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Mathematica Slovaca, Vol. 29 (1979), No. 1, 25--38

Persistent URL: http://dml.cz/dmlcz/136196

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EXTREME ESSENTIAL DERIVATIVES OF BOREL AND LEBESGUE MEASURABLE FUNCTIONS

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1. It is well known ([1] and [7]) that the Dini derivatives of Borel (Lebesgue measurable) functions are Borel (Lebesgue measurable) functions. Let B_α, respectively L, denote the family of all real Borel functions of a real variable of the class α, respectively the class of all real Lebesgue measurable functions of a real variable. Let α be an ordinal and $\delta(\alpha)$ be the least upper bound of the set of all ordinals γ for which there exists a Borel function $f \in B_\alpha$ with one of the Dini derivatives in the Borel class γ and not in the Borel class δ for $\delta < \gamma$. It is known that $\alpha \leqq \delta(\alpha) \leqq \alpha + 2$ holds ([1], [5] and [7]). From an example of J. Staniszewska ([8]) one can easily see that $\delta(0) = 2$. For $\alpha > 0$ we do not know whether the equality $\delta(\alpha) = \alpha + 2$ holds. In [5] we have proved actually that the upper, respectively lower, Dini derivatives of a Borel function of the class α are upper, respectively lower, semi-Borel functions of the class $\alpha + 1$.

Let α be an ordinal and $\delta_{ess}(\alpha)$, respectively $\bar{\delta}_{ess}(\alpha)$, be the least upper bound of the set of all ordinals γ for which there exists a Borel function $f \in B_\alpha$ with one of the extreme unilateral, respectively bilateral, essential derivatives in the Borel class γ and not in the Borel class δ for $\delta < \gamma$. Recently ([6]) we have proved that $2 \leqq \delta_{ess}(0) \leqq 3$. From the cited example of J. Staniszewska and from corollary 2 in our paper [4] (Folgerung 2, p. 158) we get that $2 \leqq \bar{\delta}_{ess}(0)$. The inequality $\delta_{ess}(0) \leqq 3$ gives that also $\bar{\delta}_{ess}(0) \leqq 3$ holds. In the presented paper the proof is given that for $\alpha > 0$ the upper (lower) unilateral essential derivatives of Borel functions of the class α are the lower (upper) semi-Borel functions of the class $\alpha + 2$. Therefore $\delta_{ess}(\alpha) \leqq \alpha + 3$ holds and $\bar{\delta}_{ess}(\alpha) \leqq \alpha + 3$. It is also proved that the extreme unilateral essential derivatives of Lebesgue measurable functions are Lebesgue measurable too.

In [3] O. Hájek proved that extreme bilateral derivatives of an arbitrary function are in the Borel class two. A similar theorem for extreme bilateral essential derivatives of functions does not hold. For any ordinal α there holds $\alpha \leqq \delta_{ess}\alpha)$ and $\alpha \leqq \bar{\delta}_{ess}(\alpha)$. There are Lebesgue measurable functions having extreme unilateral and also bilateral essential derivatives which are not Borel.

2. The set of all real numbers is denoted by R, the set of all positive integers is denoted by N. In the sequel α will mean an ordinal of the first two classes. A real function φ of a real variable is a lower (upper) semi-Borel function of the class α iff the sets $\{x \in R: \varphi(x) > \beta\}$ ($\{x \in R: \varphi(x) < \beta\}$) are of the Borel additive class α for all $\beta \in R$. The system of all lower (upper) semi-continuous functions is the system of all lower (upper) semi-Borel functions of the class zero.

We will denote by f a real function of a real variable, by x and β real numbers, by r a real number strictly between zero and one, by ω and η real numbers which satisfy the inequality $0 \leqq \omega < \eta$, by n and k positive integers and by $|A|$ the Lebesgue outer measure of the set A.

We set:

$$A_n(x; \beta; \omega, \eta) = \{h: \omega < h \leqq \eta, |f(x+h)| \leqq n, \frac{f(x+h)-f(x)}{h} > \beta\},$$
$$B_n(x; \beta; \omega, \eta) = \{h: \omega < h \leqq \eta, |f(x+h)| \leqq n, f(x+h)-f(x) > \beta\},$$
$$C_n(x; \beta; \omega, \eta) = \{h: \omega < h \leqq \eta, |f(x+h)| \leqq n, f(x+h) > \beta\},$$
$$A(x; \beta; \omega, \eta) = \left\{h: \omega < h \leqq \eta, \frac{f(x+h)-f(x)}{h} > \beta\right\},$$
$$\varphi_{n,r}(x; \omega, \eta) = \sup\{\beta: |A_n(x; \beta; \omega, \eta)| > r(\eta - \omega)\},$$
$$\psi_{n,r}(x; \omega, \eta) = \sup\{\beta: |B_n(x; \beta; \omega, \eta)| > r(\eta - \omega)\},$$
$$\chi_{n,r}(x; \omega, \eta) = \sup\{\beta: |C_n(x; \beta; \omega, \eta)| > r(\eta - \omega)\},$$
$$\varphi_r(x; \omega, \eta) = \sup\{\beta: |A(x; \beta; \omega, \eta)| > r(\eta - \omega)\},$$
$$\varphi_{n,k}(x) = \sup\left\{\varphi_{1/(k+1)}(x; 0, \eta): 0 < \eta \leqq \frac{1}{n}\right\}.$$

It is obvious that $\varphi_r(x; \omega, \eta) \leqq \varphi_{r'}(x; \omega, \eta)$ for $0 < r' \leqq r < 1$, $\varphi_{n,k}(x) \leqq \varphi_{n,k+1}(x)$, $\varphi_{n+1,k}(x) \leqq \varphi_{n,k}(x)$ for all $x \in R$ and $n, k \in N$. Therefore there exists $\lim_{n\to\infty} \varphi_{n,k}(x)$ for every $k \in N$. For all $k \in N$ we denote the limit $\lim_{n\to\infty} \varphi_{n,k}(x)$ by $\varphi_k(x)$. There holds $\varphi_k(x) \leqq \varphi_{k+1}(x)$ for all $x \in R$ and $k \in N$.

Let now $0 < \omega$, $\omega = \omega_0 < \omega_1 < \omega_2 < \ldots < \omega_k = \eta$ and $r_1, r_2, \ldots, r_k \in \langle 0, 1)$. Then we set:

$\Phi_n(x; \omega_0, \omega_1, \ldots, \omega_k : r_1\ r_2, \ldots, r_k) = \min\{\varphi_{n,r_i}(x; \omega_{i-1}, \omega_i): r_i > 0, i = 1, 2, \ldots, k\}$,

$\Psi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) = \min\{\min\left(\frac{\psi_{n,r_i}(x; \omega_{i-1}, \omega_i)}{\omega_{i-1}}, \frac{\psi_{n,r_i}(x; \omega_{i-1}, \omega_i)}{\omega_i}\right): r_i > 0, i = 1, 2, \ldots, k\}$,

$$v_k = \max\left\{\frac{\omega_i - \omega_{i-1}}{\omega_{i-1}\omega_i}: r_i > 0, i = 1, 2, \ldots, k\right\}.$$

Let $\{\eta_i\}_{i=1}^{\infty}$ be a decreasing sequence of positive numbers with the limit equal to zero, i. e. $0 < \eta_{i+1} < \eta_i$ for each $i \in N$ and $\lim_{i\to 0} \eta_i = 0$. Let $\{r_i\}_{i=1}^{\infty}$ be such a sequence of non-negative numbers less than one that the set $\{i \in N: r_i > 0\}$ is finite. Then we set:

$\Phi(x; \{\eta_i\}_{i=1}^{\infty}; \{r_i\}_{i=1}^{\infty}) \doteq \min\{\varphi_{r_i}(x; \eta_{i+1}, \eta_i): r_i > 0, i = 1, 2, \ldots\}$.

We recall the definition of the upper right essential derivative of a function of a real variable in a point. The upper right essential derivative $\bar{f}^{+}_{\text{ess}}(x)$ of a real function f of a real variable in a point x is the least upper bound of the set of all such numbers β for which the set $\left\{h \in R: h > 0, \dfrac{f(x+h)-f(x)}{h} > \beta\right\}$ has in 0 positive upper outer density.

3. **Proposition 1.** *$\chi_{n,r}(x; \omega, \eta) = \psi_{n,r}(x; \omega, \eta) + f(x)$ and $|\chi_{n,r}(x; \omega, \eta)| \leqq n$ if $\chi_{n,r}(x; \omega, \eta) > -\infty$.*

Proof. If $\chi_{n,r}(x; \omega, \eta) = -\infty$, then $|C_n(x; \beta; \omega, \eta)| \leqq r(\eta-\omega)$ for all $\beta \in R$. But $B_n(x; \beta; \omega, \eta) = C_n(x; \beta + f(x); \omega, \eta)$ for all $\beta \in R$. Therefore $|B_n(x; \beta; \omega, \eta)| \leqq r(\eta-\omega)$ for all $\beta \in R$. This implies that $\psi_{n,r}(x; \omega, \eta) = -\infty$ and the equality $\chi_{n,r}(x; \omega, \eta) = \psi_{n,r}(x; \omega, \eta) + f(x)$ holds.

Let $\chi_{n,r}(x; \omega, \eta) > -\infty$, Then $|\{h: \omega < h \leqq \eta, |f(x+h)| \leqq n, f(x+h) \geqq -n\}| > r(\eta-\omega)$ as the sets $\{h: \omega < h \leqq \eta, |f(x+h)| \geqq n, f(x+h) < -n\}$ and $\{h: \omega < h \leqq \eta, |f(x+h)| \leqq n, f(x+h) > n\}$ are empty. From this we see that there holds: $-n \leqq \chi_{n,r}(x; \omega, \eta) \leqq n$. Since $B_n(x; \beta; \omega, \eta) = C_n(x; \beta + f(x); \omega, \eta)$ for all $\beta \in R$, it is obvious that the inequality $|B_n(x; \beta; \omega, \eta)| > r(\eta-\omega)$ holds iff the inequality $|C_n(x; \beta + f(x); \omega, \eta)| > r(\eta-\omega)$ holds. Therefore $\chi_{n,r}(x; \omega, \eta) = \sup\{\beta: |C_n(x; \beta; \omega, \eta)| > r(\eta-\omega)\} = f(x) + \sup\{\gamma: |B_n(x; \gamma; \omega, \eta)| > r(\eta-\omega)\} = \psi_{n,r}(x; \omega, \eta) + f(x)$.

Proposition 2. *The function $\chi_{n,r}(x; \omega, \eta)$ is lower semicontinuous and consequently $\chi_{n,r}(x; \omega, \eta) \in B_1$.*

Proof. Let $\beta \in R$ and $\chi_{n,r}(x; \omega, \eta) > \beta$. Then there exists such a $\gamma \in R$ that $\chi_{n,r}(x; \omega, \eta) > \gamma > \beta$ and $|C_n(x; \gamma; \omega, \eta)| > r(\eta-\omega)$. It is obvious that there exists such a positive δ for which $|C_n(x; \gamma; \omega+\delta, \eta-\delta)| > r(\eta-\omega)$.

Let $u \in (x-\delta, x+\delta)$. Then for $h \in C_n(x; \gamma; \omega+\delta, \eta-\delta)$ there holds: $\omega+\delta < h \leqq \eta-\delta$, $|f(x+h)| \leqq n$ and $f(x+h) > \gamma$. There exists such a $v \in (-\delta, \delta)$ that $u = x+v$. Then there holds: $\omega = (\omega+\delta) - \delta < h - v \leqq \eta - \delta + \delta = \eta$, $|f(u+h-v)| \leqq n$ and $f(u+h-v) > \gamma$. We have proved that $h - v \in C_n(u; \gamma; \omega, \eta)$ and therefore $-v + C_n(x; \gamma; \omega+\delta, \eta-\delta) \subset C_n(u; \gamma; \omega, \eta)$. From this it follows: $|C_n(u; \gamma; \omega, \eta)| \geqq |-v + C_n(x; \gamma; \omega+\delta, \eta-\delta)| = |C_n(x; \gamma; \omega+\delta, \omega-\delta)| > r(\eta-\omega)$. Therefore $\chi_{n,r}(u; \omega, \eta) \geqq \gamma > \beta$. Therewith we have proved that $\chi_{n,r}(x; \omega, \eta)$ is lower semi-continuous. Consequently $\chi_{n,r}(x; \omega, \eta) \in B_1$.

Proposition 3. *Let $\alpha > 0$. If $f \in B_\alpha$, then $\psi_{n,r}(x; \omega, \eta) \in B_\alpha$; if f is a Lebesgue measurable function, then $\psi_{n,r}(x; \omega, \eta)$ is also a Lebesgue measurable function.*

If $\psi_{n,r}(x; \omega, \eta) > -\infty$, then $|\psi_{n,r}(x; \omega, \eta)| \leqq |f(x)| + n$ hold.

Proof. According to proposition 1 $\psi_{n,r}(x;\omega,\eta) = \chi_{n,r}(x;\omega,\eta)-f(x)$ and according to proposition 2 $\psi_{n,r}(x;\omega,\eta)\in B_\alpha$ if $f\in B_\alpha$, respectively $\psi_{n,r}(x;\omega,\eta)$, is Lebesgue measurable if f is Lebesgue measurable.

If $\psi_{n,r}(x;\omega,\eta)>-\infty$, there is also $\chi_{n,r}(x;\omega,\eta)>-\infty$ and from proposition 1 we get that $|\psi_{n,r}(x;\omega,\eta)| \leqq |f(x)|+n$.

Proposition 4. *Let* $0<\omega<\eta$ *and* $x\in R$. *Then there holds:*

$A_n\left(x;\frac{\beta}{\omega};\omega,\eta\right)\subset B_n(x;\beta;\omega,\eta)\subset A_n\left(x;\frac{\beta}{\eta};\omega,\eta\right)$ *for* $\beta>0$,

$A_n(x;0;\omega,\eta)=B_n(x;0;\omega,\eta)$,

$A_n\left(x;\frac{\beta}{\eta};\omega,\eta\right)\subset B_n(x;\beta;\omega,\eta)\subset A_n\left(x;\frac{\beta}{\omega};\omega,\eta\right)$ *for* $\beta<0$.

Proof. Let $\beta>0$.

For each $h\in A_n\left(x;\frac{\beta}{\omega};\omega,\eta\right)$ we have: $\omega<h\leqq\eta$, $|f(x+h)|\leqq n$, $f(x+h) - f(x)>\frac{\beta}{\omega}h>\beta$. Therefore $h\in B_n(x;\beta;\omega,\eta)$. Consequently

$A_n\left(x;\frac{\beta}{\omega};\omega,\eta\right)\subset B_n(x;\beta;\omega,\eta)$.

For each $h\in B_n(x;\beta;\omega,\eta)$ we have: $\omega<h\leqq\eta$, $|f(x+h)|\leqq n$, $\frac{f(x+h)-f(x)}{h} > \frac{\beta}{h} = \frac{\beta}{\eta}\frac{\eta}{h}\geqq\frac{\beta}{\eta}$ and therefore $h\in A_n\left(x;\frac{\beta}{\eta};\omega,\eta\right)$. Thus $B_n(x;\beta;\omega,\eta) \subset A_n\left(x;\frac{\beta}{\eta};\omega,\eta\right)$.

As, for $\omega<h\leqq\eta$, the inequality $\frac{f(x+h)-f(x)}{h}>0$ holds iff $f(x+h) - f(x)>0$ holds, we have that $A_n(x;0;\omega,\eta) = B_n(x;0;\omega,\eta)$.

The relations for the case $\beta<0$ are proved analogously as those for $\beta>0$.

Proposition 5. *Let* $0<\omega<\eta$.
For $\psi_{n,r}(x;\omega,\eta)>-\infty$ *there holds:*

$$\min\left(\frac{\psi_{n,r}(x;\omega,\eta)}{\omega},\frac{\psi_{n,r}(x;\omega,\eta)}{\eta}\right)\leqq\varphi_{n,r}(x;\omega,\eta)\leqq$$
$$\leqq\max\left(\frac{\psi_{n,r}(x;\omega,\eta)}{\omega},\frac{\psi_{n,r}(x;\omega,\eta)}{\eta}\right)$$

and

$$\max\left(\frac{\psi_{n,r}(x;\omega,\eta)}{\omega},\frac{\psi_{n,r}(x;\omega,\eta)}{\eta}\right)-$$
$$-\min\left(\frac{\psi_{n,r}(x;\omega,\eta)}{\omega},\frac{\psi_{n,r}(x;\omega,\eta)}{\eta}\right)\leqq(|f(x)|+n)\frac{\eta-\omega}{\eta\omega}.$$

If $\psi_{n,r}(x;\omega,\eta)=-\infty$, *then* $\varphi_{n,r}(x;\omega,\eta)=-\infty$.

Proof. Let $\psi_{n,r}(x;\omega,\eta)>0$.

Let β be such a real number which satisfies $0<\beta<\psi_{n,r}(x;\omega,\eta)$. Then there exists such a γ that $\beta<\gamma<\psi_{n,r}(x;\omega,\eta)$ and $|B_n(x;\gamma;\omega,\eta)|>r(\eta-\omega)$. From this and from propostion 4 we get that $\left|A_n\left(x;\frac{\gamma}{\eta};\omega,\eta\right)\right| \geqq |B_n(x;\gamma;\omega,\eta)| > r(\eta-\omega)$. Thus $\varphi_{n,r}(x;\omega,\eta) \geqq \frac{\gamma}{\eta}>\frac{\beta}{\eta}$. Therefore $\varphi_{n,r}(x;\omega,\eta) \geqq \sup\left\{\frac{\beta}{\eta}: 0<\beta<\psi_{n,r}(x;\omega,\eta))\right\} = \frac{\psi_{n,r}(x;\omega,\eta)}{\eta}$.

Let $\psi_{n,r}(x;\omega,\eta)<\beta$. Then $|B_n(x;\beta;\omega,\eta)| \leqq r(\eta-\omega)$ and, according to proposition 4, this implies that $\left|A_n\left(x;\frac{\beta}{\omega};\omega,\eta\right)\right| \leqq |B_n(x;\beta;\omega,\eta)| \leqq r(\eta-\omega)$. From this $\varphi_{n,r}(x;\omega,\eta) \leqq \frac{\beta}{\omega}$. Thus $\varphi_{n,r}(x;\omega,\eta) \leqq \inf\left\{\frac{\beta}{\omega}: \psi_{n,r}(x;\omega,\eta) < \beta\right\}$ $= \frac{\psi_{n,r}(x;\omega,\eta)}{\omega}$.

The inequality $0<\frac{\psi_{n,r}(x;\omega,\eta)}{\omega} - \frac{\psi_{n,r}(x;\omega,\eta)}{\eta} = \frac{\eta-\omega}{\eta\omega}\psi_{n,r}(x;\omega,\eta) \leqq (|f(x)| + n)\frac{\eta-\omega}{\eta\omega}$ finishes the proof of the assertion of proposition 5 for $\psi_{n,r}(x;\omega,\eta)>0$.

Let $\psi_{n,r}(x;\omega,\eta)\leqq 0$.

Then for every β less than $\psi_{n,r}(x;\omega,\eta)$ there exists such a number γ that $\beta<\gamma<\psi_{n,r}(x;\omega,\eta)$ and $|B_n(x;\gamma;\omega,\eta)| > r(\eta-\omega)$. From propostiion 4 and from the last inequality we get that $\left|A_n\left(x;\frac{\gamma}{\omega};\omega,\eta\right)\right| \geqq |B_n(x;\gamma;\omega,\eta)| > r(\eta-\omega)$. Thus $\varphi_{n,r}(x;\omega,\eta) \geqq \frac{\gamma}{\omega}>\frac{\beta}{\omega}$ for each β less than $\psi_{n,r}(x;\omega,\eta)$. Therefore $\varphi_{n,r}(x;\omega,\eta) \geqq \sup\left\{\frac{\beta}{\omega}: \beta<\psi_{n,r}(x;\omega,\eta)\right\} = \frac{\psi_{n,r}(x;\omega,\eta)}{\omega}$.

Let now $\psi_{n,r}(x;\omega,\eta)=0$. Then we have: $|B_n(x;\beta;\omega,\eta)| \leqq r(\eta-\omega)$ if $\psi_{n,r}(x;\omega,\eta)<\beta$. This and propostiion 4 imply that $\left|A_n\left(x;\frac{\beta}{\omega};\omega,\eta\right)\right| \leqq |B_n(x;\beta;\omega,\eta)| \leqq r(\eta-\omega)$ if $\psi_{n,r}(x;\omega,\eta) < \beta$. Thus $\varphi_{n,r}(x;\omega,\eta) \leqq \inf\left\{\frac{\beta}{\omega}: \psi_{n,r}(x;\omega,\eta)<\beta\right\} = \frac{\psi_{n,r}(x;\omega,\eta)}{\omega} = \frac{\psi_{n,r}(x;\omega,\eta)}{\eta}$. As now also $\frac{\psi_{n,r}(x;\omega,\eta)}{\eta} - \frac{\psi_{n,r}(x;\omega,\eta)}{\omega} = 0 \leqq (|f(x)|+n)\frac{\eta-\omega}{\eta\omega}$, the assertion of proposition 5 is proved for $\psi_{n,r}(x;\omega,\eta)=0$.

Let $-\infty<\psi_{n,r}(x;\omega,\eta)<0$. Then $|B_n(x;\beta;\omega,\eta)| \leqq r(\eta-\omega)$ if

$\psi_{n,r}(x;\omega,\eta) < \beta < 0$. Consequently, by proposition 4, there holds: $\left|A_n\left(x;\frac{\beta}{\eta};\omega,\eta\right)\right| \leqq |B_n(x;\beta;\omega,\eta)| \leqq r(\eta-\omega)$ if $\psi_{n,r}(x;\omega,\eta) < \beta < 0$. Therefore $\varphi_{n,r}(x;\omega,\eta) \leqq \inf\left\{\frac{\beta}{\eta}: \psi_{n,r}(x;\omega,\eta) < \beta < 0\right\} = \frac{\psi_{n,r}(x;\omega,\eta)}{\eta}$.

As $0<\frac{\psi_{n,r}(x;\omega,\eta)}{\eta} - \frac{\psi_{n,r}(x;\omega,\eta)}{\omega} = -\psi_{n,r}(x;\omega,\eta)\frac{\eta-\omega}{\eta\omega} \leqq (|f(x)|+n)\frac{\eta-\omega}{\eta\omega}$, the assertion of proposition 5 is proved for $-\infty< \psi_{n,r}(x;\omega,\eta)<0$.

It remains only to prove that $\varphi_{n,r}(x;\omega,\eta)=-\infty$ if $\psi_{n,r}(x;\omega,\eta)=-\infty$. But this is a consequence of proposition 4 and the inequality $|B_n(x;\beta;\omega,\eta)| \leqq r(\eta-\omega)$, which holds for all $\beta<0$ if $\psi_{n,r}(x;\omega,\eta)=-\infty$.

4. Let $0<\omega=\omega_0<\omega_1<\omega_2<\ldots<\omega_{k-1}<\omega_k=\eta$. Let $0<r<1$. We set $A = \{(r_1, r_2, \ldots, r_k): 0\leqq r_i<1$, r_i is a rational number for $i=1, 2, \ldots, k$ and $\sum_{i=1}^{k} r_i(\omega_i-\omega_{i-1})>r(\eta-\omega)\}$.

Proposition 6. *Let* $0<\omega<\eta$.

1. *Then for each* $(r_1, r_2, \ldots, r_k)\in A$ *there holds:*

a) $\psi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k) \leqq \Phi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k)$.

b) *If* $\Phi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k) > -\infty$, *then* $\Phi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k) - \Psi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k) \leqq (|f(x)|+n)v_k$.

c) *If* $f\in B_\alpha$, *then* $\Psi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k) \in B_\alpha$.

d) *If* f *is Lebesgue measurable, then* $\Psi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k)$ *is Lebesgue measurable.*

2. *We have:*

a) $\varphi_{n,r}(x;\omega,\eta) = \sup\{\Phi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k):(r_1,r_2,\ldots,r_k)\in A\}$.

b) $\Psi_n(x) = \sup\{\Psi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k): (r_1,r_2,\ldots,r_k)\in A\} \leqq \varphi_{n,r}(x;\omega,\eta)$.

c) *If* $\varphi_{n;r}(x;\omega,\eta)>-\infty$, *then* $\varphi_{n,r}(x;\omega,\eta)-\psi_n(x) \leqq (|f(x)|+n)v_k$.

Proof. 1. a) The assertion in a) is a direct consequence of proposition 5.

b) Let $\Phi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k)>-\infty$. Then $\min\{\varphi_{n,r_i}(x;\omega_{i-1},\omega_i): r_i>0,\ i=1, 2, \ldots, k\}>-\infty$. Thus we have: $\varphi_{n,r_i}(x;\omega_{i-1},\omega_i)>-\infty$ for each $i=1,2,\ldots,k$ for which $r_i>0$. From proposition 5 it follows that $\psi_{n,r_j}(x;\omega_{j-1},\omega_j)>-\infty$ and $\varphi_{n,r_j}(x;\omega_{j-1},\omega_j)-\min\left(\frac{\psi_{n,r_j}(x;\omega_{j-1},\omega_j)}{\omega_{j-1}}, \frac{\psi_{n,r_j}(x;\omega_{j-1},\omega_j)}{\omega_j}\right) \leqq (|f(x)|+n)\frac{\omega_j-\omega_{j-1}}{\omega_{j-1}\omega_j} \leqq (|f(x)|+n)v_k$ for each $j\in\{1, 2, \ldots, k\}$ which satisfies $r_j>0$. From this $\Phi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k) - \Psi_n(x;\omega_0,\omega_1,\ldots,\omega_k;r_1,r_2,\ldots,r_k) \leqq (|f(x)|+n)v_k$.

c) Let $f \in B_\alpha$. It follows from proposition 3 that $\psi_{n,r_i}(x; \omega_{i-1}, \omega_i)$ is a Borel function of the class α if $r_i > 0$ and $i = 1, 2, \ldots, k$. Since $\Psi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) = \min\left\{\min\left(\frac{\psi_{n,r_i}(x; \omega_{i-1}, \omega_i)}{\omega_{i-1}}, \frac{\psi_{n,r_i}(x; \omega_{i-1}, \omega_i)}{\omega_i}\right) : r_i > 0, i = 1, 2, \ldots k\right\}$, it is obvious that $\Psi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) \in B_\alpha$.

d) This is also an immediate consequence of proposition 3.

2. a) First it is obvious that $\bigcup_{i=1}^{k} A_n(x; \beta; \omega_{i-1}, \omega_i) = A_n(x; \beta; \omega, \eta)$ for each real number β.

Let $(r_1, r_2, \ldots, r_k) \in A$ and $\beta < \Phi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k)$. Then $\beta < \min\{\varphi_{n,r_i}(x; \omega_{i-1}, \omega_i) : r_i > 0, i = 1, 2, \ldots, k\}$. Therefore $|A_n(x; \beta; \omega_{i-1}, \omega_i)| > r_i(\omega_i - \omega_{i-1})$ for each $i = 1, 2, \ldots, k$ for which $r_i > 0$. From this $|A_n(x; \beta; \omega, \eta)| \geqq \Sigma\{|A_n(x; \beta; \omega_{i-1}, \omega_i)| : r_i > 0, i = 1, 2, \ldots, k\} = \sum_{i=1}^{k} r_i(\omega_i - \omega_{i-1}) > r(\eta - \omega)$. Therefore $\beta \leqq \varphi_{n,r}(x; \omega, \eta)$. Thus we have that $\Phi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) \leqq \varphi_{n,r}(x; \omega, \eta)$ and therefore $\sup\{\Phi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) : (r_1, r_2, \ldots, r_k) \in A\} \leqq \varphi_{n,r}(x; \omega, \eta)$.

There holds $\sup\{\Phi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) : (r_1, r_2, \ldots, r_k) \in A\} = \varphi_{n,r}(x; \omega, \eta)$ if $\varphi_{n,r}(x; \omega, \eta) = -\infty$.

Let $\varphi_{n,r}(x; \omega, \eta) > -\infty$. Let $\varphi_{n,r}(x; \omega, \eta) > \beta$. Then $|A_n(x; \beta; \omega, \eta)| > r(\eta - \omega)$. For $i = 1, 2, \ldots, k$, we denote by q_i the number $\frac{1}{\omega_i - \omega_{i-1}} |A_n(x; \beta; \omega_{i-1}, \omega_i)|$. If $q_i = 0$, we set $r_i = 0$. It is obvious that $\Sigma\{q_i(\omega_i - \omega_{i-1}) : q_i > 0, i = 1, 2, \ldots, k\} = \Sigma\{|A_n(x; \beta; \omega_{i-1}, \omega_i)| : q_i > 0, i = 1, 2, \ldots, k\} = |A_n(x; \beta; \omega, \eta)| > r(\eta - \omega)$. Therefore, for each $i = 1, 2, \ldots, k$ satisfying $q_i > 0$, there exists such a positive rational number r_i that $r_i < q_i$ and $\Sigma\{r_i(\omega_i - \omega_{i-1}) : r_i > 0, i = 1, 2, \ldots, k\} > r(\eta - \omega)$. Thus $(r_1, r_2, \ldots, r_k) \in A$. For $r_i > 0$ we have: $|A_n(x; \beta; \omega_{i-1}, \omega_i)| = q_i(\omega_i - \omega_{i-1}) > r_i(\omega_i - \omega_{i-1})$. From this it follows that $\varphi_{n,r_i}(x; \omega_{i-1}, \omega_i) \geqq \beta$ if $r_i > 0$ and $\beta \leqq \min\{\varphi_{n,r_i}(x; \omega_{i-1}, \omega_i) : r_i > 0, i = 1, 2, \ldots, k\} = \Phi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) \leqq \sup\{\Phi_n(x; \omega_0, \omega_1, \ldots, \omega_k; s_1, s_2, \ldots, s_k) : (s_1, s_2, \ldots, s_k) \in A\}$. Therefore there holds: $\varphi_{n,r}(x; \omega, \eta) \leqq \sup\{\Phi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) : (r_1, r_2, \ldots, r_k) \in A\}$.

b) This is an immediate consequence of 1 a) and 2 a).

c) Let $\varphi_{n,r}(x; \omega, \eta) > -\infty$ and $\varepsilon > 0$. Then, according to 2 a), there exists such a $(r_1, r_2, \ldots, r_k) \in A$ that $\Phi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) > \varphi_{n,r}(x; \omega, \eta) - \varepsilon$. Since, according to 1 b), $\Psi_n(x) \geqq \Psi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) \geqq \Phi_n(x; \omega_0, \omega_1, \ldots, \omega_k; r_1, r_2, \ldots, r_k) - (|f(x)| + n)v_k$, we have $\Psi_n(x) > \varphi_{n,r}(x; \omega, \eta) - \varepsilon - (|f(x)| + n)v_k$. As ε is any positive number, there is $\varphi_{n,r}(x; \omega, \eta) - \Psi_n(x) \leqq (|f(x)| + n)v_k$.

Let $0<\omega<\eta$, $0<r<1$ and k be a positive integer. We set $\omega_{i,k} = \omega+\frac{i}{2^k}(\eta-\omega)$ for $i=0, 1, 2, \ldots, 2^k$. Let, for $k=1, 2, 3, \ldots$, $A_k = \{(r_1, r_2, \ldots, r_{2^k})\colon 0\leqq r_i<1$ and r_i is a rational number for $i=0, 1, 2, \ldots, 2^k$ $\sum_{i=1}^{2^k} r_i \frac{\eta-\omega}{2^k} > r_r(\eta-\omega)\}$. We denote by $\Phi_{n,k}(x; r_1, r_2, \ldots, r_{2^k})$ the function $\min\{\varphi_{n,r_i}(x; \omega_{i-1,k}, \omega_{i,k})\colon r_i>0,\ i=1, 2, \ldots, 2^k\}$, by $\Psi_{n,k}(x; r_1, r_2, \ldots, r_{2^k})$ the function

$$\min\left\{\min\left(\frac{\psi_{n,r_i}(x; \omega_{i-1,k}, \omega_{i,k})}{\omega_{i-1,k}}, \frac{\psi_{n,r_i}(x; \omega_{i-1,k}, \omega_{i,k})}{\omega_{i,k}}\right)\colon r_i>0,\ i=1, 2, \ldots, 2^k\right\}$$

and by F the system $\{\Psi_{n,k}(x; r_1, r_2, \ldots, r_{2^k})\colon (r_1, r_2, \ldots, r_{2^k})\in A_k,\ k=1, 2, 3, \ldots\}$. We remark that the system F is obviously countable.

Theorem 1. *Let $0<\omega<\eta$ and $0<r<1$. If $f\in B_\alpha$, then the function $\varphi_{n,r}(x; \omega, \eta)$ is a lower semi-Borel function of the class α; if f is a Lebesgue measurable function, then $\varphi_{n,r}(x; \omega, \eta)$ is a Lebesgue measurable function.*

Proof. Now, from propostition 6 1a) and 2a), it follows that $\Psi_{n,k}(x; r_1, r_2, \ldots, r_{2^k}) \leqq \varphi_{n,r}(x; \omega, \eta)$ for $k=1, 2, 3, \ldots$ and $(r_1, r_2, \ldots, r_{2^k})\in A_k$. From this $\sup\{g(x)\colon g\in F\} \leqq \varphi_{n,r}(x; \omega, \eta)$.

If $\varphi_{n,r}(x; \omega, \eta)=-\infty$, the equality $\sup\{g(x)\colon g\in F\} = \varphi_{n,r}(x; \omega, \eta)$ holds.

Let $\varphi_{n,r}(x; \omega, \eta)>-\infty$ and $\varepsilon>0$. We choose such a positive integer k that $(|f(x)|+n)\frac{\eta-\omega}{\omega^2 2^k}<\varepsilon$. By proposition 6.2. c), $\varphi_{n,r}(x; \omega, \eta) - \sup\{\Psi_{n,k}(x; r_1, r_2, \ldots, r_{2^k})\colon (r_1, r_2, \ldots, r_{2^k}) \in A_k\} \leqq (|f(x)|+n) \max\left\{\frac{\eta-\omega}{\omega_{i,k}\omega_{i-1,k}2^k}\colon r_i>0,\ i = 1, 2, \ldots, 2^k\right\} \leqq (|f(x)|+n)\frac{\eta-\omega}{\omega^2 2^k}<\varepsilon$. Hence we get that $\varphi_{n,r}(x; \omega, \eta) - \sup\{g(x)\colon g\in F\}<\varepsilon$. The last inequality holds for all positive ε and therefore $\sup\{g(x)\colon g\in F\} = \varphi_{n,r}(x; \omega, \eta)$.

Let now $f\in B_\alpha$. By proposition 6.1. c), every function $g\in F$ is in B_α and therefore the set $\{x\in R\colon g(x)>\beta\}$ is a set of the Borel additive class α for each $g\in F$ and each $\beta\in R$. Since $\{x\in R\colon \varphi_{n,r}(x; \omega, \eta)>\beta\} = \cup\{\{x\in R\colon g(x) > \beta\}\colon g\in F\}$ and since the system F is countable, the set $\{x\in R\colon \varphi_{n,r}(x; \omega, \eta)>\beta\}$ is of the Borel additive class α. This proves that the function $\varphi_{n,r}(x; \omega, \eta)$ is a lower semi-Borel function of the class α.

Analogously, we prove that the function $\varphi_{n,r}(x; \omega, \eta)$ is a Lebesgue measurable function if f is a Lebesgue measurable function.

Proposition 7. *Let $0<\omega<\eta$ and $0<r<1$. Then for $n=1, 2, 3, \ldots$, $\beta\in R$ and $x\in R$ there holds:*

a) $A_n(x; \beta; \omega, \eta)\subset A_{n+1}(x; \beta; \omega, \eta)$,

b) $\varphi_{n,r}(x; \omega, \eta)\leqq\varphi_{n+1,r}(x; \omega, \eta)$,

c) $\varphi_r(x; \omega, \eta) = \lim_{n\to\infty} \varphi_{n,r}(x; \omega, \eta)$,

d) *The function* $\varphi_r(x; \omega, \eta)$ *is a lower semi-Borel function of the class* α *if* $f \in B_\alpha$.

e) *The function* $\varphi_r(x; \omega, \eta)$ *is a Lebesgue measurable function if* f *is a Lebesgue measurable function.*

Proof. a) This follows at once from the definition.

b) From a) it follows that $|A_{n+1}(x; \beta; \omega, \eta)| > r(\eta - \omega)$ if $|A_n(x; \beta; \omega, \eta)| > r(\eta - \omega)$. Therefore $\beta < \varphi_{n+1,r}(x; \omega, \eta)$ if $\beta < \varphi_{n,r}(x; \omega, \eta)$. Thus $\varphi_{n,r}(x; \omega, \eta) \leqq \varphi_{n+1,r}(x; \omega, \eta)$.

c) Since $A_n(x; \beta; \omega, \eta) \subset A(x; \beta; \omega, \eta)$ for $n = 1, 2, 3, \ldots$ and $\beta \in R$, one can easily prove that $\varphi_{n,r}(x; \omega, \eta) \leqq \varphi_r(x; \omega, \eta)$ for $n = 1, 2, 3, \ldots$. Thus $\lim_{n\to\infty} \varphi_{n,r}(x; \omega, \eta) \leqq \varphi_r(x; \omega, \eta)$.

Let now $\beta < \varphi_r(x; \omega, \eta)$. Then there exists such a γ that $\beta < \gamma < \varphi_r(x; \omega, \eta)$ and $|A(x; \gamma; \omega, \eta)| > r(\eta - \omega)$. Since $\{A_n(x; \gamma; \omega, \eta)\}_{n=1}^{\infty}$ is a non decreasing sequence of sets converging to the set $A(x; \gamma; \omega, \eta)$, there exists such a positive integer n that $|A_n(x; \gamma; \omega, \eta)| > r(\eta - \omega)$. But this gives that $\varphi_{n,r}(x; \omega, \eta) \geqq \gamma > \beta$. Therefore $\lim_{n\to\infty} \varphi_{n,r}(x; \omega, \eta) = \varphi_r(x; \omega, \eta)$.

d) By theorem 1, for $n = 1, 2, 3, \ldots$, the function $\varphi_{n,r}(x; \omega, \eta)$ is a lower semi-Borel function of the class α. Therefore, for $n = 1, 2, 3, \ldots$ and $\beta \in R$, the set $\{x \in R: \varphi_{n,r}(x; \omega, \eta) > \beta\}$ is of the Borel additive class α. Since $\{x \in R: \varphi_r(x; \omega, \eta) > \beta\} = \bigcup_{n=1}^{\infty} \{x \in R: \varphi_{n,r}(x; \omega, \eta) > \beta\}$ for each $\beta \in R$, the set $\{x \in R: \varphi_r(x; \omega, \eta) > \beta\}$ is of the Borel additive class α for each $\beta \in R$. Therefore the function $\varphi_r(x; \omega, \eta)$ is a lower semi-Borel function of the class α.

e) Using theorem 1, we prove easily that $\varphi_r(x; \omega, \eta)$ is a Lebesgue measurable function if f is a Lebesgue measurable function.

Let now $0 < \eta$ and $\{\eta_i\}_{i=1}^{\infty}$ be a decreasing sequence of positive numbers which converge to zero and $\eta_1 = \eta$, i. e. $\eta = \eta_1 > \eta_2 > \eta_3 > \ldots > 0$ and $\lim_{i\to\infty} \eta_i = 0$. Let A be the system of all such sequences $\{r_i\}_{i=1}^{\infty}$ of rational numbers that $0 \leqq r_i < 1$ for $i = 1, 2, 3, \ldots$, the set $\{i \in N: r_i > 0\}$ is finite and $\sum_{i=1}^{\infty} r_i(\eta_i - \eta_{i+1}) > r\eta$. Let F be the system $\{\Phi(x; \{\eta_i\}_{i=1}^{\infty}; \{r_i\}_{i=1}^{\infty}): \{r_i\}_{i=1}^{\infty} \in A\}$. We remark that it is obvious that the system F is countable.

Theorem 2. *Let* $\eta > 0$ *and* $0 < r < 1$. *Then there holds:*

a) $\varphi_r(x; 0, \eta) = \sup \{g(x): g \in F\}$.

b) *The function* $\varphi_r(x; 0, \eta)$ *is a lower semi-Borel function of the class* α *if* $f \in B_\alpha$.

c) *The function* $\varphi_r(x; 0, \eta)$ *is a Lebesgue measurable function if* f *is a Lebesgue measurable function.*

Proof. a) Let $g \in F$. Then there exists such a sequence $\{r_i\}_{i=1}^{\infty} \in A$ that $g(x) = \Phi(x; \{\eta_i\}_{i=1}^{\infty}; \{r_i\}_{i=1}^{\infty})$. Let now $\beta \in R$ and $\beta < g(x)$. Then $\beta < \varphi_{r_i}(x; \eta_{i+1}, \eta_i)$ for each $i \in N$ for which $r_i > 0$. Thus $|A(x; \beta; \eta_{i+1}, \eta_i)| > r_i(\eta_i - \eta_{i+1})$ for each $i \in N$ for which $r_i > 0$. Since $A(x; \beta; 0, \eta) = \bigcup_{i=1}^{\infty} A(x; \beta; \eta_{i+1}, \eta_i)$, there holds: $|A(x; \beta; 0, \eta)| \geqq \sum_{i=1}^{\infty} r_i(\eta_i - \eta_{i+1}) > r\eta$. Therefore $\beta \leqq \varphi_r(x; 0, \eta)$. From this $g(x) \leqq \varphi_r(x; 0, \eta)$. Hence we get that $\sup\{g(x): g \in F\} \leqq \varphi_r(x; 0, \eta)$.

Let $\beta \in R$ and $\beta < \varphi_r(x; 0, \eta)$. Then $|A(x; \beta; 0, \eta)| > r\eta$. Obviously there exists such an η_s that $|A(x; \beta; \eta_s, \eta)| > r\eta$. For each $i \geqq s$ we choose $r_i = 0$. Since $\sum_{i=1}^{s-1} \frac{|A(x; \beta; n_{i+1}, \eta_i)|}{\eta_i - \eta_{i+1}} (\eta_i - \eta_{i+1}) = |A(x; \beta; \eta_s, \eta)| > r\eta$, there exist such rational numbers $r_1, r_2, \ldots, r_{s-1}$ that, for $i = 1, 2, \ldots, s-1$, there holds: $r_i = 0$ if $|A(x; \beta; \eta_{i+1}, \eta_i)| = 0$, $0 < r_i < \frac{|A(x; \beta; \eta_{i+1}, \eta_i)|}{\eta_i - \eta_{i+1}}$ if $|A(x; \beta; \eta_{i+1}, \eta_i)| > 0$ and $\sum_{i=1}^{s-1} r_i(\eta_i - \eta_{i+1}) > r\eta$. Obviously $\{r_i\}_{i=1}^{\infty} \in A$. Thus $\Phi(x; \{\eta_i\}_{i=1}^{\infty}; \{r_i\}_{i=1}^{\infty}) \in F$. As for each $i \in N$ for which $r_i > 0$ the inequality $|A(x; \beta; \eta_{i+1}, \eta_i)| > r_i(\eta_i - \eta_{i+1})$ holds we have $\beta < \varphi_{r_i}(x; \eta_{i+1}, \eta_i)$ for each $i \in N$ for which $r_i > 0$. Therefore $\beta \leqq \Phi(x; \{\eta_i\}_{i=1}^{\infty}; \{r_i\}_{i=1}^{\infty}) = g(x)$. From this $\varphi_r(x; 0, \eta) \leqq g(x) \leqq \sup\{h(x): h \in F\}$.

Thus we have proved that $\varphi_r(x; 0, \eta) = \sup\{g(x): g \in F\}$.

b) By proposition 7 d), each function $\varphi_{r_i}(x; \eta_{i+1}, \eta_i)$ is a lower semi-Borel function of the class α. As each function of the system F is a minimum of a finite set of functions $\varphi_{r_i}(x; \eta_{i+1}, \eta_i)$ for some appropriate i, each function of F is a lower semi-Borel function of the class α. As the system F is countable and $\{x \in R: \varphi_r(x; 0, \eta) > \beta\} = \cup\{\{x \in R: g(x) > \beta\}: g \in F\}$ for each $\beta \in R$, the function $\varphi_r(x; 0, \eta)$ is a lower semi-Borel function of the class α.

c) This is a consequence of the countability of the system F, of the equation $\varphi_r(x; 0, \eta) = \sup\{g(x): g \in F\}$ and the Lebesgue measurability of each function $\varphi_{r_i}(x; \eta_{i+1}, \eta_i)$.

5. **Proposition 8.** *Let n and k be positive integers.*

a) *Then $\varphi_{n,k}(x) = \sup\{\varphi_{1/(k+1)}(x; 0, \eta): 0 < \eta \leqq \frac{1}{n}, \eta$ is a rational number$\}$.*

b) *If $f \in B_\alpha$, then $\varphi_{n,k}(x)$ is a lower semi-Borel function of the class α.*

c) *If f is a Lebesgue measurable function, then $\varphi_{n,k}(x)$ is a Lebesgue measurable function, too.*

Proof. a) Since $\{\varphi_{1/(k+1)}(x; 0, \eta): 0 < \eta \leqq \frac{1}{n}, \eta$ is a rational number$\} \subset$

$\left\{\varphi_{1/(k+1)}(x;0,\eta): 0<\eta\leqq\frac{1}{n}\right\}$ it holds $\sup\{\varphi_{1/(k+1)}(x;0,\eta): 0<\eta\leqq\frac{1}{n}$, η is a rational number$\}$ $\leqq$ $\sup\left\{\varphi_{1/(k+1)}(x;0,\eta): 0<\eta\leqq\frac{1}{n}\right\}$ $=$ $\varphi_{n,k}(x)$.

Let now $\beta<\varphi_{n,k}(x)$. Then there exists such a δ that $0<\delta\leqq\frac{1}{n}$ and $\varphi_{1/(k+1)}(x;0,\delta)>\beta$. Hence $|A(x;\beta;0,\delta)|>\frac{\delta}{k+1}$. It is obvious that there exists such a rational number ε that $0<\varepsilon\leqq\delta$ and $|A(x;\beta;0,\varepsilon)|>\frac{1}{k+1}\delta\geqq\frac{1}{k+1}\varepsilon$. From this $\varphi_{1/(k+1)}(x;0,\varepsilon)\geqq\beta$ and then also $\sup\{\varphi_{1/(k+1)}(x;0,\eta): 0<\eta\leqq\frac{1}{n}$, η is a rational number$\}$ $\geqq$ $\varphi_{1/(k+1)}(x;0,\varepsilon)\geqq\beta$. But this proves that $\sup\{\varphi_{1/(k+1)}(x;0,\eta): 0<\eta\leqq\frac{1}{n}$, η is a rational number$\}$ $\geqq$ $\varphi_{n,k}(x)$.

Thus we have proved that $\varphi_{n,k}(x)$ $=$ $\sup\{\varphi_{1/(k+1)}(x;0,\eta): 0<\eta\leqq\frac{1}{n}$, η is a rational number$\}$.

b) Let $f\in B_\alpha$. Since the system $\{\varphi_{1/(k+1)}(x;0,\eta): 0<\eta\leqq\frac{1}{n}$, η is a rational number$\}$ is a countable and since each function $\varphi_{1/(k+1)}(x;0,\eta)$, according to theorem 2 b), is a lower semi-Borel function of the class α, the function $\varphi_{n,k}$ is the least upper bound of the countable system of lower semi-Borel functions of the class α and therefore it is a lower semi-Borel function of the class α.

c) If f is a Lebesgue measurable function, then the function $\varphi_{n,k}$ is the least upper bound of a countable system of Lebesgue measurable functions and therefore it is Lebesgue measurable.

Theorem 3. a) *There holds:* $\bar{f}^+_{\text{ess}}(x)$ $=$ $\lim\limits_{k\to\infty}(\lim\limits_{n\to\infty}\varphi_{n,k}(x))$.

b) *If* $f\in B_\alpha$, *then* $\bar{f}^+_{\text{ess}}$ *is a lower semi-Borel function of the class* $\alpha+2$ *and thus it is a Borel function of the class* $\alpha+3$.

c) *If* f *is a Lebesgue measurable function, then* $\bar{f}^+_{\text{ess}}$ *is a Lebesgue measurable function.*

Proof. a) Let $\beta<\bar{f}^+_{\text{ess}}(x)$. Then there exists such a positive integer p that the upper outer density of the set $\left\{h: h>0, \frac{f(x+h)-f(x)}{h}>\beta\right\}$ in the point 0 is greater than $\frac{1}{p+1}$. Therefore, for each positive integer n, there exists such a number η that $0<\eta\leqq\frac{1}{n}$ and $|A(x;\beta;0,\eta)|$ $=$ $|\{h: 0<h\leqq\eta, \frac{f(x+h)-f(x)}{h}>$

$\beta\}| > \frac{1}{p+1}\eta$. Since for all positive integers n and k there holds: $\varphi_{n,k}(x) \geqq \varphi_{n+1,k}(x)$ and $\varphi_{n,k}(x) \leqq \varphi_{n,k+1}(x)$, we have $\lim_{n\to\infty}\varphi_{n,j}(x)\geqq\beta$ for $j\geqq p$. Thus $\lim_{k\to\infty}(\lim_{n\to\infty}\varphi_{n,k}(x))\geqq\beta$. As $\lim_{k\to\infty}(\lim_{n\to\infty}\varphi_{n,k}(x))\geqq\beta$ if $\beta<\bar{f}^{+}_{\text{ess}}(x)$, there holds: $\bar{f}^{+}_{\text{ess}}(x) \leqq \mathit{lim}_{k\to\infty}(\lim_{n\to\infty}\varphi_{n,k}(x))$.

If $\beta<\lim_{k\to\infty}(\mathit{lim}_{n\to\infty}\varphi_{n,k}(x))$, then, for each $n=1, 2, 3, \ldots$, there exists such a number η_n that $0<\eta_n\leqq\frac{1}{n}$ and $\varphi_{1/(k+1)}(x;\, 0,\, \eta_n)>\beta$. From this $0<\eta_n\leqq\frac{1}{n}$ and $|A(x;\,\beta;\, 0,\,\eta_n)| > \frac{1}{k+1}\eta_n$ for $n=1, 2, 3, \ldots$. But this implies that the set $\left\{h: h>0, \frac{f(x+h)-f(x)}{h}>\beta\right\}$ has in 0 the upper outer density not less than $\frac{1}{k+1}$. Therefore $\beta\leqq\bar{f}^{+}_{ess}(x)$. Hence we have proved that $\lim_{k\to\infty}(\lim_{n\to\infty}\varphi_{n,k}(x)) \leqq \bar{f}^{+}_{\text{ess}}(x)$. Thus the equality $\bar{f}^{+}_{\text{ess}}(x) = \lim_{k\to\infty}(\lim_{n\to\infty}\varphi_{n,k}(x))$ is valid.

b) Let $f\in B_\alpha$. Since for each $k\in N$, $\lim_{n\to\infty}\varphi_{n,k}(x)$ is the limit of a non-increasing sequence of lower semi-Borel functions of the class α, the limit $\lim_{n\to\infty}\varphi_{n,k}(x)$ is, for each $k\in N$, an upper semi-Borel function of the class $\alpha+1$. Since $\lim_{n\to\infty}\varphi_{n,k}(x) \leqq \lim_{n\to\infty}\varphi_{n,k+1}(x)$ for each $k\in N$, the function $\bar{f}^{+}_{\text{ess}}$ is the limit of a non-decreasing sequence of upper semi-Borel functions of the class $\alpha+1$. Therefore $\bar{f}^{+}_{\text{ess}}$ is a lower semi-Borel function of the class $\alpha+2$ and thus $\bar{f}^{+}_{\text{ess}}$ is a Borel function of the class $\alpha+3$.

c) This is a consequence of the equality $\bar{f}^{+}_{\text{ess}}(x) = \lim_{k\to\infty}(\lim_{n\to\infty}\varphi_{n,k}(x))$ and proposition 8 c).

6. **Theorem 4.** a) *There holds:* $\alpha\leqq\delta_{\text{ess}}(\alpha)$ *and* $\alpha\leqq\delta_{\text{ess}}(\alpha)$ *for* $\alpha\geqq0$.

b) *There exists a Lebesgue measurable function the upper right essential derivative and the upper bilateral essential derivative of which are not Borel functions.*

Proof. a) For $\alpha=0$ this is obvious.

Let C be the Cantor set in $\langle 0, 1\rangle$. The characteristic function c_C of the Cantor set is a Borel function of the class one and its upper right essential derivative, and also its upper bilateral essential derivative are Borel functions of the class one, since $\bar{c}_{C}{}^{+}_{\text{ess}}(x) = -\infty$, $\bar{c}_{C\,\text{ess}}(x)=\infty$ for $x\in C$ and $\bar{c}_{C}{}^{+}_{\text{ess}}(x) = \bar{c}_{C\,\text{ess}}(x)=0$ for $x\notin C$. Therefore $1\leqq\delta_{\text{ess}}(1)$ and $1\leqq\bar{\delta}_{\text{ess}}(1)$.

It is obvious that for $\alpha>1$ it suffices to prove this only for a non-limit α.

Let $\alpha>1$ and non-limit. From the existence theorem (Theorem I. in [2], p. 182) we get: For the Cantor set C there exists a subset A for which there holds:

(1) A is a Borel set in C of the additive class $\alpha-1$,

(2) A is not a Borel set in C of the additive class less than $\alpha-1$,

(3) $C-A$ is not a Borel set in C of the additive class $\alpha-1$.

It is obvious that the set A is a Borel set in $(-\infty,\infty)$ of the additive class $\alpha-1$ and not of the additive class less than $\alpha-1$, the set $(-\infty,\infty)-A$ is a Borel set in $(-\infty,\infty)$ of the additive class α and not of the additive class $\alpha-1$.

The characteristic function c_A is therefore a Borel function of the class α and its upper right essential derivative and its upper bilateral essential derivative are Borel functions of the class α, as $\bar{c}_{A}{}^{+}_{\text{ess}}(x) = -\infty$, $\bar{c}_{A\,\text{ess}}(x)=\infty$ for $x\in A$ and $\bar{c}_{A}{}^{+}_{\text{ess}}(x) = \bar{c}_{A\,\text{ess}}(x)=0$ for $x\notin A$. Thus we have proved that $\alpha\leqq\delta_{\text{ess}}(\alpha)$ and $\alpha\leqq\bar{\delta}_{\text{ess}}(\alpha)$ *for* $\alpha>1$ and the proof is finished.

b) Let A be a non Borel subset of the Cantor set C. Then c_A is Lebesgue measurable. As $\bar{c}_{A}{}^{+}_{\text{ess}}(x) = -\infty$, $\bar{c}_{A\,\text{ess}}(x)=\infty$ for $x\in A$ and $\bar{c}_{A}{}^{+}_{\text{ess}}(x) = \bar{c}_{A\,\text{ess}}(x)=0$ for $x\notin A$, the functions $\bar{c}_{A}{}^{+}_{\text{ess}}$ and $\bar{c}_{A\,\text{ess}}$ are Lebesgue measurable functions, but not Borel functions.

7. We add another remark.

S. Banach in [1] gives the following two theorems:

If the set of all numbers in which one of Dini's derivatives of a function f is infinite is at most countable, then the function f is a Borel function of the class 2.

If one of Dini's derivatives of a function f is almost everywhere finite, then f is a Lebesgue measurable function.

Are there any analogies to these theorems? Is the following assertion true: *If the extreme unilateral essential derivative of a function f is almost everywhere finite, is then f a Lebesgue measurable function?*

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Received February 25, 1977

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ЕКСТРАМАЛЬНЫЕ СУЩЕСТВЕННЫЕ ПРОИЗВОДНЫЕ БОРЕЛЕВСКИХ И ЛЕБЕГОВСКИХ ИНЗМЕРИМЫХ ФУНКЦИЙ

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Резюме

В этой работе доказывается, что $\alpha \leqslant \delta_{ess}(\alpha) \leqslant \alpha + 3$ и $\alpha \leqslant \bar{\delta}_{ess}(\alpha) \leqslant \alpha + 3$ для каждого порядково числа α из первых двух классов, когда $\delta_{ess}(\alpha) = \sup\{\gamma$: существует борелевская функция класса α, которой одна экстрамальная односторонняя существенная производная принадлежит борелевскому классу γ и не принадлежит борелевскому классу δ для $\delta < \gamma\}$ и $\bar{\delta}_{ess}(\alpha) = \sup\{\gamma$: существует борелевская функция класса α, которой одна экстрамальная двустронняя вущественная произбодная принадлежит борелевскому классу γ и не принадлежит борелевскому классу δ для $\delta < \gamma\}$. Каждая экстремальная существенная производная борелевской (лебеговской измеримой) функции — борелевскя (лебеговская измеримая).