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A REMARK ON THE ČEBYŠEV PROPERTY

BOHDAN ZELINKA

At the Fifth Hungarian Colloquium on Combinatorics in Keszthely in 1976 B. Uhrin has proposed the following problem [1]:

Let $A \subset R^n$ be a set of finite cardinality $|A| = m \geqq n+1$. The set A is said to have the Čebyšev ($T-$) property if the points of A can be indexed (i.e. if A can be written in the form $A = \{a_i\}_{i=1}^m$) so that the condition

$$\operatorname{sgn} \det [a_{i_1}, a_{i_2}, \ldots, a_{i_n}] = \text{const} \neq 0$$

for all $\{i_k\}_{k=1}^n$, $1 \leqq i_1 \leqq i_2 \leqq \ldots \leqq i_n \leqq m$ holds.

Problem: Find some (fairly simple) sufficient (and, or) necessary conditions for A to have the T-property.

He we shall solve this problem for the particular case of $n = 2$. We shall always use the term the Čebyšev property, not the T-property.

If $[a_1, a_2]$, $[b_1, b_2]$ are two elements of $R \times R$ (where R denotes the set of all real numbers), we write $[a_1, a_2] \rhd [b_1, b_2]$ if and only if

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} > 0 .$$

The relation $\rhd$ is evidently irreflexive and antisymmetric; it is not transitive on $R \times R$.

Suppose that A is a subset of $R \times R$ with the property described in the text of the problem. We put $[a_1, a_2] \succ [b_1, b_2]$ if and only if $[a_1, a_2]$ and $[b_1, b_2]$ are elements of A and the element $[a_1, a_2]$ has a greater index than $[b_1, b_2]$ in the described indexing. The relation $\succ$ is a linear ordering and must coincide with the restriction of $\rhd$ onto A. Therefore the restriction of $\rhd$ onto A must be transitive. On the other hand, if the restriction of $\rhd$ onto A is transitive, it is a linear ordering and A can be indexed according to that ordering and this indexing has the required property. Thus we need to find all subsets A of $R \times R$ with the property that the restriction of $\rhd$ onto A is transitive.

The set R of all real numbers can be partitioned into three sets P, N, $\{0\}$, where P is the set of all positive real numbers and N is the set of all negative real numbers. On the set $R \times R$ we have a partition

$$\mathcal{S} = \{P \times P, P \times N, P \times \{0\}, N \times P, N \times N, N \times \{0\}, \{0\} \times P, \{0\} \times N, \{0\} \times \{0\}\}$$

Table 1

	$P \times P$	$P \times N$	$P \times \{0\}$	$N \times P$	$N \times N$	$N \times \{0\}$	$\{0\} \times P$	$\{0\} \times N$	$\{0\} \times \{0\}$
$P \times P$	$a_1\ a_2 > b_1\ b_2$	never	never	always	$a_1/a_2 < b_1/b_2$	always	always	never	never
$P \times N$	always	$a_1/a_2 > b_1/b_2$	always	$a_1/a_2 < b_1/b_2$	always	never	always	never	never
$P \times \{0\}$	always	never	never	always	never	never	always	never	never
$N \times P$	never	$a_1/a_2 < b_1/b_2$	never	$a_1/a_2 > b_1/b_2$	always	always	never	always	never
$N \times N$	$a_1/a_2 < b_1/b_2$	never	always	never	$a_1/a_2 > b_1/b_2$	never	never	always	never
$N \times \{0\}$	never	always	never	never	always	never	never	always	never
$\{0\} \times P$	never	never	never	always	always	always	never	never	never
$\{0\} \times N$	always	always	always	never	never	never	never	never	never
$\{0\} \times \{0\}$	never	never	never	never	never	never	never	never	never

The sets from $\mathscr{S}$ correspond to the rows and the columns of Table 1. If $S_1 \in \mathscr{S}$, $S_2 \in \mathscr{S}$, then at the intersection of the row corresponding to S_1 and the column corresponding to S_2 it is written, when for an element $[a_1, a_2] \in S_1$ and for an element $[b_1, b_2] \in S_2$ we have $[a_1, a_2] \vartriangleright [b_1, b_2]$. The reader may verify the correctness of these data himself.

Evidently A cannot contain any pair of linearly dependent elements; in this case the determinant of this pair would be equal to zero. In Table 1 and in the following text we shall tacitly suppose that A does not contain such pairs.

Now if A contains some elements from $P \times P$ and some elements of $N \times N$ such that either $a_1/a_2 < b_1/b_2$ for each $[a_1, a_2] \in A \cap N \times N$ and each $[b_1, b_2] \in A \cap P \times P$, or $a_1/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A \cap N \times N$ and each $[b_1, b_2] \in A \cap P \times P$, then the restriction of $\vartriangleright$ onto $A \cap (P \times P \cup N \times N)$ is transitive. If A contains elements $[a_1, a_2] \in P \times P$, $[b_1, b_2] \in N \times N$, $[c_1, c_2] \in P \times P$ such that $a_1/a_2 < b_1/b_2 < c_1/c_2$, then $[a_1, a_2] \vartriangleright [b_1, b_2]$, $[b_1, b_2] \vartriangleright [c_1, c_2]$, $[c_1, c_2] \vartriangleright [a_1, a_2]$ and the restriction of $\vartriangleright$ onto A is not transitive. Analogously in the case when A contains $[a_1, a_2] \in N \times N$, $[b_1, b_2] \in P \times P$, $[c_1, c_2] \in N \times N$ and $a_1/a_2 < b_1/b_2 < c_1/c_2$.

Similarly if A contains some elements from $P \times N$ and some elements from $N \times P$, then the restriction of $>$ onto $A \cap (P \times N \cup N \times P)$ is transitive if and only if either $a_1/a_2 < b_1/b_2$ for each $[a_1, a_2] \in A \cap P \times N$ and each $[b_1, b_2] \in A \cap N \times P$, or $a_1/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A \cap P \times N$ and each $[b_1, b_2] \in A \cap N \times P$.

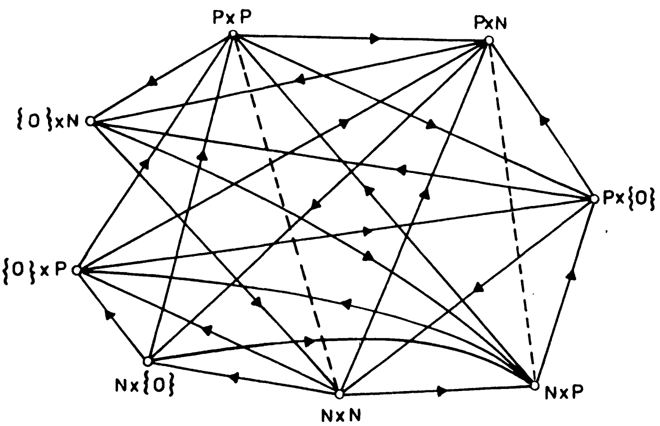

Fig. 1

Now for two elements S_1, S_2 from $\mathscr{S}$ we write $S_1 \to S_2$ if and only if $[a_1, a_2] \vartriangleright [b_1, b_2]$ for each $[a_1, a_2] \in S_2$ and each $[b_1, b_2] \in S_1$. We construct a mixed graph G whose vertex set is $\mathscr{S} - \{\{0\} \times \{0\}\}$ and in which there is a directed edge from S_1 into S_2 if and only if $S_1 \to S_2$ and there are undirected edges joining $P \times P$ with $N \times N$ and $P \times N$ with $N \times P$. The graph G is in Fig. 1; the undirected edges are drawn by dashed lines.

We omit the set $\{0\} \times \{0\}$, because every determinant containing its element as a row is equal to zero.

The pairs $P \times \{0\}$, $N \times \{0\}$ and $\{0\} \times P$, $\{0\} \times N$ are not joined by an edge, because the determinants from elements of sets of any of these pairs are equal to zero.

Now let $A \subset R \times R$. By $G(A)$ denote the subgraph of G induced by the set of all vertices corresponding to the sets with which A has non-empty intersections.

Let $\mathcal{T} = \{\{P \times P, P \times N, N \times \{0\}\}, \{P \times P, \{0\} \times N, N \times \{0\}\}, \{P \times P, \{0\} \times N, N \times P\}, \{P \times N, N \times \{0\}, \{0\} \times P\}, \{P \times N, \{0\} \times N, N \times N\}, \{P \times \{0\}, N \times N, \{0\} \times P\}\}$. This is the set of all triples of vertices of G which induce directed circuits. Evidently A must have the property that in each triple from T there exists at least one set disjoint with A; otherwise the restriction of $\rhd$ onto A would not be transitive.

Now consider the undirected edge of $\check{G}$ joining $P \times P$ with $N \times N$. There are three directed paths of the length 2 from $N \times N$ to $P \times P$; they go through the vertices $N \times P$, $N \times \{0\}$, $\{0\} \times P$. Further there are two directed paths of the length 2 from $P \times P$ to $N \times N$; they go through the vertices $P \times \{0\}$, $\{0\} \times N$. Therefore if A has non-empty intersections with $P \times P$, $N \times N$ and at least one of the sets $N \times P$, $N \times \{0\}$, $\{0\} \times P$, then A must be disjoint with $P \times \{0\}$ and $\{0\} \times N$ and $a_1/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A \cap P \times P$ and each $[b_1, b_2] \in A \cap N \times N$. If A has non-empty intersections with $P \times P$, $N \times N$ and at least one of the sets $P \times \{0\}$, $\{0\} \times N$, then A must be disjoint with $N \times P$, $N \times \{0\}$ and $\{0\} \times P$ and $a_1/a_2 < b_1/b_2$ for each $[a_1, a_2] \in A \cap P \times P$ and each $[b_1, b_2] \in A \cap N \times N$.

Similarly, if A has non-empty intersections with $P \times N$, $N \times P$ and at least one of the sets $P \times P$, $P \times \{0\}$, $\{0\} \times P$, then it must be disjoint with $N \times \{0\}$ and $\{0\} \times N$ and $a_1/a_2 < b_1/b_2$ for each $[a_1, a_2] \in A \cap P \times N$ and each $[b_1, b_2] \in A \cap N \times P$; if A has non-empty intersections with $P \times N$, $N \times P$ and at least one of the sets $N \times \{0\}$, $\{0\} \times N$, then it must be disjoint with $P \times P$, $P \times \{0\}$ and $\{0\} \times P$ and $a/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A \cap P \times N$ and each $[b_1, b_2] \in A \cap N \times P$.

We have listed some necessary conditions for A to have the Čebyšev property. Now suppose that A fulfills these conditions. Then the graph $G(A)$ contains no directed circuit. If $G(A)$ contains $P \times P$ and $N \times N$ and we have $a_1/a_2 > b_1/b_2$ (*or* $a_1/a_2 < b_1/b_2$) for each $[a_1, a_2] \in A \cap P \times P$ and each $[b_1, b_2] \in A \cap N \times N$, we direct the edge joining $P \times P$ with $N \times N$ towards $P \times P$ (or $N \times N$ respectively). If $G(A)$ contains $P \times N$ and $N \times P$ and we have $a_1/a_2 < b_1/b_2$ (or $a_1/a_2 > b_1/b_2$) for each $[a_1, a_2] \in A \cap P \times N$ and each $[b_1, b_2] \in A \cap N \times P$, we direct the edge joining $P \times N$ with $N \times P$ towards $P \times N$ (or $N \times P$ respectively). Evidently we obtain an acyclic digraph. As $\rhd$ is transitive on each set from S, it is evidently transitive an A. Therefore our conditions are also sufficient.

Thus we have proved a theorem.

Theorem. *Let $A \subset R \times R$ be a set of finite cardinality $|A| = m \geqq 3$. The set A has the Čebyšev property if and only if the following conditions are fulfilled:*

(i) *A contains no pair of linearly dependent pairs.*

(ii) *A does not contain* $[0, 0]$.

(iii) *If* $\mathcal{T} = \{\{P\times P,\ P\times N,\ N\times\{0\}\},\ \{P\times P,\ \{0\}\times N,\ N\times\{0\}\},\ \{P\times P,\ \{0\}\times N,\ N\times P\},\ \{P\times N,\ N\times\{0\},\ \{0\}\times P\},\ \{P\times N,\ \{0\}\times N,\ N\times P\},\ \{P\times N,\ N\times\{0\},\ \{0\}\times P\},\ \{P\times N,\ \{0\}\times N,\ N\times N\},\ \{P\times\{0\},\ N\times N,\ \{0\}\times P\}\}$, *then in any element of* $\mathcal{T}$ there is at least one set disjoint with A.

(iv) *If $A\cap P\times P \neq \emptyset$, $A\cap N\times N \neq \emptyset$, then either $a_1/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times P$ and each $[b_1, b_2] \in A\cap N\times N$, or $a_1/a_2 < b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times P$ and each $[b_1, b_2] \in A\cap N\times N$.*

(v) *If $A\cap P\times N \neq \emptyset$, $A\cap N\times P \neq \emptyset$, then either $a_1/a_2 < b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times N$ and each $[b_1, b_2] \in A\cap N\times P$, or $a_1/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times N$ and each $[b_1, b_2] \in A\cap N\times P$.*

(vi) *If A has non-empty intersections with $P\times P$, $N\times N$ and at least one of the sets $N\times P$, $N\times\{0\}$, $\{0\}\times P$, then A is disjoint with $P\times\{0\}$ and $\{0\}\times N$ and $a_1/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times P$ and each $[b_1, b_2] \in A\cap N\times N$.*

(vii) *If A has non-empty intersections with $P\times P$, $N\times N$ and at least one of the sets $P\times\{0\}$, $\{0\}\times N$, then A is disjoint with $N\times P$, $N\times\{0\}$, $\{0\}\times P$ and $a_1/a_2 < b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times P$ and each $[b_1, b_2] \in A\cap N\times N$.*

(viii) *If A his non-empty intersections with $P\times N$, $N\times P$ and at least one of the sets $P\times P$, $P\times\{0\}$, $\{0\}\times P$, then A is disjoint with $N\times\{0\}$ and $\{0\}\times N$ and $a_1/a_2 < b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times N$ and each $[b_1, b_2] \in A\cap N\times P$.*

(ix) *If A has non-empty intersections with $P\times N$, $N\times P$ and at least one of the sets $N\times\{0\}$, $\{0\}\times N$, then A is disjoint with $P\times P$, $P\times\{0\}$ and $\{0\}\times P$ and $a_1/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times N$ and each $[b_1, b_2] \in A\cap N\times P$.*

This theorem is very complicated. But most of the troubles are caused by the pairs of numbers which contain zero. If we exclude them, we obtain a corollary.

Corollary. *Let $A \subset (R - \{0\}) \times (R - \{0\})$ be a set of finite cardinality $|A| = m \geqq 3$. The set A has the Čebyšev property if and only if the following conditions are fulfilled:*

(α) *A contains no pair of linearly dependent pairs.*

(β) *If $A\cap P\times P \neq \emptyset$, $A\cap N\times N \neq \emptyset$, then either $a_1/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times P$ and each $[b_1, b_2] \in A\cap N\times N$, or $a_1/a_2 < b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times P$ and each $[b_1, b_2] \in A\cap N\times N$.*

(γ) *If $A\cap P\times N \neq \emptyset$, then either $a_1/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times N$ and each $[b_1, b_2] \in A\cap N\times P$, or $a_1/a_2 < b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times N$ and each $[b_1, b_2] \in A\cap P\times N$.*

(δ) *If $A\cap P\times P \neq \emptyset$, $A\cap N\times N \neq \emptyset$, $A\cap N\times P \neq \emptyset$, then $a_1/a_2 > b_1/b_2$ for each $[a_1, a_2] \in A\cap P\times P$ and each $[b_1, b_2] \in A\cap N\times N$.*

(ε) *If* $A \cap P \times P \neq \emptyset$, $A \cap P \times N \neq \emptyset$, $A \cap N \times P \neq \emptyset$, *then* $a_1/a_2 < b_1/b_2$ *for each* $[a_1, a_2] \in A \cap P \times N$ *and each* $[b_1, b_2] \in A \cap N \times P$.

The subgraph of G induced by the vertex set $\{P \times P, P \times N, N \times P, N \times N\}$ is in Fig. 2.

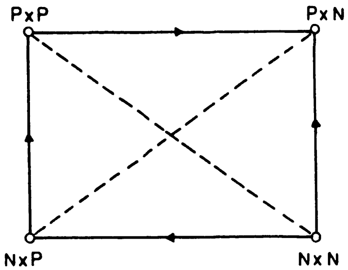

Fig. 2

REFERENCE

[1] Proceedings of the Fifth Hungarian Colloquium on Combinatorics held in Keszthely, June 28—July 3, 1976 (to appear).

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Katedra matematiky
Vysoké školy strojní a textilní
Komenského 2
461 17 Liberec

ЗАМЕТКА О СВОЙСТВЕ ЧЕБЫШЕВА

Богдан Зелинка

Резюме

Пусть $A \subset R^n$ есть множество конечной мощности $|A| = m \geq n + 1$. Мы говорим, что A обладает свойством Чебышева, если A может быть написано как $A = \{a_i\}_{i=1}^m$ так, что условие

$$\operatorname{sgn} \det [a_{i_1}, a_{i_2}, \ldots, a_{i_n}] = \text{const} \neq 0$$

выполнено для всех

$$\{i_k\}_{k=1}^n, \qquad 1 \leq i_1 < i_2 < \ldots < i_n \leq m .$$

Приведены необходимые и достаточные условия для того, чтобы множество обладало свойством Чебышева в случае $n = 2$. Это является частичным решением проблемы Б. Урина.