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DECOMPOSITION OF COMPLETE BIPARTITE GRAPHS INTO FACTORS WITH GIVEN RADII

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Introduction

The author of paper [5] studied the problem of the existence of decompositions of the complete bipartite graphs $K_{m,n}$ into factors with given diameters. In the present paper we study a similar problem for the radii. In paper [4] the decomposition of the complete graphs into factors with given radii is studied. Some of the results are concerned with q-partite graphs. The main aim of this paper is to determine the necessary and sufficient conditions for the existence of a decomposition of $K_{m,n}$ into two factors with given radii.

All graphs in the present paper are undirected, without loops and multiple edges. Let an integer $q \geqslant 2$ be given. A graph G with the vertex set V is called q-partite if V can be partitioned into q mutually disjoint, nonempty subsets $V_1, V_2, \ldots, V_q$, which are called parts of G such that every edge of G joins the vertices of two different parts of G. If G contains every edge joining the vertices of two different parts of G, then G is said to be a complete q-partite graph and we write $G = K_{m_1, m_2, \ldots, m_q}$, where $m_1, m_2, \ldots, m_q$ are the cardinalities of the parts $V_1, V_2, \ldots, V_q$, respectively (2-partite graphs are called bipartite).

By a factor of a graph G we mean a subgraph of G containing all the vertices of G. By a decomposition of a graph G into factors we mean a system $\mathcal{S}$ of factors of G such that every edge of G is contained in exactly one factor of $\mathcal{S}$. The eccentricity $e(v)$ of a vertex v is sup $\varrho_G(u, v)$, for all $u \in V_G$, where $\varrho_G(u, v)$ denotes the distance between two vertices $u, v \in V_G$ in G. The radius $r(G)$ of a graph G is defined as $r(G) = \min e(v)$ and the diameter $d(G)$ of a graph G as $d(G) = \max e(v)$. A vertex v is a centre of G if $e(v) = r(G)$. The radius $r(G)$ is ∞ if G is a disconnected graph or if G is a connected but $e(v)$ is infinite for all v. The remaining terms are used in the usual sense [1, 2, 3, 4, 5].

Let natural numbers p, $q \geqslant 2$, r_i (or symbol ∞) for $1 \leqslant i \leqslant p$ and non-zero cardinal numbers m_j for $1 \leqslant j \leqslant q$ be given. Our aim is to determine the conditions for the existence of a decomposition of the graph $K_{m_1, m_2, \ldots, m_q}$ into p factors with given radii $r_1, r_2, \ldots, r_p$.

1. General case

Let $q \geq 2$ and p be natural numbers, m_i $(i=1, 2, \ldots, q-1)$ — cardinal numbers ≥ 1, r_j $(j=1, 2, \ldots, p)$ — natural numbers or symbol ∞. Denote by $C_{m_1, m_2, \ldots, m_{q-1}}$ $(r_1, r_2, \ldots, r_p)$ the smallest cardinal number m_q such that the graph $K_{m_1, m_2, \ldots, m_q}$ can be decomposed into p factors with the radii $r_1, r_2, \ldots, r_p$. If such a number does not exist, we shall write $C_{m_1, m_2, \ldots, m_{q-1}}(r_1, r_2, \ldots, r_p) = \infty$.

The importance of the function $C_{m_1, m_2, \ldots, m_{q-1}}$ can be seen from the next theorem.

Theorem 1. *If the graph $K_{m_1, m_2, \ldots, m_q}$ is decomposable into p factors with the radii $r_1, r_2, \ldots, r_p$ (where $r_i \geq 2$ for $i=1, 2, \ldots, p$), then the graph $K_{M_1, M_2, \ldots, M_q}$ (where $M_1 \geq m_1$, $M_2 \geq m_2$, $\ldots$, $M_q \geq m_q$) is also decomposable into p factors with the radii $r_1, r_2, \ldots, r_p$.*

The proof of this theorem is analogous to that of Theorem 1 in paper [5].

From this theorem it follows:

Corollary. *The graph $K_{m_1, m_2, \ldots, m_q}$ can be decomposed into p factors with the radii $r_1, r_2, \ldots, r_p$ (where $r_i \geq 2$, $i=1, 2, \ldots, p$) if and only if*

$$m_q \geq C_{m_1, m_2, \ldots, m_{q-1}}(r_1, r_2, \ldots, r_p).$$

2. Decompositions of $K_{m,n}$ into p factors

In the graph $K_{m,n}$ (where m, n are natural numbers such that $2 \leq m \leq n$) there evidently exists a factor with an arbitrary radius r for $2 \leq r \leq m$, and a factor with another finite radius in $K_{m,n}$ does not exist. If $m=1$, then in the graph $K_{m,n}$ there exists a factor with the radii 1 or ∞ only.

Lemma 1. *Let natural numbers p, m, n be given. If the graph $K_{m,n}$ is decomposable into p factors with finite radii, then*

$$p \leq \left[\frac{mn}{m+n-1}\right].$$

Proof. The graph $K_{m,n}$ has mn edges. It is clear that the number of edges of a factor with a finite radius is at least $m+n-1$. Therefore

$$p(m+n-1) \leq mn$$

and the required inequality easily follows.

Theorem 2. *Let $m \geq 2$, $p \geq 3$ and $r_2 = r_3 = \ldots = r_p = \infty$. Then*

$$C_m(r_1, r_2, \ldots, r_p) = \begin{cases} r_1, & \text{if } 2 \leq r_1 \leq m; \\ \infty, & \text{if } m < r_1 < \infty; \\ 1, & \text{if } r_1 = \infty, m \geq 2; \\ 2, & \text{if } r_1 = \infty, m = 1; \\ 1, & \text{if } r_1 = 1. \end{cases}$$

Proof. The last four relations are evident. To prove the first relation it is sufficient to construct a decomposition of the graph K_{m,r_1} (which is easy to do).

3. Decomposition of $K_{m,n}$ into two factors

In the following we shall consider the case $p=2$. The case $p=1$ is trivial, as we obviously have

$$C_m(r) = \begin{cases} 1, & \text{if } r=1; \\ 2, & \text{if } r=2, \ m \geq 2; \\ \infty, & \text{otherwise.} \end{cases}$$

A vertex v of a bipartite graph is said to be saturated if by adding an edge incident with v there always arises a graph that is not bipartite.

Lemma 2. *If a bipartite graph G has radius $r=2$, then G contains a saturated vertex.*

Proof. It is clear that the centre of the graph G with the radius $r=2$ is a saturated vertex.

Lemma 3. *If the graph $K_{m,n}$ is decomposable into two factors F_1, F_2 with radii $r(F_1)=r$, $r(F_2)=s$, then $4 \leq r < \infty$ implies $s \leq 5$.*

Proof. *Let $s \geq 6$.* Let us consider two cases:

I. Let $s<\infty$. Then there exists in F_2 a vertex x with a finite eccentricity $s \geq 6$. Let A_i $(i=0, 1, 2, \ldots)$ be the set of all vertices from F_2, with the distance i from x. It is clear that $A_i \neq \emptyset$ for $0 \leq i \leq 6$. Choose $y \in A_3$. It is easy to show that the eccentricity of y in F_1 equals 3, which is a contradiction to the condition $r \geq 4$.

II. Let $s=\infty$. If F_2 is connected, then choose an arbitrary vertex x and we use the method from the case I. Let F_2 be disconnected. Then F_1 is connected (otherwise $r=\infty$) and it does not contain a saturated vertex (otherwise $r=2$). Denote by P one of the components of F_2 and by Q the union of the others. In the factor F_1 there exist between P and Q all the edges between the vertices of different parts of $K_{m,n}$. P and Q contain vertices from both parts, because F_1 does not contain a saturated vertex. Between vertices of two different parts in P (or Q, otherwise F_1 is disconnected) there must exist at least one edge and an arbitrary vertex of this edge is a centre of F_1. It is easy to show that $r \leq 3$, which is a contradiction to the assumption $r \geq 4$.

Lemma 4. *If the graph $K_{m,n}$ is decomposable into two factors with the radii r and s, where $r=5$, then $s=3$.*

Proof. Let F_1 be a factor of $K_{m,n}$ with the radius $r=5$. Then there exists in F_1 a vertex x (the centre of F_1) with a finite eccentricity 5. The vertex set of F_1 can be decomposed into the subsets $A_i=\{w: \varrho(x, w)=i\}$ for $i=1, 2, ..., 5$. Evidently $A_i \neq \emptyset$ for $1 \leqslant i \leqslant 5$, and every vertex from the vertex set of $K_{m,n}$ different from x belongs to exactly one of the sets A_i. In the factor F_1 there are all edges joining the vertex x with vertices from A_1 and some edges joining vertices of consecutive subsets A_i, A_{i+1} ($i=1, 2, 3, 4$). If F_1 has the radius $r=5$, then F_2 must contain all the edges which join:

1. x with the vertices from the sets A_3 and A_5;
2. the vertices from A_2 with the vertices from A_5;
3. the vertices from A_1 with the vertices from A_4.

If F_2 does not contain any other edges, then $s=\infty$ and $r \leqslant 3$ (Lemma 3). If F_2 contains only edges joining the vertices of A_2 with the vertices of A_3, then $s=\infty$, which contradicts Lemma 3. It follows that F_2 contains edges joining the vertices of A_1 with the vertices of A_2, or the vertices of A_3 with the vertices of A_4, or the vertices of A_4 with the vertices of A_5. But in all the three cases $s=3$. Q.E.D.

Theorem 3. *Let $m>0$ be a cardinal number, r, s natural numbers or symbols ∞ ($r \leqslant s$). Denote by $C_m(r, s)$ the smallest cardinal number n such that the graph $K_{m,n}$ can be decomposed into two factors with radii r and s. If such a number does not exist, we shall write $C_m(d, e)=\infty$. Then*

$$C_m(r, s)=\begin{cases} r & \text{if } r\leqslant 3,\ s=\infty,\ m\geqslant r;\\ 2 & \text{if } r=s=\infty,\ m=1;\\ 1 & \text{if } r=s=\infty,\ m\geqslant 2;\\ 4 & \text{if } r=s=3,\ m=3, \text{ or } r=s=4,\ m\geqslant 4;\\ 3 & \text{if } r=s=3,\ m\geqslant 4;\\ s & \text{if } r=3,\ 4\leqslant s<\infty,\ m\geqslant s;\\ \infty & \text{otherwise.} \end{cases}$$

Proof. The first three relations are obvious. To prove the fourth relation it is sufficient to construct a decomposition of the graph $K_{3,4}$ into two factors with the radii $r=s=3$ (this is impossible for $K_{3,3}$) — see Fig. 1 — and the graph $K_{m,4}$ ($m \geqslant 4$) into two factors with the radii $r=s=4$. According to Theorem 1 it is sufficient to construct the corresponding decomposition of $K_{4,4}$ into two factors with the radii $r=s=4$ (see Fig. 2).

The graph $K_{4,3}$ is decomposable into two factors with the radii 3 (Fig. 1). From this and from Theorem 1 the fifth relation follows. To prove the sixth statement it is sufficient to decompose the graph $K_{s,s}$ into two factors with the radii $r=3$ and s and to use Theorem 1. From Lemmas 3 and 4 the seventh statement follows.

Corollary. *The bipartite graph $K_{m,n}$ is decomposable into two factors with radii r and s* $(2 \leq r \leq s \leq \infty)$ *if and only if* $n \geq C_m(r, s)$, *where* $C_m(r, s)$ *is given in Theorem 3.*

Proof follows from Theorems 1 and 3.

From Theorems 1 and 3 the next Theorem 4 follows. In this Theorem all the couples of cardinal numbers m, n $(m \leq n)$ are given for which the graph $K_{m,n}$ is decomposable into two factors with given radii.

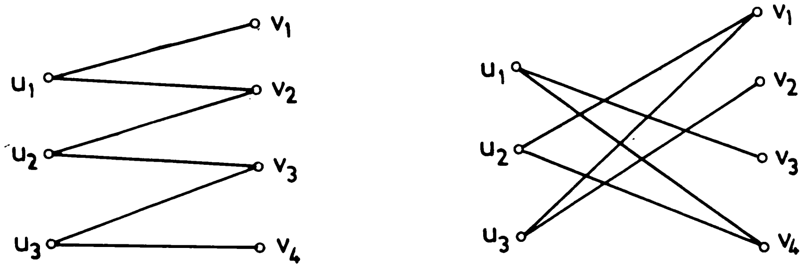

Fig. 1

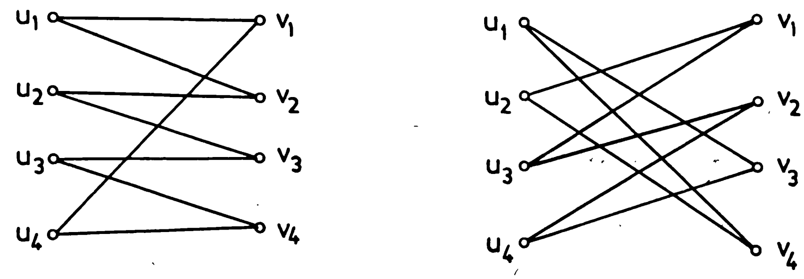

Fig. 2

Theorem 4. *Let r, s be positive integers or symbols* ∞ *and m, n be cardinal numbers such that* $r \leq s$ *and* $m \leq n$ *holds. The bipartite graph* $K_{m,n}$ *is decomposable into two factors with the radii r and s if and only if one of the following cases occurs:*

(1) $r = s = \infty$, $m \geq 1$, $n \geq 2$.

(2) $r = 1$, $s = \infty$, $m = 1$.

(3) $r = 2$, $s = \infty$, $m \geq 2$.

(4) $r = 3$, $s = \infty$, $m \geq 3$.

(5) $r = s = 3$, $m \geq 3$, $n \geq 4$.

(6) $r = 3$, $4 \leq s < \infty$, $m \geq s$.

(7) $r = 4$, $m \geq 4$, $s = 4$.

The next Corollary shows for which radii it is possible to decompose a bipartite graph.

Corollary. *Let natural numbers r and s* ($r \leq s$) *be given. A complete bipartite graph decomposable into two factors with the radii r and s exists if and only if one of the following cases occurs:*

(1) $r=3$.

(2) $r=s=4$.

Proof. If $r<3$, then no bipartite graph can be decomposed into two factors with finite radii r and s. From Lemmas 3 and 4 it follows that no bipartite graphs decomposable into two factors with other radii than those given in (1) and (2) exist. According to Theorem 3 bipartite graphs which are decomposable into two factors with radii given in the Corollary do exist.

Table I

There are shown for given r and s all couples (m, n), where $m \leq n$, such that $C_m(r, s)=n$ and $C_M(r, s)=N$ does not hold for any $M \leq m$, $N \leq n$, $(M, N)=(m, n)$.

r \ s	∞	1	2	3	4	5	6	7	...
∞	(1,2)	(1,1)	(2,2)	(3,3)		In this area no decomposition			
1	(1,1)					exists for any $K_{m,n}$			
2	(2,2)								
3	(3,3)			(3,4)	(4,4)	(5,5)	(6,6)	(7,7)	...
4				(4,4)	(4,4)	In this area no decomposition			
5				(5,5)		exists for any $K_{m,n}$			
⋮				⋮					

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РАЗЛОЖЕНИЯ ПОЛНЫХ ДВУДОЛЬНЫХ ГРАФОВ НА ФАКТОРЫ С ДАННЫМИ РАДИУСАМИ

Элишка Томова

Резюме

Рассматривается проблема разложения полных двудольных графов на факторы с данными радиусами. Здесь находятся все пары чисел (m, n), для которых возможно разложить полный двудольный граф на два фактора с данными радиусами.