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OSCILLATION CRITERIA FOR FOURTH-ORDER LINEAR DIFFERENTIAL EQUATIONS

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The present paper is a study of the oscillation of the differencial equation

$$\text{(L)} \qquad y^{(4)}+P(t)y''+Q(t)y=0\,,$$

where $P(t)$, $Q(t)$ are continuous functions on the interval $I=[a, \infty)$, $-\infty<a<\infty$. We shal assume throughout that

$$\text{(A)} \qquad P(t)\leqq 0, \quad Q(t)\leqq 0$$

and $Q(t)$ not identically zero in any subinterval of I.

The terminology introduced in [10] remains valid.

Oscillation criteria for equation (L) will be obtained by an application of the theory developed in [10] and by the oscillation of linear differential equation of the third order.

An equation of this type had been considered earlier by V. Pudei [8]. This author used for this equation a theory of conjugate points and obtained some oscillation and nonoscillation theorems of a somewhat different character from those described in this paper.

We remark that the proofs of the theorems in [8] and also in [7] are more complicated than those used in this article.

1.

Lemma 1.1. *Let $f(t)\in C^2[c, \infty)$ and $f(t)>0$, $f'(t)>0$, $f''(t)<0$ in $[c, \infty)$, $c\geqq a$. Then*

$$f(t)-(t-c)f'(t)>0$$

for $t\in[c, \infty)$.

Proof. Let $\varphi(t)=f(t)-(t-c)f'(t)$ for $t\geqq c$. Since $\varphi(c)=f(c)>0$ and $\varphi'(t)=-(t-c)f''(t)>0$ in (c, ∞), then $\varphi(t)>0$ for $t\in[c, \infty)$.

Remark 1.1. Let there be a function $\omega(t)\in C^3[c, \infty)$ and let $\omega(t)>0$, $\omega'(t)>0$, $\omega''(t)>0$ and $\omega'''(t)<0$ in $[c, \infty)$, $c\geqq a$.

Set $f(t)=\omega'(t)$ for $t\in[c,\infty)$. It follows from Lemma 1.1 that

(1) $$\omega'(t)-(t-c)\omega''(t)>0$$

for $t\in[c,\infty)$. Integration of (1) from c to t yields

(1′) $$\omega(t)>\frac{t-c}{2}\,\omega'(t)$$

for $t\in[c,\infty)$. From the above inequalities (1), (1′) we obtain

$$\omega(t)>\frac{(t-c)^2}{2}\,\omega''(t)$$

for $t\in[c,\infty)$.

Now we will state the relevant theorems without proof.

Lemma 1.2. [2]. *Let $f(t)$ be a real valued function defined in $[t_0,\infty)$ for some real number $t_0\geqq 0$. Suppose that $f(t)>0$ and that $f'(t)$ and $f''(t)$ exist for $t\geqq t_0$. Suppose also that if $f'(t)\geqq 0$ eventually, then $\lim_{t\to\infty} f(t)=A<\infty$. Then*

$$\liminf_{t\to\infty} |t^\alpha f''(t)-\alpha t^{\alpha-1}f'(t)|=0$$

for any $\alpha\leqq 2$.

Theorem 1.1 [10]. *Suppose that* (A) *holds. Then* (L) *is nonoscillatory on I if and only if there exists a number $t_0\in I$ and a solution $y(t)$ of* (L) *such that either*

$$y(t)>0,\quad y'(t)>0,\quad y''(t)<0,$$

or

$$y(t)>0,\quad y'(t)>0,\quad y''(t)>0,\quad y'''(t)<0$$

for all $t\geqq t_0$.

The preceding results will now be used to prove the following assertion.

Theorem 1.2. *Suppose that* (A) *holds and let*

$$\int_{\tau_0}^{\infty} t^{2+\alpha}Q(t)\,dt=-\infty,\quad \tau_0\geqq \max\{a,0\},\quad 0\leqq\alpha<1.$$

Then (L) *is nonoscillatory if and only if there exists a solution $y(t)$ of* (L) *and a number $t_0\in I$ such that $y(t)>0$, $y'(t)>0$ and $y''(t)<0$ for all $t\geqq t_0$.*

Proof. The sufficient condition follows from Theorem 1.1. In order to prove the necessary condition, we will show that (L), by the above assumptions, has no solution $y(t)$ with the properties $y(t)>0$, $y'(t)>0$, $y''(t)>0$, $y'''(t)<0$ in the interval $[t_0,\infty)$, $t_0>\max\{a,0\}$. The assertion then again follows from Theorem 1.1.

Suppose on the contrary that such a solution $y(t)$ exists. Applying Lemma 1.1 and Remark 1.1 to the solution $y(t)$, we obtain

$$\frac{y''(t)}{y(t)} < \frac{2}{(t-t_0)^2}$$

for $t > t_0$ and hence for $t \geqq 2t_0$

$$\frac{y''(t)}{y(t)} < \frac{8}{t^2}. \tag{2}$$

Multiplication of (L) by $\frac{t^{2+\alpha}}{y}$, $0 \leqq \alpha < 1$ and integration from $2t_0$ to t yields

$$\begin{aligned} & t^{2+\alpha}\frac{y'''(t)}{y(t)} - (2+\alpha)t^{1+\alpha}\frac{y''(t)}{y(t)} + \int_{2t_0}^{t} \frac{y'(s)y'''(s)}{y^2(s)} s^{2+\alpha}\, ds - \\ & -(2+\alpha)\int_{2t_0}^{t} \frac{y'(s)y''(s)}{y^2(s)} s^{1+\alpha}\, ds + (2+\alpha)\,(1+\alpha)\int_{2t_0}^{t} \frac{y''(s)}{y(s)} s^{\alpha}\, ds + K \geqq \\ & \geqq -\int_{2t_0}^{t} s^{2+\alpha} Q(s)\, ds, \end{aligned} \tag{3}$$

where K is a finite constant. It follows from (2) and the fact that $0 \leqq \alpha < 1$ that the integral

$$\int_{2t_0}^{\infty} \frac{y''(s)}{y(s)} s^{\alpha}\, ds$$

is finite. Therefore the left-hand side of (3) consists of bounded or negative terms while the right-hand side of (3) tends to ∞ as $t \to \infty$. This is a contradiction. Theorem 1.2 is proved.

Theorem 1.3. *Suppose that* (A) *holds and let* $-2t^{-2} \leqq P(t)$ *for* $t > t_0 \geqq \max\{a, 0\}$. *Then there does not exist a solution* $y(t)$ *of* (L) *such that* $y(t) > 0$, $y'(t) > 0$, $y''(t) < 0$ *for* $t > t_1 \geqq t_0$.

Proof. Suppose on the contrary that there exists a solution $y(t)$ with the properties $y(t) > 0$, $y'(t) > 0$, $y''(t) < 0$ for $t \geqq t_1$. It follows from (L) on account of the hypothesis $-2t^{-2} \leqq P(t)$ that

$$y^{(4)} - \frac{2}{t^2} y'' + Q(t)y \geqq 0, \qquad t \geqq t_1. \tag{4}$$

Since $y'''(t) < 0$ eventually is impossible ($y' > 0$ and $y'' < 0$), pick $t_1' > t_0$ such that $y'''(t_1') \geqq 0$. Now multiply (4) by t^2 and integrate by parts from t_1' to t to obtain

$$t^2 y'''(t)) - 2ty''(t) \geqq t_1'^2 y'''(t_1') - 2t_1' y''(t_1') - \int_{t_1'}^{t} s^2 Q(s) y(s)\, ds.$$

The right-hand side of (5) is positive and increasing for $t \geqq t_1'$. However, by Lemma 1.2, with $\alpha = 2$, it follows that the lim inf of the left-hand side of (5) is zero. This contradiction proves the theorem.

The following example shows that in general the condition $-2t^{-2} \leqq P(t)$ cannot be replaced by $-\mathrm{D}t^{-2} \leqq P(t)$, $\mathrm{D} > 2$.

Example. The equation

$$y^{(4)} - \frac{\mathrm{D}}{t^2} y'' + Q(t)y = 0, \qquad t > 1,$$

where

$$Q(t) = -\frac{\mathrm{D}-2}{t^4 \ln t} - \frac{2(\mathrm{D}+1)}{t^4 \ln^2 t} - \frac{12}{t^4 \ln^3 t} - \frac{24}{t^4 \ln^4 t},$$

has the solution $y(t) = t(\ln t)^{-1}$ with $y(t) > 0$, $y'(t) > 0$ and $y''(t) < 0)$ eventually, say for $t_1 > 1$. If $\mathrm{D} > 2$, then there is a $t_2 > 1$ such that $Q(t) < 0$ for $t > t_2$. Hence for $\mathrm{D} > 2$ and $t \geqq \max\{t_1, t_2\} y(t) > 0$, $y'(t) > 0$, $y''(t) < 0$ and $Q(t) < 0$.

Theorem 1.4. *Suppose that* (A) *holds and let*

$$\int_{t_0}^{\infty} sP(s)\,\mathrm{d}s > -\infty, \qquad t_0 > \max\{a, 0\}.$$

Then there is not a solution $y(t)$ of (L) *with the properties $y(t) > 0$, $y'(t) > 0$ and $y''(t) < 0$ for $t \geqq t_0$.*

The proof is obtained similarly to that of Theorem 2.4 [2], and will be omitted.

Example. The equation

$$y^{(4)} - \frac{1}{2(1+e^t)} y'' - \frac{e^{t}}{t^2 + 2e^{-t}} y = 0, \qquad t \geqq 0,$$

has the solution $y = \frac{1}{2}t^2 + e^{-t}$ with the properties $y > 0$, $y' > 0$, $y'' > 0$, $y''' < 0$ for $t \geqq t_0 \geqq 0$.

Since

$$\int_{t_0}^{t} sP(s)\,\mathrm{d}s = -\frac{1}{2}\int_{t_0}^{t} \frac{s}{1+e^s}\,\mathrm{d}s > -\infty$$

as $t \to \infty$, the above differential equation cannot have a solution y with the properties $y > 0$, $y' > 0$, $y'' < 0$ for $t \geqq t_0$.

Theorem 1.5. *Suppose that* (A) *holds and let* $\int_{\tau_0}^{\infty} t^2 Q(t)\,\mathrm{d}t = -\infty$, $\tau_0 \in I$. *If in addition* $-\infty < -M \leqq t^2 P(t)$ *for* $t \geqq \tau_0$, *then the equation* (L) *is oscillatory.*

Proof. Suppose on the contrary that (L) is nonoscillatory. Then, by

Theorem 1.2 there exists a solution $y(t)$ of (L) and a number $t_0 \in I$ such that $y(t)>0$, $y'(t)>0$, $y''(t)<0$ for all $t \geqq t_0$. Multiply (L) by t^2 and integrate from t_0 to t to obtain

$$\begin{aligned}[s^2y'''(s)]_{t_0}^t - 2[sy''(s)]_{t_0}^t + (2-M)\int_{t_0}^t y''(s)\,ds + \\ + \int_{t_0}^t s^2Q(s)y(s)\,ds \geqq 0.\end{aligned} \tag{6}$$

Note that the term $(2-M)\int_{t_0}^t y''(s)\,ds$ is bounded as $t\to\infty$ since $y'(t)$ has a finite limit. Therefore (6) can be written

$$t^2y'''(t) - 2ty''(t) \geqq K - \int_{t_0}^t s^2Q(s)y(s)\,ds,$$

where K is a finite constant. It follows from the assumptions of the Theorem and the fact that $y(t)$ is a positive and increasing function that the right-hand side of (7) tends to ∞ as $t\to\infty$. However, by Lemma 1.2, with $\alpha=2$ it follows that

$$\liminf_{t\to\infty} |t^2y'''(t) - 2ty''(t)| = 0.$$

This contradiction proves the Theorem.

Theorem 1.6. *Suppose that* (A) *holds and let* $\int_{t_0}^\infty Q(t)\,dt = -\infty$, $t_0 \in I$. *If in addition* $-\infty < -m \leqq P(t)$ *for* $t \geqq t_0$, *then the equation* (L) *is oscillatory.*

Proof. Suppose on the contrary that (L) is nonoscillatory. Then, by Theorem 1.2 the exists a solution $y(t)$ of (L) and a number $t_0 \in I$ *such that* $y(t)>0$, $y'(t)>0$ and $y''(t)<0$ for $t \geqq t_0$. From (L) we get on account of the hypothesis $-m \leqq P(t)$

$$(y''' - my')' + Q(t)y \geqq 0, \qquad t \geqq t_0.$$

Integration of the last inequality from t_0 to t yields

$$y'''(t) - my'(t) \geqq K - \int_{t_0}^t Q(s)y(s)\,ds,$$

where K is a finite constant. Since $y(t)$ is a positive and increasing function and $\int_{t_0}^\infty Q(s)\,ds = -\infty$, the right-hand side of (8) tends to ∞ as $t\to\infty$. It then follows from (8) that $y'''(t)$ tends to ∞ as $t\to\infty$. Hence there exists a number $t_1 \geqq t_0$ such that $y''(t)>0$ for $t \geqq t_1$. This contradicts the fact that $y''(t)<0$ for $t \geqq t_0$. The Theorem is proved.

Theorem 1.7. *Suppose that* (A) *holds and let*

1. $$\int_{\tau_0}^{\infty} t^{2+\alpha}Q(t)\,\mathrm{d}t = -\infty,\ \tau_0 > \max\{a, 0\}\ \textit{for some}\ 0 \leqq \alpha < 1$$

and

$$\int_{\tau_0}^{\infty} tP(t)\,\mathrm{d}t > -\infty \text{ or } -\frac{2}{t^2} \leqq P(t),\ t \geqq \tau_0,$$

or

2. $$\int_{\tau_0}^{\infty} t^2 Q(t)\,\mathrm{d}t = -\infty \text{ and } -\infty < -M \leqq t^2 P(t) \text{ for } t \geqq \tau_0,\ \tau_0 \in I,$$

or

3. $$\int_{t_0}^{\infty} Q(t)\,\mathrm{d}t = -\infty \text{ and } -\infty < -m \leqq P(t) \text{ for } t \geqq t_0,\ t_0 \in I.$$

Then equation (L) *is oscillatory and there exists a fundamental system of solutions of* (L) *such that two solutions of this system are oscillatory, other solutions of this system are nonoscillatory and one of them tends monotonically to* ∞ *as* $t \to \infty$ *and the other of them tends to zero.*

Proof. It follows from Theorem 1.5 and 1.6 that (L) is oscillatory by the assumptions 2 or 3 and on account of Theorems 1.2, 1.3, 1.4 and Theorem 6 [10] is oscillatory also by the assumption 1.

Now we prove the last statement of Theorem 1.7. It follows from Theorem 2 [10] that there exists a solution z with the properties $z > 0$, $z' < 0$, $z'' > 0$ and $z''' < 0$ for $t \geqq a$. By Theorem 3 [10]

$$\lim_{t\to\infty} z(t) = 0.$$

Let z_0, z_1, z_2, z_3 denote solutions of (L) defined on I by the initial conditions

$$z_i^{(j)}(a) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

for i, $j = 0, 1, 2, 3$. Then (L) has oscillatory solutions

$$u(t) = b_0 z_0(t) + b_3 z_3(t)$$
$$v(t) = c_2 z_2(t) + c_3 z_3(t),$$

whose construction has already been shown in the proof of Theorem 6 [10].

Note that z_3 has no zero to the right of a by Lemma 4 [10] and $\lim_{t\to\infty} z_3(t) = \infty$.

The solutions $z(t)$, $u(t)$, $v(t)$ and $z_3(t)$ form the fundamental system of (L). In fact, their Wronskian at the point a yields

$$\begin{vmatrix} z(a), & b_0, & 0, & 0 \\ z'(a), & 0, & 0, & 0 \\ z''(a), & 0, & c_2, & 0 \\ z'''(a), & b_3, & c_3, & 1 \end{vmatrix} = -b_0 c_2 z'(a) \neq 0,$$

since $z'(a)<0$ and $b_0 \neq 0$, otherwise it would be $u(t)=b_3 z_3(t)$, which would contradict the fact that $u(t)$ is oscillatory and $z_3(t)$ has not zeros to the right of a. By the same argument $c_2 \neq 0$. The proof of Theorem 1.7 is complete.

Remark 1.2. By the assumptions (A), 1 and $P(t)\equiv 0$, Theorem 1.7 is generalization of Kondratev's Theorem 2.5 [5].

Remark 1.3. In the special case equation

$$(\mathrm{L}') \qquad y^{(4)}+Q(t)y=0$$

is oscillatory if

$$\int_{t_0}^{\infty} t^{2+\alpha} Q(t)\,\mathrm{d}t = -\infty, \quad 0 \leqq \alpha < 1, \quad t_0 > \max\{a, 0\}.$$

Remark 1.4. Suppose that (A) holds and $P(t)\equiv 0$. If (L′) is nonoscillatory, then

$$(9) \qquad \int_{t_0}^{\infty} t^{2+\alpha} Q(t)\,\mathrm{d}t > -\infty, \quad t_0 > \max\{a, 0\}$$

for any $0 \leqq \alpha < 1$.

The nonoscillatory equation

$$y^{(4)} - \frac{9}{16t^4}\,y = 0, \qquad t>0$$

shows that the conclusion (9) does not hold for $\alpha=1$. On the bases of Theorem 9 [10] this result is equivalent to the result of W. Leighton and Z. Nehari [7]. Negation of this implication has not been proved in [7].

2.

M. Gera [3] considered the linear differential equation of the third order

$$K[y] \equiv y''' + p(t)y'' + q(t)y' + r(t)y = 0,$$

'where $p(t)$, $q(t)$, $r(t) \in C(I)$.

Let

$$F(y, z) \equiv zy'' + (p(t)z - z')y' + [q(t)z + (z' - p(t)z)']y.$$

He proved the following theorem and corollaries.

Theorem 2.1. *A necessary and sufficient condition for the equation $K[y]=0$ to be disconjugate in $[t_0, \infty)$, $t_0 \in I$ is the existence of the functions $w_1(t)$, $w_2(t) \in C^3[t_0, \infty)$ with the properties*

$$w_1(t)>0, \quad w_2(t)>0, \quad K[w_1]\leqq 0, \quad K[w_2]\geqq 0$$

for $t \in [t_0, \infty)$ such that the differential equations of the second order

$$F(w_1, z)=0, \qquad F(w_2, z)=0$$

are disconjugate in (t_0, ∞).

Corollary 2.1. *Let there be a function $w(t) \in C^3(t_0, \infty)$ with the properties $w(t)>0$, $w'(t)>0$ (<0), $K[w]\leqq 0$ $(\geqq 0)$ in (t_0, ∞), $t_0 \in I$ and let $r(t)\geqq 0$ $(\leqq 0)$ for $t \in [t_0, \infty)$. Then the differential equation $K[y]=0$ is disconjugate in $[t_0, \infty)$.*

Corollary 2.2. *Let there be a function $w(t) \in C^3(t_0, \infty)$ with the properties $w(t)>0$, $w''(t)\geqq 0$, $K[w]\leqq 0$ in (t_0, ∞) and let $q(t)\leqq 0$, $r(t)\leqq 0$ for $t \in [t_0, \infty)$. Then the differential equation $K[y]=0$ is disconjugate in $[t_0, \infty)$.*

Theorem 2.2 [10]. *Suppose that* (A) *holds. Then* (L) *is nonoscillatory on I if and only if there exists a number $t_0 \in I$ and a solution $y(t)$ of* (L) *such that either*

$$y(t)>, \quad y'(t)>0, \quad y''(t)<0,$$

or

$$y(t)>0, \quad y'(t)>0, \quad y''(t)>0, \quad y'''(t)<0$$

for all $t \geqq t_0$.

The preceding results will now be used in order to prove the following theorem.

Theorem 2.3. *Suppose that* (A) *holds. Let $\mu(t)$ be a positive and continuous function in (T, ∞), $T\geqq a$, such that*

$$\liminf_{t\to\infty} \frac{t-t_0}{\mu(t)} \geqq 2 \tag{10}$$

for arbitrary $t_0\geqq a$ and let the differential equation of the third order

$$x''' + P(t)x' + \Theta\mu(t)Q(t)x = 0$$

for some $\Theta \in (0, 1)$ be oscillatory. Then (L) *also is oscillatory.*

Remark. Examples of functions $\mu(t)$ for which the condition (10) is satisfied are as follows:

$$v(t-a),\ v \in (0, \tfrac{1}{2}]; \quad k(t-a)^{\varepsilon},\ k>0,\ \varepsilon<1; \quad \ln(t-a),\ t>T,\ T\geqq a+1.$$

Proof. Suppose on the contrary that (L) is nonoscillatory. Then, by Theorem 2.2 there exists a number $t_0 \in I$ $(t_0 \geqq T)$ and a solution y of (L) such that either $y>0$, $y'0$, $y''>0$ and $y'''<0$, or $y>0$, $y'>0$ and $y''<0$ for $t \geqq t_0$. Applying Lemma 1.1 and Remark 1.1 to the solution y, we obtain

$$y > \frac{t-t_0}{2} y'$$

for $t \geqq t_0$. It then follows from (L) that

$$y^{(4)} + P(t)y'' + \frac{t-t_0}{2} Q(t)y' \geqq 0.$$

If $\liminf_{t\to\infty} \frac{t-t_0}{\mu(t)} \geqq 2$, for $\Theta \in (0, 1)$, then there exists a number $\tau > t_0 (\tau > T)$ such that

$$\frac{t-t_0}{\mu(t)} > 2\Theta \quad \text{for} \quad t > \tau$$

and hence

$$\frac{t-t_0}{2} > \Theta\mu(t)$$

for $t > \tau$. Setting $y' = z$, we obtain

$$z''' + P(t)z' + \Theta\mu(t)Q(t)z \geqq 0$$

and $z>0$, $z'>0$, $z''<0$ or $z>0$, $z'<0$ on (τ, ∞). It follows from the above inequalities and Corollaries 2.1 and 2.2 of Theorem 2.1 that the linear differential equation

$$x''' + P(t)x' + \Theta\mu(t)Q(t)x = 0$$

is disconjugate on $[\tau, \infty)$. This contradicts the hypothesis of the theorem. Theorem is established.

Theorem 2.3 is the main result of this paper. By combining Theorem 2.3 with the known oscillation criteria for the third-order equation

$$\text{(K')} \qquad x''' + q(t)x' + r(t)x = 0$$

we obtain oscillation criteria for (L).

Theorem 2.4 [6]. *Suppose that $q(t) \in C^1(I)$, $q(t) \leqq 0$, $q'(t) - r(t) > 0$ in I and*

$$\int_a^\infty [3\sqrt{3}(q'(t) - r(t)) - 2(-q(t))^{3/2}]\, dt = \infty.$$

Then (K') *is oscillatory.*

Theorem 2.4'. *Let $\mu(t)$ be a positive and continuous function in (T, ∞), $T \geqq a$ such that*

$$\liminf_{t\to\infty} \frac{t-t_0}{\mu(t)} \geqq 2$$

for arbitrary $t_0 \geqq a$. *Suppose that* (A) *holds and let* $P(t) \in C^1[T, \infty)$, $P'(t) - \Theta\mu(t)Q(t) > 0$ *in* $[T, \infty)$ *and*

$$\int_T^\infty [3\sqrt{3}(P'(t) - \Theta\mu(t)Q(t)) - 2(-P(t))^{3/2}]\, dt = \infty$$

for some $\Theta \in (0, 1)$. *Then* (L) *is oscillatory.*

Theorem 2.5 [11]. *Suppose that the coefficients of* (K′) *satisfy the assumptions*

$$q(t) \in C^1(I),\ q(t) \geqq \sigma,\ |2r(t) - q'(t)| \geqq \delta,$$

where $\sigma \leqq 0$ *and* $\delta > \frac{4}{3\sqrt{3}}(-\sigma)^{3/2}$, σ *and* δ *are both constants, or the assumptions*

$$q(t) \in C^1(I),\ q(t) \geqq \frac{\sigma}{t^2},\ |2r(t) - q'(t)| \geqq \frac{\delta}{t^3},$$

where $\sigma < 1$ *and* $\delta > \frac{4}{3\sqrt{3}}(1-\sigma)^{3/2}$, σ *and* δ *are both constants. Then* (K′) *is oscillatory..*

Theorem 2.5′. *Let* $\mu(t)$ *be a positive and continuous function in* (T, ∞), $T \geqq a$ *such that*

$$\liminf_{t\to\infty} \frac{t - t_0}{\mu(t)} = \geqq 2$$

for arbitrary $t_0 \geqq a$. *Suppose that* (A) *holds and* $P(t) \in C^1[T, \infty)$. *Let*

$$P(t) \geqq \sigma,\ |2\Theta\mu(t)Q(t) - P'(t)| > \delta \text{ for some } \Theta \in (0, 1),$$

where $\sigma \leqq 0$ *and* $\delta > \frac{4}{3\sqrt{3}}(-\sigma)^{3/2}$, σ *and* δ *are both constants, or let*

$$P(t) \geqq \frac{\sigma}{t^2},\ |2\Theta\mu(t)Q(t) - P'(t)| \geqq \frac{\delta}{t^3}$$

for some $\Theta \in (0, 1)$, *where*

$$\sigma \leqq 0 \text{ and } \delta > \frac{4}{3\sqrt{3}}(1-\sigma)^{3/2},$$

σ *and* δ *are both constants. Then* (L) *is oscillatory.*

Theorem 2.6. *Let* $-\frac{2}{t^2} \leqq q(t) \leqq 0$ *or* $q(t) \leqq 0$ *and* $\int_{t_0}^\infty tq(t)\, dt > -\infty$ *and let* $r(t) \leqq 0$ *for* $t \geqq t_0$, $t_0 > \max\{1, a\}$.

If

$$\int_{t_0}^{\infty} r(t)f(t)\,dt = -\infty,$$

where $f(t)$ is one of the functions

$$t^{1+\alpha},\ t^2(\ln t)^{\alpha-2},\ t^2(\ln t)^{-1}(\ln(\ln t))^{\alpha-2},$$

for some $0<\alpha<1$, then (K′) *is oscillatory.*

The proof of Theorem 2.6 follows from Theorem 1.1 [9] and from Theorems 2.4, 2.5 and 2.8 [2].

Theorem 2.6′. *Let $\mu(t)$ be a positive and continous function in (T, ∞), $T>$* max $\{1, a\}$ *such that*

$$\liminf_{t\to\infty} \frac{t-t_0}{\mu(t)} \geqq 2$$

for arbitrary $t_0 \geqq a$. Suppose that (A) *holds and let*

$$-\frac{2}{t^2} \leqq P(t) \text{ for } t \geqq T \text{ or } \int_T^{\infty} tP(t)\,dt > -\infty$$

and suppose that

$$\int_T^{\infty} \mu(t)Q(t)f(t)\,dt = -\infty,$$

where $f(t)$ is one of the function

$$t^{1+\alpha},\ t^2(\ln t)^{\alpha-2},\ t^2(\ln t)^{-1}(\ln(\ln t))^{\alpha-2},\ 0<\alpha<1.$$

Then (L) *is oscillatory.*

Remark 2.1. The oscillation criterion which was proved in Theorem 1.7 is special case of Theorem 2.6′ for $\mu(t)=vt$, $v \in (v, \frac{1}{2}]$ and $f(t)=t^{1+\alpha}$, $0<\alpha<1$.

Now we consider the equation

$$\text{(K}''\text{)} \qquad y''' + r(t)y = 0.$$

The following theorem is proved in [4].

Theorem 2.7. *Suppose that $r(t) \in C(I)$ and let $r(t) \leqq \left(-\frac{2\sqrt{3}}{9} - \varepsilon(t)\right)\frac{1}{t^3}$ for $t \geqq \tau_0 >$* max $\{a, 0\}$, *where $\varepsilon(t) \geqq 0$, $\int_{\tau_0}^{\infty} \frac{\varepsilon(t)}{t}\,dt = \infty$.*

Then equation (K″) *is oscillatory.*

On account of Theorem 2.7 we obtain the following statement.

Theorem 2.7′. *Let $\mu(t)$ be a positive and continuous function in (T, ∞), $T > \max\{a, 0\}$, such that*

$$\liminf_{t \to \infty} \frac{t - t_0}{\mu(t)} \geqq 2$$

for arbitrary $t_0 \geqq a$. If

$$\Theta \mu(t) Q(t) \leqq \left(-\frac{2\sqrt{3}}{9} - \varepsilon(t) \right) \frac{1}{t^3}$$

for $t \geqq T$ and some $\Theta \in (0, 1)$, where $\varepsilon(t) \geqq 0$, $\int_T^\infty \frac{\varepsilon(t)}{t} \mathrm{d}t = \infty$, then equation $y^{(4)} + Q(t) y = 0$ is oscillatory.

We finally show how Theorem 2.2, Lemma 1.1 and Corollaries 2.1 and 2.2 of Theorem 2.1 may be used in order to obtain sufficient criteria for the nonoscillation of the linear differential equation of the form

$$(\mathrm{K}''') \qquad x''' + P(t) x' + Q(t) x = 0.$$

Theorem 2.8 [5]. *If* (A) *holds and if*

$$(10) \qquad P(t) \in C^2(I), \ \tfrac{1}{2} P''(t) + Q(t) \geqq 0 \ \text{ for } t \in I,$$

then (L) *is nonoscillatory.*

Theorem 2.8′. *Suppose that* (A) *and* (10) *hold. Then* (K′′′) *is nonoscillatory on I.*

Proof. It follows from Theorem 2.8 that (L) is nonoscillatory. Hence by Theorem 2.2, there exists a number $t_0 \in I$ and a solution $y(t)$ of (L) such that $y(t) > 0$, $y'(t) > 0$, $y''(t) < 0$ or $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$, $y'''(t) < 0$ for all $t \geqq t_0$. Applying Lemma 1.1 and Remark 1.1 to the solution $y(t)$, we obtain

$$y(t) > \frac{t - t_0}{2} y'(t)$$

for $t \geqq t_0$. It then follows from (L) that

$$y^{(4)} + P(t) y'' \geqq -\frac{t - t_0}{2} Q(t) y', \quad t \geqq t_0.$$

If $t \geqq 2 + t_0 = t_1$, then

$$y^{(4)} + P(t) y'' + Q(t) y' \geqq 0, \quad t \geqq t_1.$$

Setting $y' = z$, we obtain

$$z''' + P(t) z' + Q(t) z \geqq 0, \quad t \geqq t_1$$

and $z>0$, $z'<0$ or $z>0$, $z'>0$, $z''<0$ in $[t_1, \infty)$. It follows from the above inequalities and Corollaries 2.1 and 2.2 that equation

$$x''' + P(t)x' + Q(t)x = 0$$

is disconjugate on $[t_1, \infty)$ and hence nonoscillatory on $[a, \infty)$. The theorem is proved.

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ОСЦИЛЛЯЦИОННЫЕ КРИТЕРИИ ДЛЯ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЪНЫХ УРАВНЕНИЙ ЧЕТВЕРТОГО ПОРЯДКА

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Резюме

В работе приведены критерии для осцилляции уравнения (L). Некоторые из этих критерий являются обобщением некоторых известных результатов.

Например, утверждение обобщающее результат В. А. Кондрвтьера [3]:

Пусть (A) выполнено и пусть

$$\int_{\tau_0}^{\infty} t^{2+\alpha} Q(t)\,\mathrm{d}t = -\infty, \quad \tau_0 > \max\{a, 0\}, \quad 0 \leqq \alpha < 1.$$

Пусть, кроме того

$$\int_{\tau_0}^{u} tP(t)\,\mathrm{d}t > -\infty \left(\frac{2}{t^2} \leqq P(t)\right), \quad t \geqq \tau_0.$$

Тогда уравнение (L) осцилляционное и существует фундаментальная система решений уравнения (L), такая, что два из входящих в нее решений колеблются а остальные два не колеблются, причем одно из низ монотонно стремится к нулю, а второе монотонно стремится к $+\infty$.