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PSEUDOPROJECTIVE MODULES

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The concept of a pseudoinjective module is well known (see e.g. [7], [9]). As almost everything in modules, the concept possesses its dual, pseudoprojective module. In this paper we intend to make a comprehensive exposition of the concept which has been scarcely studied so far.

Preliminaries

In the following, R stands for an associative ring with unit. The category of all unital left R-modules is denoted by R-mod.

Let r be a preradicel for R-mod (i.e. r is a subfunctor of the identity functor). Then r is said to be

— idempotent if $r(r(M))=r(M)$ for every module M,

— a radical if $r(M/r(M))=0$ for every module M,

— hereditary if $r(N)=N\cap r(M)$ for every submodule N of a module M,

— cohereditary if $r(M/N)=(r(M)+N)/N$ for every submodule N of a module M,

— costable if $r(R)$ is a direct summand in R,

— centrally splitting if it is cohereditary and $r(R)$ is a ring direct summand in R,

— splitting if $r(M)$ is a direct summand in M for every module M.

As it is easy to see, r is cohereditary iff $r(M)=r(R)\cdot M$ for every module M. Conversely, if I is an ideal and $r(M)=IM$, then r is a cohereditary radical. In this case, r is idempotent iff I is so.

Let r and s be two preradicals. Put $(r\circ s)(M) = r(s(M))$ for every module M. It is visible that $r\circ s$ is a preradical. Further, $T_r=\{M|r(M)=M\}$ and $F_r=\{M|r(M)=0\}$. The modules from T_r and F_r are called r-torsion and r-torsionfree, resp.

Let A be a non-empty class of modules. For every module M, define $p_A(M)=\Sigma\mathrm{Im}f$, $f\in\mathrm{Hom}(X,M)$, $X\in A$ and $q_A(M) = \cap\mathrm{Ker}f$, $f\in\mathrm{Hom}(M,X)$, $X\in A$. Obviously, p_A is an idempotent preradical and q_A is a radical. If r is a preradical, we put $\bar{r} = p_{T_r}$. If A is the class of all simple modules, then $p_A = J$. Further, $\mathrm{id} = p_{R-\mathrm{mod}}$ and $\mathrm{zer} = q_{R-\mathrm{mod}}$.

If M is a module then $E(M)$ denotes the injective hull of M. A module is called cocyclic if it is an essential extension of a simple module. Clearly, a module is cocyclic iff it is subdirectly irreducible. An epimorphism $f: P \to M$ is said to be a projective cover of M if P is projective and $\mathrm{Ker} f$ is small in P. The ring R is said to be left perfect if every module has a projective cover. A module M is said to be cofaithfull if $p_{\{M\}}(E(R)) = E(R)$.

Let M_i, $i \in I$ be a family of modules. Then $\perp\!\perp M_i$ denotes the direct sum of the family.

As for details concerning the preliminary text, the reader is referred to [2], [3], [4], [5] and [8].

1. Preradical Y

For every module M, let $Y(M) = \cap N$, where N runs through all submodules N of M such that M/N is cocyclic and small in its injective hull.

1.1 Lemma. *Let N be a small submodule of a module M. Then N is small in $E(N)$.*

Proof. Easy.

1.2 Lemma. *Let r be a preradical such that $\bar{r}$ is cohereditary. Then $r(M)/\bar{r}(M)$ is small in $M/\bar{r}(M)$ for every module M.*

Proof. Let $\bar{r}(M) \subseteq A \subseteq M$ and $r(M) + A = M$. Then M/A is both r-torsion and r-torsionfree. Hence $A = M$.

1.3 Proposition.

(i) *Y is a radical and $Y = q_C$, where C is the class of all modules which are small submodules in some modules.*

(ii) *If R is left perfect, then Y is cohereditary.*

(iii) *If no non-zero simple module is injective, then $Y \subseteq J$.*

(iv) *If R is left herditary, then every injective module is Y-torsion.*

(v) *If $r \subseteq Y$ is a preradical such that r is cohereditary, then $\bar{r} = r \circ r$.*

(vi) *If R is left perfect, then $\bar{Y} = Y \circ Y$.*

Proof.

(i) The assertion is an immediate consequence of 1.1.

(ii) Clearly, the class C is closed under homomorphic images. Now, taking into account that R is left perfect, it is easy to check that Y is cohereditary.

(iii) If M is a non-injective simple module, then M is small in $E(M)$.

(iv) R is left hereditary, i.e. injective modules are closed under factors, and the situation is clear.

(v) This is an easy consequence of 1.2.
(vi) Apply (v) and (ii).

1.4 Proposition.
(i) *Let $r \subseteq Y$ be a preradical such that $R/r(R)$ has a projective cover and $r(r(R)) = r(R)$. Then r is costable.*
(ii) *Let $r \subseteq Y$ be a preradical such that $r(r(R)) = r(R) \subseteq J(R)$. Then $r(R) = 0$.*
Proof.
(i) Let $f: P \to R/r(R)$ be a projective cover and h be the natural homomorphism of R onto $R/r(R)$. Then $h = fg$ for some epimorphism g of R onto P. Further, $r(\mathrm{Ker} f) \subseteq Y(\mathrm{Ker} f)$, $Y(\mathrm{Ker} f) = 0$ by 1.3 (i), $g(r(R)) \subseteq \mathrm{Ker} f$ and $r(r(R)) = r(R)$. Hence $g(r(R)) = 0$ and P is isomorphic to $R/r(R)$.
(ii) Since $r(R) \subseteq J(R)$, $R \to R/r(R)$ is a projective cover.

1.5 Corollary. *$Y(R) \cap J(R)$ contains no non-trivial idempotent left ideal.*

1.6 Proposition. *The following conditions are equivalent:*
(i) $Y = \mathrm{id}$.
(ii) *Y is centrally splitting.*
(iii) *Y is hereditary.*
(iv) *R is a left V-ring (i.e. every simple module is injective).*
Proof. Only the implication (iii) implies (iv) needs a proof. Suppose that $M \neq E(M)$ for a simple module M. Then M is small in $E(M)$, and so $Y(M) = 0$. Further, if $N \subseteq E(M)$ and $E(M)/N$ is small in $E(E(M)/N)$, then $N \neq 0$, so that $M \subseteq N$. Thus $M \subseteq Y(E(M))$ and $Y(M) = M$. Consequently, $M = 0$, a contradiction.

1.7 Proposition. *Suppose that $R = Y(R) + S$ (ring direct sum). Then $Y(R)$ is a left V-ring and $Y(S) = 0$.*
Proof. Put $T = Y(R)$. Every simple T-module is Y-torsion as both an R-module and a T-module. The rest is obvious.

1.8 Corollary. *Let R be a left duo-ring (i.e. every left ideal is a two-sided ideal). Then the following conditions are equivalent:*
(i) *Y is costable.*
(ii) *$R = T \dot{+} S$, where T is a regular ring and $Y(S) = 0$.*

2. Pseudoprojective modules

A module Q is said to be pseudoprojective with respect to a homomorphism $g: B \to C$ if for every $0 \neq f: Q \to C$ there exist $k: Q \to Q$ and $h: Q \to B$ such that $0 \neq gh = fk$. If Q is pseudoprojective with respect to every epimorphism, then we shall say that Q is pseudoprojective.

2.1 Proposition. *Let Q be a module and $r = p_{\{Q\}}$. The following statements are equivalent:*

(i) *Q is pseudoprojective.*

(ii) *Q is pseudoprojective with respect to every epimorphism $P \to C$ with P projective and C cocyclic.*

(iii) *If C is cocyclic, $f: B \to C$ is epi and* $\mathrm{Hom}_R(Q, C) \neq 0$, *then* $\mathrm{Im}\, g$ *is not contained in* $\mathrm{Ker} f$ *for some $g: Q \to B$.*

(iv) *If $r(B) \subseteq A \subseteq B$ and B/A is cocyclic, then $r(B/A) = 0$.*

(v) *r is an idempotent cohereditary radical.*

(vi) *If $f: P \to Q$ is a projective presentation, then P is equal to* $\mathrm{Ker} f + r(P)$.

(vii) *There exists a projective presentation $f: P \to Q$ such that $P =$* $\mathrm{Ker} f + r(P)$.

Proof. (i) implies (iii). Let $k: P \to C$ be a projective presentation and $q: Q \to C$ be non-zero. There are $h: Q \to P$ and $t: Q \to Q$ such that $0 \neq qt = kh$. On the other hand, $k = fp$ for some $p: P \to B$ and $fph = kh = qt \neq 0$.

(iv) implies (v). Every module is isomorphic to a submodule of a direct product of cocyclic factormodules.

(vii) implies (i). Let $p: A \to B$ be epi and $g: Q \to B$ be non-zero. Let a projective module P satisfy the condition (vii). There is $k: P \to A$ with $pk = gf$. Put $q = f|r(P)$ and $h = k|r(P)$. Then $gq = ph$ and q is an epimorhism onto Q. Since $0 \neq g$, $0 \neq gq = ph$. Further, let $m = \perp\!\perp Q_i$, where $0 \neq i: Q \to P$ and $Q_i = Q$. There is an epimorphism $t: M \to r(P)$ and $pht \neq 0 \neq phtd$ for some $d: Q \to M$. Now, $qtd: Q \to Q$, $htd: Q \to A$ and $0 \neq phtd = gqtd$. The remaining implications are obvious.

The following proposition is clear.

2.2 Proposition. *The following conditions are equivalent for every module Q:*

(i) *There exists a projective module P such that $p_{\{P\}} = p_{\{Q\}}$.*

(ii) *There is a projective presentation $P \to Q$ such that $p_{\{P\}} = p_{\{Q\}}$.*

(iii) *There is a projective presentation $P \to Q$ such that P is a homomorphic image of a direct sum of copies of Q.*

Every module satisfying the equivalent conditions of 2.2 will be called strongly pseudoprojective. It is evident that every strongly pseudoprojective module is pseudoprojective.

The following proposition is an aesy consequence of 2.1, 2.2.

2.3 Proposition. *Let Q be a pseudoprojective module and $r = p_{\{Q\}}$. Then Q is strongly pseudoprojective, provided at least one of the following conditions is satisfied:*

(i) *Q has a projective cover.*

(ii) *r is costable.*

(iii) *R is left hereditary.*

(iv) *Q has a projective presentation $P \to Q$ such that $r(P)$ is projective.*

2.4 Proposition. *Let Q, M be two modules and $p_{\{Q\}}(M)=M$. Then Q is (strongly) pseudoprojective iff $Q+M$ is so.*

Proof. Obviously, $p_{\{Q+M\}}=p_{\{Q\}}$ and we can apply 2.1, 2.2.

2.5 Corollary. *Let A be a submodule of a (strongly) pseudoprojective module Q. Then the outer direct sum $Q+(Q/A)$ is (strongly) pseudoprojective.*

2.6 Corollary. *Every module is a direct summand of a strongly pseudoprojective module.*

2.7 Proposition. *Let Q be a pseudoprojective module and M be a simple $p_{\{Q\}}$-torsion module. Then:*

(i) *M is a homomorphic image of Q.*

(ii) *If M has a projective cover $P\to M$, then P is a homomorphic image of Q.*

(iii) *If R is left hereditary, then there is a projective presentation $P\to M$ such that P is a homomorphic image of Q.*

Proof. Let $0\neq M$, $f\colon Q\to M$ be epi and $h\colon P\to M$ be a projective presentation. There are $k\colon Q\to Q$ and $g\colon Q\to P$ with $0\neq fk=hg$. Thus fk is an epimorphism and the rest of the proof is clear.

The following proposition is obvious.

2.8 Proposition.

(i) *Every simple pseudoprojective module is projective.*

(ii) *A module, every non-zero factormodule of which has a nonzero projective homomorphic image, is strongly pseudoprojective.*

2.9 Corollary. *The following conditions are equivalent:*

(i) *Every module is pseudoprojective.*

(ii) *Every strongly pseudoprojective module is projective.*

(iii) *R is completely reducible (i.e. R is artinian and $J(R)=0$).*

2.10 Proposition. *The following conditions are equivalent for every module Q:*

(i) *Q is a generator (i.e. $p_{\{Q\}}=\mathrm{id}$).*

(ii) *Q is strongly pseudoprojective and every simple module is a homomorphic image of Q.*

(iii) *Q is pseudoprojective and every simple module is a homomorphic image of Q.*

(iv) *Q is cofaithful and $p_{\{Q\}}$ is hereditary. Moreover, if R is either semiperfect or left hereditary, then these conditions are equivalent to:*

(v) *Every simple module has a projective presentation which is a homomorphic image of Q.*

Proof. (iv) implies (i). Suppose, on the contrary, that $p_{\{Q\}}$ is not equal to id. Since $p_{\{Q\}}$ is cohereditary, there is a non-zero cocyclic module C with $0=p_{\{Q\}}(C)$. On the other hand, $0\neq \mathrm{Soc}(C)$ and therefore there exists a non-zero homomorphism h of Q into C, a contradiction.

(iii) implies (v) by 2.7 and (v) implies (iii) by 2.1 (iii). The remaining implications are obvious.

2.11 Proposition. *Let Q be a module satisfying at least one of the following conditions:*

(1) *$Y(Q)=Q$ and $R/p_{\{Q\}}(R)$ has a projective cover.*

(2) *$p_{\{Q\}}$ is costable.*

Then the following assertions are equivalent:

(i) *If* $\mathrm{Hom}_R(Q, B)=0$, *then* $\mathrm{Hom}_R(Q, B/A)=0$ *for every submodule A of B.*

(ii) *$p_{\{Q\}}$-torsionfree modules are closed under factormodules.*

(iii) *Q is pseudoprojective.*

(iv) *Q is strongly pseudoprojective*

(v) *There is a projective presentation $P \to Q$ such that* $\mathrm{Hom}_R(Q, B)=0$ *implies* $\mathrm{Hom}_R(P, B)=0$.

Proof. By 1.4 (i), (1) implies (2). For the proof of the proposition itself it suffices to apply 2.1, 2.2 and 2.3.

2.12 Proposition. *Let R be a ring without non-trivial idempotent two-sided ideals. Then the following conditions are equivalent for a module Q:*

(i) *Q is strongly pseudoprojective.*

(ii) *Q is pseudoprojective.*

(iii) *Q is a generator.*

Proof. (ii) implies (iii). Let $0 \neq Q$ and $r = p_{\{Q\}}$. Then $r \neq \mathrm{zer}$ and r is an idempotent cohereditary radical. Consequently $0 \neq r(R)$ is an idempotent ideal and $r(R)=R$. Thus $r = \mathrm{id}$ and Q is a generator.

2.13 Proposition. *Let Q be a cofaithful pseudoprojective module with $Y(Q)=Q$. Then $Y = p_{\{Q\}}$.*

Proof. $Y(Q)=Q$ implies $r = p_{\{Q\}} \subseteq Y$. To prove the inverse inclusion it is enough to show that $r(M)=0$ implies $Y(M)=0$. Suppose, on the contrary, that $r(M)=0$ and $Y(M)\neq 0$ for a module M. There is a cocyclic factormodule N of M such that $r(N)=0\neq Y(N)$. In particular, N is not small in $E(N)$, and hence $N+K=E(N)$ for a proper submodule K of $E(N)$. The non-zero module $E(N)/K$ is a homomorphic image of N, so that $r(E(N)/K)=0$. On the other hand, Q is cofaithful, and therefore $r(E(N))=E(N)$ and $r(E(N)/K)=E(N)/K$, which is a contradiction.

2.14 Corollary. *Let R be a left perfect ring such that $E(M)=Y(E(M))$ for every simple module M. Then Y is costable idempotent cohereditary radical.*

Proof. Put $A = \perp\perp E(M)$, where $M \in S$ and S is a representative set of simple modules. Let $P \to A$ be a projective cover. Then $Y(A)=A$, and hence $Y(P)=P$ (because Y is cohereditary). Now we can use 2.13 and its proof.

2.15 Proposition. *Let R be a left perfect, Q be a cofaithful module and $r = p_{\{Q\}}$. The following conditions are equivalent:*

(i) *Q is pseudoprojective and $Y(Q) = Q$.*

(ii) *$r = Y$.*

(iii) *$T_r = T_Y$.*

(iv) *Y is cohereditary and $F_Y = F_r$.*

Proof. (i) implies (ii) by 2.13, (ii) implies (iii) trivially and (ii) implies (iv) by 1.3 (ii).

(iii) implies (i). Clearly, $Y(Q) = Q$. Since Y is cohereditary, T_Y is closed under covers, and consequently r is cohereditary. (See [3], Proposition 4.2.)

(iv) implies (i). Since $F_Y = F_r$ and Y is a radical, r is contained in Y. From this, $Y(Q) = Q$ and r is costable. On the other hand, F_r is closed under factormodules and we can use 2.11.

3. Pseudoprojective injective modules

3.1 Proposition. *Pseudoprojective (strongly pseudoprojective) modules are closed under direct sums.*

Proof. It suffices to observe the equality $p_{\{\perp\perp A_i\}} = \Sigma p_{\{A_i\}}$ for every family $\{A_i\}$ of modules.

3.2 Proposition. *Let I be an idempotent two-sided ideal. Then:*

(i) *$p_{\{I\}}(M) = IM$ for every module M.*

(ii) *I is pseudoprojective (as a module).*

Proof. (i) Let $s(M) = IM$ for every module M. Then s is an idempotent cohereditary radical. Since $s(I) = I$, $p_{\{I\}} \subseteq s$. On the other hand, if N is a module and $m \in N$, the mapping $I \to N$ defined by $i \to im$ for every $i \in I$ is a homomorphism. Consequently, $s(N) = IN \subseteq p(N)$ and we have proved that $s = p_{\{I\}}$. (ii) This is obvious from (i) and 2.1.

3.3 Proposition. *Let Q be a pseudoprojective module and $I = p_{\{Q\}}(R)$.* Then:

(i) *I is an idempotent two-sided ideal.*

(ii) *$IQ = Q$ and $p_{\{Q\}} = p_{\{I\}}$.*

(iii) *Q is a homomorphic image of a direct sum of copies of I.*

(iv) *If I is projective, then Q is strongly pseudoprojective.*

Proof. Put $r = p_{\{Q\}}$. Then r is an idempotent cohereditary radical (see 2.1). Hence $I = r(R)$ is an idempotent two-sided ideal and $r = p_{\{I\}}$. By 3.2, $q = r(Q) = IQ$. Now, (i), (ii) and (iii) follow easily. Finally, (iv) is an immediate consequence of (ii) and 2.2.

3.4 Corollary. *Let R be a commutative noetherian ring. Then every pseudoprojective module is strongly pseudoprojective.*

Proof. Every idempotent finitely generated ideal of a commutative ring is a direct summand.

3.5 Proposition. *Let Q be a module and $I = p_{\{Q\}}(R)$. Then Q is pseudoprojective if and only if $IQ = Q$.*

Proof. The "only if" part follows from 3.3. As for the "if" part, first observe that $p_{\{Q\}} = p_{\{I\}}$. Further, there is a module N which is a homomorphic image of a direct sum of copies of Q and an epimorphism $f: N \to I$. Hence $N = IN$ and $I^2 = f(IN) = f(N) = I$. Thus both I and Q are pseudoprojective (see 3.2).

3.6 Corollary. *Let Q be pseudoprojective module and A be a submodule of Q such that $\mathrm{Hom}_R(A, R) = 0$. Then $N = Q/A$ is pseudoprojective.*

Proof. According to the hypothesis, $p_{\{N\}}(R) = I = p_{\{Q\}}(R)$. By 3.5, $IQ = Q$, $IN = N$ and N is pseudoprojective.

3.7 Proposition. *Let Q be a module and $I = p_{\{Q\}}(R)$. Suppose that at least one of the following conditions holds:*
(1) *Q is injective and R is left hereditary.*
(2) *Q is injective and the singular ideal $Z(R)$ is equal to* 0.
(3) $\mathrm{Im} f$ *is an injective module for every homomorphism f of Q into R.*
(4) $\mathrm{Im} f$ *is an idempotent left ideal for every $f: Q \to R$*
(5) *R is a regular ring.*
(6) *R is a left V-ring.*
(7) *R is a simple ring.*
Then the module IQ is pseudoprojective.

Proof. Clearly, each of the conditions (1), (2), (3), (5), (6) and (7) implies the condition (4). Hence we shall assume that (4) is satisfied. Let $K = p_{\{IQ\}}(R)$, $f: Q \to R$ be a homomorphism and $L = \mathrm{Im} f$. Then LQ is contained in IQ and we have $f(LQ) = Lf(Q) = L^2 = L$. Thus $f(IQ) = L$ and we see that I is contained in K. Finally, I is an idempotent ideal and $IQ = I(IQ) \subseteq K(IQ)$, and so IQ is pseudoprojective by 3.5.

3.8 Proposition. *Let Q be an injective module such that for every $q \in Q$ there exists a homomorphism $f: Q \to R$ with $0 = Rq \cap \mathrm{Ker} f$. Then Q is pseudoprojective.*

Proof. Let $I = p_{\{Q\}}R$, $q \in Q$ and $f: Q \to R$ be such that $0 = Rq \cap \mathrm{Ker} f$. Denote by K the set of all $a \in R$ with $f(aq) = 0$. Then, obviously, $0 = Kq$, and therefore $q \in f(q) \cdot Q$. Thus $Q = IQ$ and we can apply 3.5.

3.9 Proposition. *Let $Z(R) = 0$ and Q be an injective module containing no infinite direct sum of submodules. Suppose that for every $q \in Q$ there exists a homomorphism $f: Q \to R$ such that $0 \neq f(q)$. Then Q is strongly pseudoprojective.*

Proof. With respect to the hypothesis and 3.1, we can assume that Q is directly indecomposable. Let $0 \neq q \in Q$ and f be a homomorphism of Q into R such that

$0 \neq f(q)$. Since $0 = Z(R)$, $\mathrm{Ker} f$ is a direct summand in Q. Consequently, $0 = \mathrm{Ker} f$ and Q is isomorphic to $\mathrm{Im} f$. The rest is clear.

3.10 Proposition. *Let R be left noetherian and $Z(R) = 0$. Let Q be an injective module such that for every $0 \neq q \in Q$ there is a homomorphism $f: Q \to R$ with $0 \neq f(q)$. Then Q is strongly pseudoprojective.*

Proof. The proof is in fact the same as that of 3.9.

3.11 Proposition. *Let R be a regular ring, Q be a pseudoprojective module and N be a submodule of Q such that $p_{\{N\}}(R) = I = p_{\{Q\}}(R)$. Then N is pseudoprojective.*

Proof. Since Q is pseudoprojective and the right module R/I is flat, $IQ = Q$ and $IN = N \cap IQ = N$. It suffices to use 3.5.

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ПСЕВДОПРОЕКТИВНЫЕ МОДУЛИ

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Резюме

Статья посвящена изучению псевдопроективных модулей. Модуль Q называется пвсевдопроективным, если идемпотентный прерадикал порожденный Q является конаследсвенным. В частности, изучены псевдопроективные инъективные модули.