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SOMEWHAT CONTINUITY ON LINEAR TOPOLOGICAL SPACES IMPLIES CONTINUITY

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I. Several mathematicians have generalized the classical results of Banach and Steinhaus. A strictly topological version of those theorems can be found in [8] as Theorems 2.5 and 2.11.

In [2] there is given an extension of the Closed Graph and the Open Mapping Theorems for topological spaces complete in the sense of Čech by means of nearly continuity; namely, it is whown that under some additional assumptins nearly continuity becomes continuity.

We will show that the condition for a transformation to be continuous in the theorems mentioned above can be weakened to somewhat continuity in the sense of Gentry and Hoyle (see [4]).

A mapping $f: X \to Y$ is called nearly continuous if for every open subset A of Y, $f^{-1}(A) \subset \operatorname{Int} \operatorname{Cl} f^{-1}(A)$.

We say that $f: X \to Y$ is somewhat continuous if for every open subset A of Y, $f^{-1}(A) \neq \emptyset$ implies that $\operatorname{Int} f^{-1}(A) \neq \emptyset$. Easy examples of functions from reals into reals show that somewhat continuity neither implies nor is implied by nearly continuity. The relation between these various kinds of almost continuity is as follows:

$$\begin{pmatrix}\text{somewhat}\\ \text{continuous}\end{pmatrix} \Leftarrow \begin{pmatrix}\text{almost continuous in}\\ \text{the sense of Frolik}\end{pmatrix} \Leftarrow (\text{continuous}) \Rightarrow \begin{pmatrix}\text{nearly}\\ \text{continuous}\end{pmatrix}$$

None of the above implications can be inverted (see [7], Proposition 1). We start from the following, rather technical.

Theorem 1. *If g is a somewhat continuous, somewhat open mapping from X onto Y, f is a mapping from Y into Z, then: f is somewhat continuous if and only if $f \circ g$ is somewhat continuous.*

Proof. Necessity. It follows from Theorem 2.6 p. 320 of [1].

Sufficiency. Let U be an open subset of Z, such that $f^{-1}(U) \neq \emptyset$. It will be shown that $\operatorname{Int} f^{-1}(U) \neq \emptyset$. By "onto" assumption, $g^{-1}(f^{-1}(U)) \neq \emptyset$. From the somewhat continuity of the composition $f \circ g$ it follows that $\operatorname{Int} g^{-1}(f^{-1}(U)) \neq \emptyset$. By the somewhat openess of g we have that the following set $\operatorname{Int} g(\operatorname{Int} g^{-1}(f^{-1}(U))) \neq \emptyset$. This set is an open subset of $f^{-1}(U)$, and therefore it is contained in $\operatorname{Int} f^{-1}(U)$. Thus the proof is finished.

Corollary. *If g is a continuous, open mapping from X onto Y, f is a mapping from Y into Z, then: f is somewhat continuous if and only if $f \circ g$ is somewhat continuous.*

II. If F is a linear subspace of a linear topological space X, the linear topological quotient space is the quotient space X/F with a topology such that the set U in X/F is open if and only if $Q^{-1}(U)$ is open in X, where Q is the quotient map; that is $Q(x) = x + F$. This topology for X/F is the quotient topology and this topology is a vector topology. For all notions used but undefined here see [5], pp. 9, 34 and 39.

Theorem 2 ([5], Theorem 5.7 p. 39). *Let F be a linear topological subspace of a linear topological space X, let X/F be the quotient space and let Q be the quotient map. Then the map Q is linear, continuous and open.*

The following theorem which is a consequence of Corollary and Theorem 2 generalizes the second part of Theorem 5.7 of [5].

Theorem 3. *A function T on X/F is somewhat continuous if and only if the composition $T \circ Q$ is somewhat continuous.*

Now we will show our main result

Theorem 4. *If T is a somewhat continuous linear transformation from a lineqr topological space X to a linear topological space Y, then T is continuous.*

Proof. Let 0_X and 0_Y denote the zeros in X and Y, respectively. We will show that T is continuous at 0_X. Our transformation is linear, thus $T(0_X) = 0_Y$. Let U be an open subset around 0_Y in Y. Then there exists an open set V which is symmetric, such that $0_Y \in V$ and $V + V \subset U$, because there is a fundamental system of symmetric open sets around 0_Y. By somewhat continuity, $\operatorname{Int} T^{-1}(V) \neq \emptyset$, since $T^{-1}(V)$, containing 0_X, is non-empty.

Let $x \in \operatorname{Int} T^{-1}(V)$. Hence 0_X is in $-x + \operatorname{Int} T^{-1}(V)$, which is an open set as a homeomorphic image of $\operatorname{Int} T^{-1}(V)$. Thus we have: $T(-x + \operatorname{Int} T^{-1}(V)) = -T(x) + T(\operatorname{Int} T^{-1}(V)) \subset -T(x) + V$ and $-T(x) \in V$, since $T(x) \in V$. Therefore $T(-x + \operatorname{Int} T^{-1}(V)) \subset V + V \subset U$ and thus T is continuous.

Corollary 5. *If f is a somewhat continuous functional on a linear topological space, then f is continuous.*

Since a transformation T on the quotient space X/F is linear if and only if the composition $T \circ Q$ is linear, it follows by Theorems 3 and 4 that T is continuous if $T \circ Q$ is linear and somewhat continuous.

III. By our Theorem 4 and Theorems 2.5 and 2.11 of [8], and p. 110 of [6] we can deduce the following corollaries:

6. *Let X be a F-space, Y — a linear topological space and T: $X \to Y$ be a somewhat continuous, linear transformation such that $T(X)$ is of the second category in Y. Then:*

(i) $T(X) = Y$
(ii) *the transformation T is open*
(iii) *Y is a F-space.*

7. *Let X, Y — linear topological spaces, S — a set of somewhat continuous linear transformations of X into Y, B — the set of such $x \in X$ that their orbits*

$$S(x) = \{T(x) \colon T \in S\}$$

be bounded in Y. If B — the set of the second category in X, then $B = X$ and the set S is uniformly continuous.

8. *If f is a real-valued linear functional on a real linear topological space, then f is somewhat continuous if and only if the set $\{x \colon f(x) = 0\}$ is closed.*

Combining our Theorem 4 with the main result of [9], Theorem 3.1 p. 283 we obtain the following corollary:

9. *The image of a linear pseudo-complete space under a linear somewhat continuous almost open mapping is complete if its completion has a continuous metric.*

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