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A GENERALIZATION OF THE FRIENDSHIP THEOREM

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Introduction. Given the integers $m \geqslant 1$ and $k \geqslant 0$, a graph with at least m points is said to be an (m, k)-graph if any m-tuple of its points has exactly k common adjacent points.

In [3] G. Higman and in [5] H. S. Wilf described (2,1)-graphs by the well-known friendship theorem. In [1] R. C. Bose and S. S. Shrikhande and in [4] J. Plesník proved that any $(2, k)$-graph is regular for $k > 1$. Further, J. Plesník in [4] proved that any (m, k)-graph is the complete graph with $m+k$ points for $m \geqslant k+2 \geqslant 3$.

In the present we shall show that any (m, k)-graph is the complete graph with $m+k$ points for $m \geqslant 3$ and $k \geqslant 1$.

In the paper we shall use all notations and definitions in the sense of [2].

If G is a graph, then we denote by $V(G)$ and $E(G)$ the set of its points and lines, respectively. Given $u \in V(G)$, $d_G(u)$ denotes the degree of the point u. Let $N_G(u) = \{v \in V(G) | uv \in E(G)\}$. It is easily seen that $|N_G(u)| = d_G(u)$. When G is a regular graph, then $d(G)$ denotes the degree of G.

Given $U \subset V(G)$, $G(U)$ denotes the induced subgraph of G with the point set U.

Results. Let m and k be integers with $m \geqslant 1$ and $k \geqslant 0$. A graph G is called an (m, k)-graph if and only if $|V(G)| \geqslant m$ and $\left|\bigcap_{i=1}^{m} N_G(v_i)\right| = k$ for any m-tuple of its distinct points $v_1, v_2, \ldots, v_m$.

Theorem 1. *Let $k > 1$. Then G is a $(3, k)$-graph if and only if $G = K_{k+3}$.*

Proof. Suppose that G is a $(3, k)$-graph and $G \neq K_{k+3}$. Therefore there are two distinct points $u, v \in V(G)$ with $uv \notin E(G)$. We put $d_G(v) = p$. The graph $G_1 = G(N_G(v))$ is a regular $(2, k)$-graph with $|V(G_1)| = d_G(v) = p$ (see Lemma 3.2 and Theorem 4.5 of [4]). Let $d(G_1) = r$. According to Theorem 4.5 of [4], we have

$$p = 1 + \frac{r(r-1)}{k}, \tag{1}$$

where

$$(2) \qquad k < r \leq k(k+1).$$

The graph $G_2 = G(N_G(u) \cap N_G(v))$ is a regular $(1, k)$-graph of the degree $d(G_2) = k$. Let $|V(G_2)| = q$. Obviously $|E(G_2)| = \frac{qk}{2}$. Let $E - \{xy \in E(G_1) \mid x \in V(G_2)$ and $y \in V(G_1) - V(G_2)\}$. Since G_1 is a regular graph of the degree r and G_2 is a regular graph of the degree k with $|V(G_2)| = q$, we receive $|E| = q(r-k)$. Denote by G_3 the graph with $V(G_3) = V(G_1) - V(G_2)$ and $E(G_3) = E(G_1) - E(G_2) - E$. If w is any point of $V(G_3)$, then w is adjacent exactly to k points of the $V(G_2)$ in G (because u, v and w have in G exactly k common adjacent points) as well as in G_1. Hence G_3 is a regular graph of the degree $d(G_3) = r - k$ and $|E(G_3)| = \frac{(p-q)(r-k)}{2}$.

Obviously $E(G_1) = E \cup E(G_2) \cup E(G_3)$ and $E \cap E(G_2) = E \cap E(G_3) = E(G_2) \cap E(G_3) = \emptyset$. Therefore

$$(3) \qquad |E(G_1)| = q(r-k) + \frac{qk}{2} + \frac{(p-q)(r-k)}{2}.$$

On the other hand, since G_1 is a regular graph of the degree $d(G\ ) = r$ with $|V(G_1)| = p$, we obtain

$$(4) \qquad |E(G_1)| = \frac{pr}{2}.$$

The equalities (3) and (4) imply

$$qr = pk.$$

Using (1) in the preceding equality we obtain

$$qr = r(r-1) + k.$$

Thus $\frac{k}{r}$ is an integer, which contradicts (2). Hence $uv \in E(G)$ for any two distinct points u, $v \in V(G)$.

As the proof of the second part of the assertion is trivial, the theorem is proved.*

Theorem 5.3 of [4] states: If there exists $m_0 \geq 2$ such that any (m_0, k)-graph is the complete graph K_{m_0+k}, then for every $m \geq m_0$, K_{m+k} is the only (m, k)-graph. Thus Theorem 1 implies:

Theorem 2. *Let $m \geq 3$ and $k \geq 1$. Then G is an (m, k)-graph if and only if $G = K_{m+k}$.*

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* Added in proof: Carstens and Kruse in J. of Comb. Th., 3, 1977, give the same.

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ОБОБЩЕНИЕ ПРИЯТЕЛЬСКОЙ ТЕОРЕМЫ

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Резюме

Пусть $m \geqslant 0$ и $k \geqslant 0$ – целъе числа. Граф, содержащий не менее m вершин (без петель и кратных ребер), мы назовем (m, k)-графом, если произвольная m-тица его вершин соединена точно с k общими вершинами. Простейшим примером (m, k)-графа является полный граф s $m+k$ вершинами.

Существование неполных (2,1)-графов (известных как приятельские графы) было показано Хигманом [3] и Вильфом [5]. Босе и Шриханд [1] и Плесник [4] доказали, что все $(2, k)$-графы для $k>1$ регулярны. Кроме этого Плесник [4] доказал несуществование неполного (m, k)-графа для $m \geqslant k+2 \geqslant 3$.

В нашей статье показано, что произвольный (m, k)-граф для $m>2$ и $k \geqslant 1$ обязательно является полным.