = \mathbf{A}\theta$, $\Sigma_\xi = \sigma^2\mathbf{H}$. *Let* $\sigma^2$ *as well as* $\mathbf{H}$ *be known. Then the true value of the parameter* $\theta$ *lies in the set* $\mathcal{L}$ *with the probability* $\operatorname{Prob}(\chi^2(r) \leq c^2)$. *Here* $\mathcal{L} = \{x + y\colon x \in \mathcal{K},\ y \in \operatorname{Ker} \mathbf{A}\}$, *where*

$$\begin{aligned}\mathcal{K} = \{y\colon (y - \mathbf{P}_o'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\xi)'[\mathbf{P}_o'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}_o]^- \times \\ \times (y - \mathbf{P}_o'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\xi)/\sigma^2 \leq c^2\} \cap \\ \cap \{\mathcal{M}(\mathbf{P}_o'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^-_{m(\mathbf{H})}\mathbf{P}_o) + \mathbf{P}_o'[(\mathbf{A}')^-_{m(\mathbf{H})}]'\xi\}\end{aligned}$$

*and*

$$r = \mathrm{R}(\mathbf{H}(\mathbf{H} + \mathbf{AA}')^{-}\mathbf{A}).$$

**Theorem 3.2.** *Let $\xi$ be a gaussian vector with $\mathrm{E}_\theta(\xi) = \mathbf{A}\theta$, $\Sigma_\xi = \sigma^2\mathbf{H}$. Let $\sigma^2$ be an unknown parameter and $\mathbf{H}$ be a known matrix, respectively. Then the true value of the parameter $\theta$ lies in the set*

$$\mathscr{W} = \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in \tilde{\mathscr{K}}, \boldsymbol{y} \in \operatorname{Ker} \mathbf{A}\}$$

*with the probablity*

$$\operatorname{Prob}(\mathrm{F}(r, \mathrm{R}(\mathbf{H}, \mathbf{A}) - \mathrm{R}(\mathbf{A})) \leqslant \mathrm{c}^2).$$

*Here*

$$\tilde{\mathscr{K}} = \{\boldsymbol{y} : \mathscr{l}(\boldsymbol{y})/\mathrm{B} \leqslant \mathrm{c}^2\} \cap \{\mathscr{M}(\mathbf{P}_o'[(\mathbf{A}')^{-}_{\mathrm{m}(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^{-}_{\mathrm{m}(\mathbf{H})}\mathbf{P}_o) + \mathbf{P}_o'[(\mathbf{A}')^{-}_{\mathrm{m}(\mathbf{H})}]'\xi$$

$$\mathscr{l}(\boldsymbol{y}) = (\boldsymbol{y} - \mathbf{P}_o'[(\mathbf{A}')^{-}_{\mathrm{m}(\mathbf{H})}]'\xi)'[\mathbf{P}_o'[(\mathbf{A}')^{-}_{\mathrm{m}(\mathbf{H})}]'\mathbf{H}(\mathbf{A}')^{-}_{\mathrm{m}(\mathbf{H})}\mathbf{P}_o]^{-} \times$$
$$\times (\boldsymbol{y} - \mathbf{P}_o'[(\mathbf{A}')^{-}_{\mathrm{m}(\mathbf{H})}]'\xi)/r,$$
$$\mathrm{B} = \boldsymbol{v}'\mathbf{H}^{-}\boldsymbol{v}/(\mathrm{R}(\mathbf{H}, \mathbf{A}) - \mathrm{R}(\mathbf{A})), \quad r = \mathrm{R}(\mathbf{H}(\mathbf{H} + \mathbf{AA}')^{-}\mathbf{A}).$$

The proofs of Theorems 3.1 and 3.2 follow from Section 2. It suffices to consider the confidence region for the projection $\vartheta_1$ of the parameter $\theta$ because of the fact that $\mathrm{E}(\hat{\vartheta}_2) = \boldsymbol{0}$.

It is easy to see that the sets $\mathscr{L}$, resp. $\mathscr{W}$, from the Theorems above consist of two parts. The first part $\mathscr{K}$ (resp. $\tilde{\mathscr{K}}$) is the set where lies the projection $\vartheta_1$ of $\theta$, which is an unbiassedly estimable parameter, with $\operatorname{Prob}(\chi^2(r) \leqslant \mathrm{c}^2)$ (resp. $\operatorname{Prob}(\mathrm{F}(r, \mathrm{R}(\mathbf{H}, \mathbf{A}) - \mathrm{R}(\mathbf{A})) \leqslant \mathrm{c}^2)$).

REFERENCES


[1] KUBÁČEK, L., ŠUJAN, Š.: Základné poznatky štatistickej teórie odhadu, Vydavateľstvo RUK, Bratislava 1975.

[2] KUBÁČEK, L., WIMMER, G., VOLAUFOVÁ, J.: Matematické metódy teórie odhadov v metronomike, Záverečná správa čiastkovej úlohy štátneho programu základného výskumu III-7-1/1, Ústav teórie merania SAV, Bratislava 1975.

[3] RAO, R. C.: Linear statistical inference and its applications, John Wiley, New York 1965.

[4] RAO, R. C., MITRA, S. K.: Generalized inverse of matrices and its applications, John Wiley, New York 1971.



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## ДОВЕРИТЕЛЬНАЯ ОБЛАСТЬ ВЕКТОРНОГО ПАРАМЕТРА

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Резюме

В рамках регрессионной модели $(\xi, \mathbf{A}\theta, \sigma^2\mathbf{H})$ решается проблема доверительной области как для несмещенно оцениваемой функции векторного параметра, так и для несмещенно неоцениваемого векторного параметра $\theta$. Предполагается, что матрицы $\mathbf{H}$ и $\mathbf{A}$ известны, а $\sigma^2$ может быть известный или неизвестный параметр.