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DECOMPOSITION OF COMPLETE BIPARTITE GRAPHS INTO FACTORS WITH GIVEN DIAMETERS

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Introduction

The authors of paper [1] study the problem of the existence of decompositions of the complete graph K_n into factors with given diameters. In the present paper we study similar problem for the complete q-partite graphs. Most of the results are concerned with the case $q=2$ of bipartite graphs.

All graphs in the present paper are undirected, without loops and multiple edges. Let an integer $q \geqslant 2$ be given. A graph G with the vertex set V is called q-partite if V can be partitioned into q mutually disjoint, nonempty subsets $V_1, V_2, \ldots, V_q$, which are called parts of G such that every edge of G joins vertices of two different parts of G. If G contains every edge joining vertices of two different parts of G, then G is said to be a complete q-partite graph and we write $G = K_{m_1, m_2, \ldots, m_q}$, where the cardinality $|v_i| = m_i$ for $i = 1, 2, \ldots, q$ (2-partite graphs are called bipartite).

By a factor of a graph G we mean a subgraph of G containing all the vertices of G. By a decomposition of a graph G into factors we mean a system $\mathscr{S}$ of factors of G such that every edge of G is contained in exactly one factor of $\mathscr{S}$. The diameter $d(G)$ of a graph G is the supremum of the set of all distances $\varrho_G(x, y)$ between two vertices x and y of G. The diameter of G can be also equal to ∞, if G is disconnected or if there does not exist the maximum of the distances, which may occur in infinite graphs. The other terms are used in the usual sense [2, 3].

Let natural numbers $q \geqslant 2$, p and non-zero cardinal numbers $m_1, m_2, \ldots, m_q$ be given. Our aim is to determine the conditions for the existence of a decomposition of the graph $K_{m_1, m_2, \ldots, m_q}$ into p factors with the given diameters $d_1, d_2, \ldots, d_p$, where each d_i $(i = 1, 2, \ldots, p)$ is a natural number or the symbol ∞.

1. The general case

Let $q \geqslant 2$ and p be natural numbers, m_i $(i = 1, 2, \ldots, q-1)$ — cardinal numbers $\geqslant 1$, d_j $(j = 1, 2, \ldots, p)$ — natural numbers or symbols ∞. Denote by $B_{m_1, m_2, \ldots, m_{q-1}}$

$(d_1, d_2, \ldots, d_p)$ the smallest cardinal number m_q such that the graph $K_{m_1, m_2, \ldots, m_q}$ can be decomposed into p factors with the diameters $d_1, d_2, \ldots, d_p$. If such a number does not exists, we shall write $B_{m_1, m_2, \ldots, m_{q-1}}(d_1, d_2, \ldots, d_p) = \infty$.

The importance of the function $B_{m_1 m_2, \ldots, m_{q-1}}$ is evident from the next theorem.

Theorem 1. *If the graph $K_{m_1, m_2, \ldots, m_q}$ is decomposable into p factors with the given diameters $d_1, d_2, \ldots, d_p$ (where $d_i \geq 2$ for $i = 1, 2, \ldots, p$), then the graph $K_{M_1, M_2, \ldots, M_q}$ (where $M_1 \geq m_1, M_2 \geq m_2, \ldots, M_q \geq m_q$) is also decomposable into p factors with the diameters $d_1, d_2, \ldots, d_p$.*

Proof. Let $K_{m_1, m_2, \ldots, m_q}$ be a complete q-partite subgraph of $K_{M_1, M_2, \ldots, M_q}$. Denote by $V_1, V_2, \ldots, V_q$ the parts of $K_{m_1, m_2, \ldots, m_q}$ and by $W_1, W_2, \ldots, W_q$ the parts of $K_{M_1, M_2, \ldots, M_q}$ where $V_1 \subseteq W_1, V_2 \subseteq W_2, \ldots, V_q \subseteq W_q$. Put $A_i = W_i - V_i$ for $i = 1, 2, \ldots, q$. Choose in each V_i an arbitrary vertex, denote it by v_i. Decompose in some way the graph $K_{m_1, m_2, \ldots, m_q}$ into p factors $F_1, F_2, \ldots, F_p$ with the diameters $d_1, d_2, \ldots, d_p$ it can be done by the assumption. Decompose the graph $K_{M_1, M_2, \ldots, M_q}$ into p factors $G_1, G_2, \ldots, G_p$ with the same diameters as follows: The factors $G_k^{(q)} = G_k$ (where $k = 1, 2, \ldots, p$) are constructed from the factors $F_k = G_k^{(0)}$ successively in such a way that in the first step we add to the set $V_1 \cup V_2 \cup \ldots \cup V_q$ all the vertices of A_1 and obtain factors $G_1^{(1)}, G_2^{(1)}, \ldots, G_p^{(1)}$ of $K_{M_1, m_2, \ldots, m_q}$, then in the second step we add all vertices from A_2 and obtain factors $G_1^{(2)}, G_2^{(2)}, \ldots, G_p^{(2)}$ of $K_{M_1, M_2, m_3, \ldots, m_q}$, and so on. Finally we add all the vertices from A_q and construct factors $G_1^{(q)}, G_2^{(q)}, \ldots, G_p^{(q)}$ of $K_{M_1, M_2, \ldots, M_q}$.

Form the factors $G_k^{(j)}$ (where $k = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$) of the graph $K_{M_1, M_2, \ldots, M_j, m_{j+1}, \ldots, m_q}$ in the following way:

(a) $G_k^{(j)}$ contains all the edges contained in $G_k^{(j-1)}$,

(b) if $u \in V_i$, $j < i \leq q$, $v \in A_j$, then the edge uv belongs to $G_k^{(j)}$ if and only if uv_j belongs to $G_k^{(j-1)}$,

(c) if $u \in W_i$, $1 \leq i < j$, $v \in A_j$, then the edge uv belongs to $G_k^{(j)}$ if and only if uv_j belongs to $G_k^{(j-1)}$.

It is easy to see that the factors $G_1^{(j)}, G_2^{(j)}, \ldots, G_p^{(j)}$ form a decomposition of $K_{M_1, M_2, \ldots, M_j, m_{j+1}, \ldots, m_q}$.

The proof of the equality $d(G_k^{(j)}) = d(G_k^{(j-1)})$ is the same as in Theorem 1 of [1]. Hence $d(G_k) = d(F_k)$ for $k = 1, 2, \ldots, p$, and this complete the proof of the theorem.

Corollary. *The graph $K_{m_1, m_2, \ldots, m_q}$ can be decomposed into p factors with the diameters $d_1, d_2, \ldots, d_p$ (where $d_i \geq 2$, $i = 1, 2, \ldots, p$) if and only if*

$$m_q \geq B_{m_1, m_2, \ldots, m_{q-1}}(d_1, d_2, \ldots, d_p).$$

Theorem 2. *Let $d, q, m_1, m_2, \ldots, m_q$ be natural numbers, where $d \geq 2$, $q \geq 2$,*

$m_1 \leqslant m_2 \leqslant \ldots \leqslant m_q$. In the graph $K_{m_1, m_2, \ldots, m_q}$ there exists a factor with the diameter d if and only if

$$d \leqslant \min \{2(m_1 + m_2 + \ldots + m_{q-1}), m_1 + m_2 + \ldots + m_q - 1\}. \tag{1}$$

Proof. If $m_1 = m_2 = \ldots = m_q = 1$, then the statement is obvious. Therefore we assume that $m_q > 1$. Denote by G the graph $K_{m_1, m_2, \ldots, m_q}$ and its parts by V_1, V_2, ..., V_q, where $|V_i| = m_i$ for $i = 1, 2, \ldots, q$. Let us consider two cases:

I. $m_q > m_1 + m_2 + \ldots + m_{q-1}$,

II. $m_q \leqslant m_1 + m_2 + \ldots + m_{q-1}$.

I. In the first case the inequality (1) has the form

$$d \leqslant 2(m_1 + m_2 + \ldots + m_{q-1}). \tag{2}$$

We shall assume that the graph G has a factor F with the diameter d. Every arc (simple path) in F contains at most $m_1 + m_2 + \ldots + m_{q-1}$ vertices not belonging to V_q; this arc has at most $m_1 + m_2 + \ldots + m_{q-1} + 1$ vertices from V_q because in this arc no two vertices from V_q are adjacent. Hence every arc in F has at most $2(m_1 + m_2 + \ldots + m_{q-1}) + 1$ vertices, i.e. $2(m_1 + m_2 + \ldots + m_{q-1})$ edges. This implies (2).

On the other hand, let (2) be true. If $d = 2$, a requested factor is the whole graph G. Let $d \geqslant 3$. In G we construct an arc $v_0 v_1 v_2 \ldots v_d$ of length d such that the vertices v_i with an even index i belong to V_q and those with an odd index i belong to the remaining parts of G. All the other vertices from V_q are joined by an edge with v_1 and each of the remaining vertices is joined by an edge with v_2. These edges with the edges of the arc $v_0 v_1 v_2 \ldots v_d$ from a factor of G with the diameter d.

II. The inequality (1) in the second case has the form

$$d \leqslant m_1 + m_2 + \ldots + m_{q-1} \tag{3}$$

Clearly, condition (3) is necessary for every factor with the diameter d in G. For the converse we assume that (3) holds. We shall construct a factor F of G with diameter d. Assume that $d \geqslant 3$ (if $d = 2$, we put $F = G$). Choose in V_q an arbitrary vertex v. Then for every vertex u of G we have:

$$\deg_G u \geqslant \deg_G v = m_1 + m_2 + \ldots + m_{q-1} \geqslant \frac{m_1 + m_2 + \ldots + m_q}{2} = \frac{p}{2},$$

were p is the number of vertices of G. The graph G has a hamiltonian circuit $v_0 v_1 \ldots v_{p-1} v_0$, which follows from [4] (see also [2], corollary 7.3. (b)). Let F be defined as follows: F contains all the edges of the arc $v_0 v_1 \ldots v_d$, the other vertices are joined by an edge with the vertex v_1 if they do not belong to the same part as v_1, otherwise they are joined with v_2. It is clear that F has the diameter d.

2. Factors of $K_{m,n}$ with given diameters and with a given number of edges

From Theorem 2 we easily obtain:

Corollary. *Let d be a natural number, m, n — cardinal numbers, where $d \geq 2$, $1 \leq m \leq n$. A factor of $K_{m,n}$ with the diameter d exists if and only if $m \geq \frac{d}{2}$, except $m = n = \frac{d}{2}$.*

Proof. If n (and also m) is finite it is sufficient to apply Theorem 2. For n (or m) infinite we use the same method as in the proof of Theorem 2.

Lemma 1. *Let m, n be cardinal numbers, $1 \leq m \leq n$. The maximal possible degree of a vertex in a factor with a finite diameter $d \geq 2$ in a graph $K_{m,n}$ (if such a factor exists) is:*

(I) $n - \frac{d+1}{2} + 2$ *if d is odd, n finite;*

(II) $n - \frac{d}{2} + 2$ *if $d > 2$ is even, $m = n$, n finite;*

(III) $n - \frac{d}{2} + 1$ *if $d > 2$ is even, $m < n$, n finite;*

(IV) n *if $d = 2$ or if n is infinite.*

Proof. Put $e = \left[\frac{d+2}{2}\right]$.

(I) If d is odd, then obviously the maximal possible degree of a vertex is $(n - e - 2) = n - \frac{d+1}{2} + 2$ (see u_{e-1} in Fig. 1). The degree of all other vertices is less than or equal to this number (equal only if $m = n$). If there exists a vertex with the degree greater than this number, it is clear that the factor cannot have the diameter d.

(II) In this case the maximal possible valency is $n - \frac{d}{2} + 2$ (see v_{e-1} in Fig. 2). The proof is the same as in the case I.

(III) The maximal possible degree is $n - \frac{d}{2} + 1$ (see u_m in Fig. 2). The proof is the same as in the case (I).

(IV) The first part is evident. The second part can be proved by a simple extension of construction from Fig. 1 and 2 for the case of an infinite n.

Theorem 3. *Let d be natural number, m, n and E be cardinal numbers, where $1 \leq m \leq n$, $m \geq \frac{d}{2}$, but there does not hold $m = n = \frac{d}{2}$. The factor of $K_{m,n}$ with the*

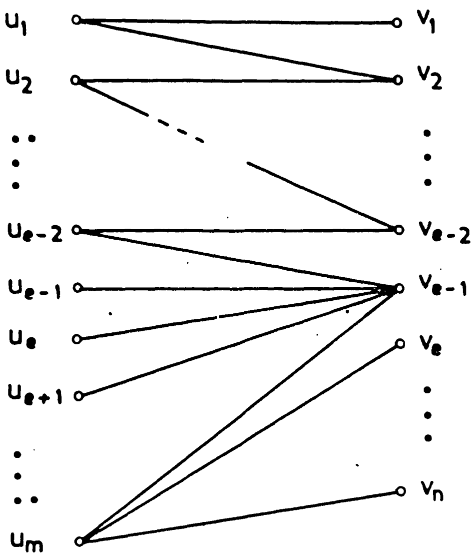

Fig. 1

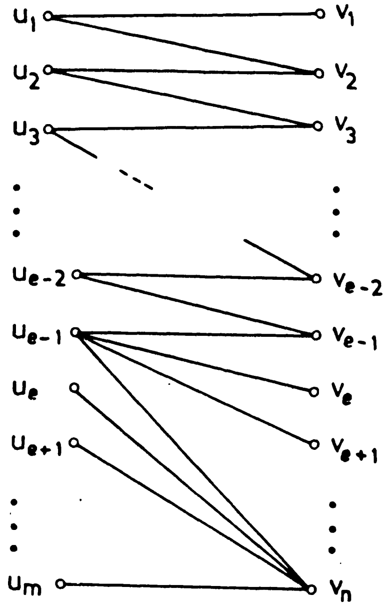

Fig. 2

diameter d and with the number E of edges exists if and only if one of the following cases occurs:

I. $E=m=n=d=1$.

II. $E=mn$, $d=2$, *n is finite.*

III. $m+n-1 \leqslant E \leqslant mn-1-\frac{d-3}{2}\left(m+n-\frac{d+1}{2}\right)$,

$d \geqslant 3$, *d is odd, n is finite.*

IV. $m+n-1 \leqslant E \leqslant m(n-1)-\frac{d-4}{2}\left(m+n-\frac{d+2}{2}\right)$,

$d \geqslant 4$, *d is even, n is finite.*

V. $E=n$, $d \geqslant 2$, *n is finite.*

Proof. We shall denote the parts of the graph $K_{m,n}$ by U and V, where $|U|=m$, $|V|=n$. Let one of the cases I—V occur. From the corollary of Theorem 2 it follows that the graph $K_{m,n}$ has a factor F with the diameter d. Let us prove that it is possible to construct the factor F such that it has E edges. It is evidnet that in the cases I, II and V the factor F cannot have another number of edges. In the cases III and IV we shall prove that it is possible to obtain the upper bound.

If $d \geqslant 3$, d is odd and n is finite, then construct the factor F in the following way: in $K_{m,n}$ we choose an arbitrary arc $v_0v_1 \dots v_d$ of length d such that $v_1, v_3, \dots, v_d \in U$ and $v_0, v_2, \dots, v_{d-1} \in V$. Other vertices from U are joined with v_0, v_2 and with other vertices of V which are joined with v_1 and v_3. It is easy to show that the factor F has exactly

$$mn - 1 - \frac{d-3}{2}\left(m + n - \frac{d+1}{2}\right)$$

edges. The factor with an arbitrary number E of edges from case III can be constructed by deleting a suitable number of edges joining the vertices of $U - \{v_1, v_5, v_7, \ldots, v_d\}$ with the vertices of $V - \{v_2, v_4, \ldots, v_{d-1}\}$.

The proof of case IV is analogous.

Let us prove now that if the graph $K_{m,n} (m \leq n)$ has a factor with E edges and with a finite diameter d, then one of the cases I—V occurs. If $n = 1$, then evidently I occurs. If n is infinite, then the case V occurs. We can suppose that n is finite. If $d = 2$, then II holds. Evidently, we may assume that $d \geq 3$. The factor F has at least $m + n - 1$ edges. We shall prove that the total number of edges of F is less than or equal to the upper bound in III or IV, respectively. Let $P = v_0 v_1 v_2 \ldots v_d$ be an arc of length d such that in F there does not exist a shorter arc connecting the vertices v_0, v_d. Each of the remaining vertices of F can be joined with at most two vertices from P and with all the vertices from the other part.

If $v_0, v_2, \ldots \in V$, $v_1, v_3, \ldots \in U$ and all the edges of the given form exist, then we get the upper bound from III or IV. If $v_0, v_2, v_4, \ldots \in U$, $v_1, v_3, v_5, \ldots \in V$ and d is odd, we have the same result as before (the upper bound in III), if d is even, we have the same result as in the case IV. This complete the proof of the theorem.

3. Decomposition of $K_{m,n}$ into p factors

In the following we shall consider bipartite graphs $K_{m,n}$ only. For the complete solution of our problem it is sufficient to determine the value of $B_m(d_1, d_2, \ldots, d_p)$ for every $(p+1)$-tuple $(m, d_1, d_2, \ldots, d_p)$.

Lemma 2. *Let the natural numbers p, m, n be given. If the graph $K_{m,n}$ is decomposable into p factors with finite diameters, then*

$$p \leq \left[\frac{mn}{m+n-1}\right].$$

Proof. The graph $K_{m,n}$ has mn edges. It is clear that the number of edges of a factor with a finite diameter is at least $m + n - 1$. Therefore

$$p(m + n - 1) \leq mn$$

and the required inequality easily follows.

Theorem 4. *Let $p \geq 3$ and $d_2 = d_3 = \ldots = d_p = \infty$. Then*

$$B_m(d_1, d_2, \ldots, d_p) = \begin{cases} \left[\frac{d_1+1}{2}\right] & \text{if } 2 \leq d_1 < 2m; \\ m+1 & \text{if } d_1 = 2m; \\ \infty & \text{if } 2m < d_1 < \infty; \\ 1 & \text{if } d_1 = \infty, \quad m \geq 2; \\ 2 & \text{if } d_1 = \infty, \quad m = 1. \end{cases}$$

Proof. The last three relations are evident. To prove the first and the second relation it is sufficient to construct a decomposition of the corresponding complete bipartite graphs into three factors with the diameters d_1, ∞, ∞. We construct an arbitrary factor F_1 with the diameter d_1 (this is possible according to the Corollary of Theorem 2). The factor F_2 contains all the edges which are incident with one fixed vertex and do not belong to F_1. The factor F_3 contains all the other of the complete bipartite graph. It is clear that the diameter of F_2 and F_3 is ∞. The same is true about the factors F_4, F_5, ..., F_p containing no edges.

4. Decomposition of $K_{m,n}$ into two factors

In the next five lemmas we shall assume that the cardinal numbers $m \geq 1$, $n \geq 1$ are given.

Lemma 3. *If the graph $K_{m,n}$ is decomposed into two factors with the diameters d_1 and d_2, where $d_1 = 6$, then $d_2 \leq 6$.*

Proof. Let F_1 be the factor with the diameter d_1 and F_2 be the factor with the diameter d_2. Let $\varrho_{F_1}(u, v) = 6$. The vertex set of F_1 can be decomposed into subsets $A_i = \{w\colon \varrho(u, w) = i\}$ for $i = 0, 1, 2, \ldots, 6$. (Fig. 3). In F_1 there are edges joining vertices of the consecutive subsets A_i, A_{i+1}. Otherwise there would exist a shorter path joining u and v. We shall show that the diameter d_2 of F_2 is less than or equal to 6. The distance of vertices in A_i ($i = 1, 2, \ldots, 6$) in F_2 is equal to 2. The distances of vertices between the subsets A_i and A_j, $i \neq j$, $i, j = 0, 1, \ldots, 6$ are equal to 1, 2, 3 or 5, except A_2 and A_4. The distance between vertices of these two subsets is equal at most to 6 (e.g. from A_2 to A_5 then to A_0, A_3, A_6, A_1 and to A_4 — see Fig. 3). Thus the diameter of F_2 is less than or equal to 6.

Lemma 4. *If the graph $K_{m,n}$ is decomposed into two factors with the diameters d_1 and d_2, where $d_1 = 7$ or 8, then $d_2 \leq 4$.*

Proof. Let $d_1 = 7$ (for $d_1 = 8$ the proof is the same). Let F_1 be the factor with the diameter $d_1 = 7$, F_2 be the factor with the diameter d_2. Let $\varrho_{F_1}(u, v) = 7$. The vertex set of F_1 can be decomposed into eight subsets $A_i = \{w\colon \varrho(u, w) = i\}$ for $i = 0, 1, \ldots, 7$. In the factor F_1 there are edges joining vertices of consecutive subsets A_i, A_{i+1}. The distance of vertices in A_i ($i = 0, 1, \ldots, 7$) in the factor F_2 is equal to 2. The distances of vertices between the subsets A_i and A_j, $i \neq j$, $i, j = 0, 1, \ldots, 7$ are equal

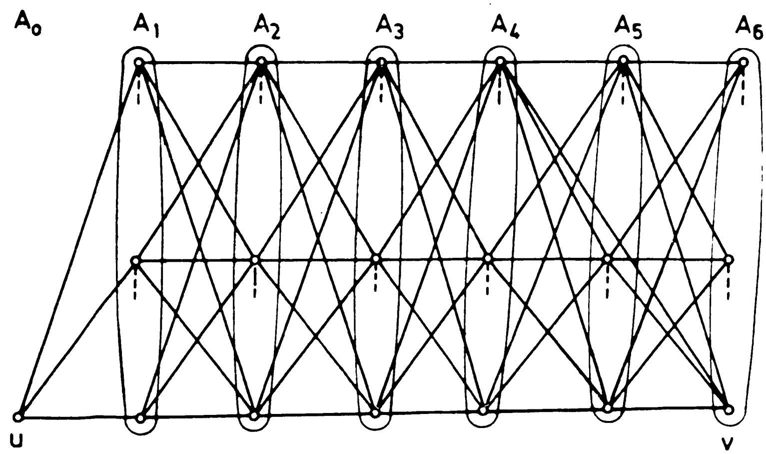

Fig. 3

to 1, 2 or 3 except in two cases: A_1 and A_5, A_2 and A_6. The distance between the vertices of these subsets is at most 4 (e.g. from A_1 to A_4, then to A_7, A_2 and to A_5). Hence the diameter of F_2 is less than or equal to 4.

Lemma 5. *If the graph $K_{m,n}$ is decomposed into two factors with the diameters d_1 and d_2, where $d_1 \geq 9$, then $d_2 = 3$.*

Proof. Let F_i $(i = 1, 2)$ be the factor with the diameter d_i. Let $\varrho_{F_1}(u, v) = d_1$. The vertex set is decomposed into subsets $A_i = \{w: \varrho(u, w) = i\}$ for $i = 0, 1, 2, \ldots, d_1$. In the factor F_1 with the diameter $d_1 \geq 9$ there are edges joining vertices of consecutive subsets A_i, A_{i+1} for $i = 0, 1, 2, \ldots, d_1 - 1$. The distance of different vertices in A_i $(i = 1, 2, \ldots, d_1)$ is equal to 2. The distance of vertices between A_i and A_j $(i \neq j, i, j = 0, 1, 2, \ldots, d_1)$ is equal to 1, 2 or 3. Thus the diameter $d_2 \leq 3$, but the case $d_2 \leq 2$ cannot occur (F_2 does not contain all the edges of $K_{m,n}$), so that the diameter of F_2 is $d_2 = 3$.

A vertex v of a bipartite graph is said to be saturated if by adding an edge incident with v there always arises a graph that is not bipartite.

Lemma 6. *Let n be an integer ≥ 3. The minimal number of edges in a factor of $K_{3,n}$ with the diameter 3, not containing a saturated vertex, is $2n$.*

Proof. Let F be a factor of $K_{3,n}$ that does not contain a saturated vertex such that $d(F) = 3$. The vertex set of $K_{3,n}$ can be partitioned into two subsets U and V, where $|U| = 3$, $|V| = n$. The distance between arbitrary vertices of U in the factor F is 2 (since $d(F) = 3$). The same holds for the vertices of V. The degrees of all the vertices from U are at most $n - 1$ and it follows that the degree of the vertices from V is at least 2. The number of edges in the factor F is at least $2n$. We shall construct such a factor. The vertex set of F is decomposed into two subsets: $U = \{u_1, u_2, u_3\}$

and $V=\{v_1, v_2, \ldots, v_n\}$. The factor F of $K_{3,n}$ consists of the following edges (Fig. 4):

(1) u_1v_i for $i=1, 2, \ldots, n-1$;

(2) u_2v_i for $i=1, n$;

(3) u_3v_i for $i=2, 3, \ldots, n$.

The diameter of F is 3. The factor F has $2n$ edges and it does not contain a saturated vertex.

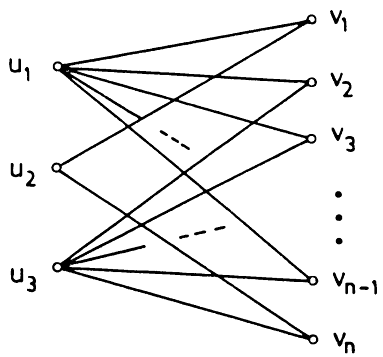

Fig. 4

Lemma 7. *Let the natural numbers $m \geqslant 3$, $n \geqslant 4$, be given. The minimal number of edges in a factor of $K_{m,n}$ with the diameter* 4, *which does not contain a saturated vertex is $m+n$.*

Proof. Let F be a factor (of $K_{m,n}$) not containing a saturated vertex and $d(F)=4$. The number of edges in F is at least $m+n-1$ (otherwise F would be disconnected). If the factor F contains exactly $m+n-1$ edges, then F is a tree. Let $abcd$ be a path of length 4 in the tree F. The factor F has the same form as in Fig. 5. The vertex c is a saturated vertex, which is contradiction to the assumption of the lemma.

We shall construct a factor F (of $K_{m,n}$) with the diameter 4, with $m+n$ edges and without saturated vertex for every natural number $m \geqslant 3$, $n \geqslant 4$. Denote by $U=\{u_1, u_2, \ldots, u_m\}$, $V=\{v_1, v_2, \ldots, v_n\}$ the parts of $K_{m,n}$. Let us define the factor F in the following way: F contains the edges (Fig. 6).

(1) u_1v_1, u_1v_2,

(2) u_2v_1, u_2v_3,

(3) u_iv_2 for $i=3, 4, \ldots, m$,

(4) u_mv_i for $i=3, 4, \ldots, n$.

It is easy to show that the factor F satisfies the conditions of the lemma.

Lemma 8. *The graph $K_{m,n}$ cannot be decomposed into two factors with finite diameters if $m \leq 2$ or $m = n = 3$.*

Proof. For a finite n the assertion follows from Lemma 2. It is easy to show that the lemma holds for finite n as well.

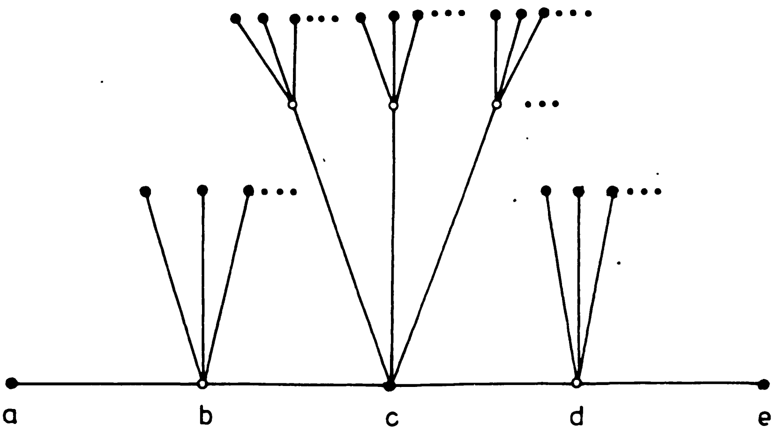

Fig. 5

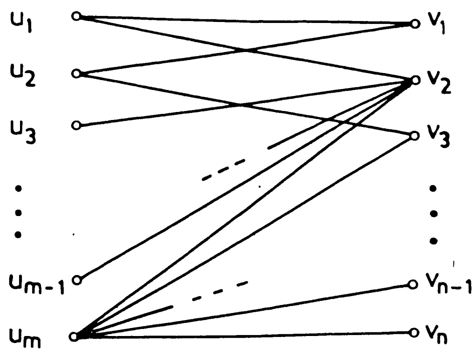

Fig. 6

Lemma 9. *If the graph $K_{m,n}$ is decomposable into two factors, F_1 with a finite diameter and F_2 with the diameter* 3, *then each vertex of F_2 has the degree at least* 2.

Proof. No vertex of F_2 can have the degree 0. If some vertex u has in F_2 the degree 1, then the vertex v adjacent to u in F_2 is saturated (since the diameter of F_2 is 3) and the diameter of F_1 is ∞.

Lemma 10. *There is no cardinal number n for which the graph $K_{5,n}$ is decomposable into two factors with the diameter* 3.

Proof. Let us suppose that such a decomposition of $K_{5,n}$ exists. Denote the parts of $K_{5,n}$ by U, V where $|U|=5$ and $|V|=n$. Let $v \in V$. Let F be such a factor, where $d_F(v) \leq 2$. From Lemma 9 it follows that $d_F(v)=2$. The vertex v is adjacent in F to the vertices u and u'. The distance between two arbitrary vertices of V in F cannot exceed 2 (the diameter of F is 3). It follows that every vertex $v' \in V$, $v' \neq v$ is adjacent to u or u'. In this case the distance between u and u' in another factor G is greater than 2. But this is a contradiction to the assumption that the diameter of G is 3.

Lemma 11. *The graph $K_{5,5}$ cannot be decomposed into two factors with the diameters* 3 *and* 4.

Proof. Denote the parts of $K_{5,5}$ by U and V. Suppose a factor F to have the diameter 3. From Lemmas 7 and 9 it follows that the degrees of vertices from V are given by certain of the following sequences: (22223), (22233), (22224), (22234), (22244), (22333), (22334), (23333), (23334) and (33333). By the systematic examination of all possibilities we can establish that the second factor F has a diameter greater than 4.

Denote by $B_m(d, e) = B_m(e, d)$ the smallest cardinal number n such that the graph $K_{m,n}$ can be decomposed into two factors with the diameters d and e. If such a number does not exists, we shall write $B_m(d, e) = \infty$.

Theorem 5. *Let $1 \leq d \leq \infty$ and $m \geq 1$ be a cardinal number, then $B_m(d, \infty)$ equals:*

(1) 2 *if* $d=\infty$, $m=1$.
(2) 1 *if* $d=\infty$, $m \geq 2$.
(3) d *if* $d=1$ *or* 2, $m=1$.
(4) $d-1$ *if* $d=2$ *or* 3, $m \geq 2$.
(5) 3 *if* $d=4$, $m=2$.
(6) $d-2$ *if* $d=4$ *or* 5, $m \geq 3$.
(7) ∞ *if* $d \geq 6$.

Proof. The first six relations are obvious. The seventh relation follows from Lemmas 3, 4 and 5.

The next Theorem 6 can be proved by using the previous results and by systematic examination of all possibilities.

Theorem 6. *Let $3 = d \leq e < \infty$ and $m \geq 1$ be a cardinal number, then $B_m(3, e)$ equals:*

(1) 6 *if* $e=3$ and $m \geq 6$.
(2) $12-m$ *if* $e=4$, $m=5$ or 6.
(3) 5 *if* $e=4$, $m \geq 7$.
(4) 5 *if* $e=6$, $m=4$.

(5) 4 *if* $e=6$, $m\geq 5$.

(6) 5 *if* $e=5$, 7, 8 *or* 9, $m\geq 5$.

(7) $\left[\frac{e+2}{2}\right]$ *if* $e\geq 10$, $m\geq\left[\frac{e+1}{2}\right]$.

(8) $\left[\frac{e+1}{2}\right]$ *if* $e\geq 10$, $m\geq\left[\frac{e+2}{2}\right]$.

(9) ∞ *otherwise.*

Theorem 7. *Let* $4=d\leq e<\infty$ *and* $m\geq 1$ *be a cardinal number, then* $B_m(4, e)$ *equals:*

(1) 6 *if* $e=4$, $m=3$.

(2) 4 *if* $e=4$, $m=4$ or 5.

(3) 3 *if* $e=4$, $m\geq 6$.

(4) 4 *if* $e=5$ *or* 7, $m\geq 4$.

(5) 5 *if* $e=6$, $m=3$ *or* 4.

(6) 3 *if* $e=6$, $m\geq 5$.

(7) 5 *if* $e=8$, $m=4$.

(8) 4 *if* $e=8$, $m\geq 5$.

(9) ∞ *otherwise.*

Proof. Let us prove the statement (1). Evidently $B_3(4,4)>3$. According to Lemma 7 $B_3(4,4)$ does not equal 4 or 5. A decomposition of $K_{3,6}$ into two factors with the diameters 4 is given in Table I, number 1.

Let us prove the statement (2). According to the statement (1) it follows that neither $B_4(4,4)$, nor $B_5(4,4)$ are equal to 3 (according to Theorem 1 they cannot be <3). A decomposition of $K_{4,4}$ into two factors with the diameters 4 is given in Table I, number 2. Therefore $B_4(4,4)=4$. From Theorem 1 and from previous results it follows that $B_5(4,4)=4$.

From the statement (1), Lemma 8 and Theorem 1 the statement (3) follows.

Let us prove the statement (4) for $e=5$. First let us prove that $B_m(4,5)\neq 3$, i.e. $K_{m,3}(m\geq 3)$ cannot be decomposed into two factors F and G with the diameters 4 and 5, respectively. From Lemma 8 it follows that such a decomposition does not exist for $m=3$. Let us assume that such a decomposition exists for $m\geq 4$. Let G contain a track* $(v_1u_1v_2u_2v_3u_3)$ of the length 5. It is easy to verify that without adding a new vertex in the part $U=\{u_1, u_2, u_3\}$ the factor F does not contain a track of the length ≤ 3 between the vertices v_3 and v_2. The decomposition of $K_{4,4}$ into two factors with the diameters 4 and 5 is given in Table I, number 3. From Theorem 1 and from previous results the statement (4) for $e=5$, $m>4$ follows. Evidently the statement (4) holds for $e=7$, $m\geq 4$. Let us prove the statement (5)

* If x and y, $x\neq y$, are vertices of a graph, a track from x to y is defined to be a path of the minimum length going from x to y (cf. [5], p. 125).

Table I

	Edges of		Diameters	
Number	F	G	$d=d(F)$	$e=d(G)$
1	11, 12, 22, 23, 14, 34, 25, 35, 36.	21, 31, 32, 13, 33, 24, 15, 16, 26.	4	4
2	11, 21, 22, 33, 43, 14, 44.	31, 41, 12, 42, 13, 23, 34, 24.	4	4
3	11, 21, 12, 32, 42, 23, 43, 44.	31, 41, 22, 13, 33, 14, 24, 34.	4	5
4	21, 31, 32, 13, 14, 24, 15, 35.	11, 12, 22, 23, 33, 34, 25.	4	6
5	11, 21, 22, 32, 33, 24.	31, 12, 13, 23, 14, 34,	5	5
6	21, 31, 41, 32, 42, 13, 14, 24, 44.	11, 12, 22, 33, 43, 34, 23.	5	6

In the Table I the edges of the factors F and G of the graph $K_{m,n}$ are given. The vertex set of $K_{m,n}$ is partitioned into two disjoint subsets (parts) $U=\{u_1, u_2, \dots, u_m\}$ and $V=\{v_1, v_2, \dots, v_n\}$. Instead of the edges u_iv_j we write only ij, where the first number means the index of the vertex u_i and the second index of the vertex v_j.

for $m=3$. An arbitrary factor (of $K_{3,4}$) with the diameter 6 has the complement with the diameter 6. Therefore $B_3(4,6)\neq 4$. A decomposition of $K_{3,5}$ into two factors with the diameters 4 and 6 is given in Table I, number 4. Let us prove the statement (5) for $m=4$. Obviously it is sufficient to prove that $B_4(4,6)\neq 4$, i.e. that $K_{4,4}$ cannot be decomposed into two factors F and G with the diameters 4 and 6, respectively. Let us admit that such a decomposition exists and a track of the length 6 in G is $(v_1u_1v_2u_2v_3u_3v_4)$. The vertex u_4 is joined in G with v_2 or v_3 (otherwise the diameter of G is 7 or 4). Let u_4 be joined with v_3. It is easy to show that the diameter of F is 5. Therefore $K_{4,4}$ cannot be decomposed into two factors with the diameters 4 and 6. The decomposition of $K_{4,5}$ into two factors with the diameters 4 and 6 is easy construct. Therefore $B_4(4,6)=5$.

From the statement (5) for $m=3$, Lemma 8 and Theorem 1 the statement (6) follows.

The statement (7) and (8) are evident.

From previous results, Lemma 5, Lemma 8 and Theorem 1 the statement (9) follows.

Theorem 8. *Let $5=d\leq e<\infty$ and $m\geq 1$ be a cardinal number. Then $B_m(5, e)$ equals:*

(1) 4 *if* $e=5$, $m=3$;
(2) $e-2$ *if* $e=5$ *or* 6, $m\geq 4$;
(3) ∞ *otherwise.*

Proof. From Lemma 8 it follows that $B_3(5,5)\geq 4$. In Table I, number 5

Table 2

d \ e	∞	1	2	3	4	5	6	7	8	9	...
∞	(1,2)	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)					
1	(1,1)			In this area no decomposition exists for any $K_{m,n}$							
2	(1,2)										
3	(2,2)			(6,6)	(5,7) (6,6)	(5,5)	(4,5)	(5,5)	(5,5)	(5,5)	...
4	(2,3)			(5,7) (6,6)	(3,6) (4,4)	(4,4)	(3,5)	(4,4)	(4,5)		
5	(3,3)			(5,5)	(4,4)	(3,4)	(4,4)				
6				(4,5)	(3,5)	(4,4)	(3,4)				
7				(5,5)	(4,4)	In this area no decomposition exists for any $K_{m,n}$					
8				(5,5)	(4,5)						
9				(5,5)							
10				(5,6)							
⋮				⋮							

There are shown for given d and e all couples (m, n) $m \leq n$, such that $B_m(d, e) = n$ and $B_M(d, e) = N$ does not hold for any $M \leq m$, $N \leq n$, $(M, N) \neq (m, n)$.

a decomposition of $K_{3,4}$ into two factors with the diameters 5 is given so that $B_3(5,5) = 4$.

From Lemma 8 and Theorem 1 the statement (2) for $e = 5$ follows. We shall prove (2) for $e = 6$, $m = 4$. Assume that $K_{3,4}$ can be decomposed into two factors F and G with the diameters 5 and 6, respectively. Let G contain a track $(v_1 u_1 v_2 u_2 v_3 u_3 v_4)$ of the length 6. It is easy to verify that either F contains a track or F is disconnected. Thus the factor F cannot have the diameter 5. The decomposition of the graph $K_{4,4}$ into two factors with the diameters 5 and 6 is given in Table I, number 6. Therefore $B_4(5,6) = 4$.

From our previous results and Theorem 1, the statement (2) for $e = 6$, $m > 4$ follows as well.

Feom Lemmas 4 and 5 and from the proofs of the statements (1) and (2) the statement (3) follows.

Theorem 9. *Let* $6 = d \leq e < \infty$ *and* $m \geq 1$ *be a cardinal number. Then* $B_m(6, e)$ *equals:*

(1) 4 *if* $e=6$, $m=3$;
(2) 3 *if* $e=6$, $m\geq 4$;
(3) ∞ *otherwise.*

Proof. The statement (1) is evident. The statement (2) follows from (1) and from Theorem 1. The statement (3) follows from (1), (2) and from Lemmas 3, 4 and 5.

Theorem 10. *Let* $1\leq d\leq e<\infty$. *Then* $B_m(d, e)=\infty$, *if one of the following cases occurs:*

(1) $d=1$ *or* 2;
(2) $d\geq 7$, $e\geq 7$.

Proof. The first relation follows from Lemma 8. The second relation follows from Lemmas 4 and 5.

Corollary. *The bipartite graph* $K_{m,n}$ *is decomposable into two factors with the diameters* d *and* $e(2\leq d\leq e\leq\infty)$ *if and only if* $n\geq B_m(d, e)$, *where* $B_m(d, e)$ *is given in Theorems 5—10.*

The proof follows from Theorems 1, 5—10.

In the next Theorem 11 there are given all couples of cardinal numbers m, n $(m\leq n)$ for which the graph $K_{m,n}$ is decomposable into two factors with given diameters. The proof is based on Theorems 5, 6, 7, 8, 9 and 10.

Theorem 11. *Let* $1\leq d\leq e\leq\infty$ *and* m, n *be cardinal numbers such that* $m\leq n$. *The bipartite graph* $K_{m,n}$ *is decomposable into two factors with the diameters* d *and* e *if and only if one of the following cases occurs:*

(1) $d=e=\infty$, $m=1$, $n\geq 2$.
(2) $d=1$, $e=\infty$, $m=1$, $n=1$.
(3) $d=2$, $e=\infty$, $m\geq 1$, $n\geq 2$.
(4) $d=3$, $e=\infty$, $m\geq 2$.
(5) $d=4$, $e=\infty$, $m\geq 2$, $n\geq 3$.
(6) $d=5$, $e=\infty$, $m\geq 3$.
(7) $d=3$, $e=3$ or 4, $m\geq 6$.
(8) $d=3$, $e=4$, $m\geq 5$, $n\geq 7$.
(9) $d=3$, $e=6$, $m\geq 4$, $n\geq 5$.
(10) $d=3$, $e=5$, 7, 8 or 9, $m\geq 5$.
(11) $d=3$, $e\geq 10$, $m\geq\left[\frac{e+1}{2}\right]$, $n\geq\left[\frac{e+2}{2}\right]$.
(12) $d=4$, $e=4$, $m\geq 3$, $n\geq 6$.
(13) $d=4$, $e=4$, 5 or 7, $m\geq 4$.
(14) $d=4$, $e=6$, $m\geq 3$, $n\geq 5$.
(15) $d=4$, $e=8$, $m\geq 4$, $n\geq 5$.
(16) $d=5$, $e=5$, $m\geq 3$, $n\geq 4$.

(17) $d=5$, $e=6$, $m \geqslant 4$.

(18) $d=6$, $e=6$, $m \geqslant 3$, $n \geqslant 4$.

The next corollary shows for which diameters it is possible to decompose a bipartite graph.

Corollary. *Let the natural numbers d, e $(d \leqslant e)$ be given. A bipartite graph decomposable into two factors with the diameters d and e exists if and only if one of the following cases occurs:*

(1) $d=3$.

(2) $d=4$, $e=4$, 5, 6, 7 or 8.

(3) $d=5$, $e=5$ *or* 6.

(4) $d=e=6$.

Proof. If $d<3$, then no bipartite graph can be decomposed into two factors with the diameters d and e. From Lemmas 3, 4 and 5 it follows that no bipartite graph decomposable into two factors with other diameters than those in (1)—(4) exists. According to Theorems 6—9 bipartite graphs which are decomposable into two factors with the diameters given in the corollary do exist.

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РАЗЛОЖЕНИЯ ПОЛНЫХ ДВУДОЛЬНЫХ ГРАФОВ НА ФАКТОРЫ С ДАННЫМИ ДИАМЕТРАМИ

Элишка Томова

Резюме

Рассматривается проблема разложения полных двудольных графов $K_{m,n}$ на факторы с данными диаметрами. Здесь находятся все пары чисел (m, n), для которых возможно разложить полный двудольный граф на два фактора с данными диаметрами.