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GENERIC BIFURCATIONS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS ON DIFFERENTIABLE MANIFOLDS

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1. Introduction. This paper describes generic properties of parametrized second order ordinary differential equations on differentiable manifolds. Generic properties of such equations without parameters have been considered by S. Shahshahani [8]. The problem of generic properties of 1-parametric dynamical systems is studied e.g. in [3], [6], [7], [9].

Let A be a compact C^r manifold and let X be a compact C^{r+1} manifold. Let $T(X)$ denotes the tangent bundle of X. Let K_i $(i=1, 2, \ldots)$ be compact subsets of $T(X)$ such that $K_i \subset K_{i+1}$ for all i and $\bigcup_{i=1}^{\infty} K_i = T(X)$. Denote by $\Gamma_{\mathrm{I}}^r(TX)$ the set of C^r vectorfields on $T(X)$. Since $T(X)$ is not compact, we endow the set $\Gamma_{\mathrm{I}}^r(TX)$ with the Whitney C^r topology. A basis for this topology is given by the sets of the form

$$B(\zeta, \delta)=\{\eta \in \Gamma_{\mathrm{I}}^r(TX) \mid d_r(\zeta/K_i\text{-int } K_{i-1},\ \eta/K_i\text{-int } K_{i-1})<\delta_i \ \text{ fol all } i\},$$

where $\zeta \in \Gamma_{\mathrm{I}}^r(TX)$, $\delta\colon T(X)\to R$ is a continuous positive-valued function with $\delta_i = \min \delta$ on K_i—K_{i-1}. The set $\Gamma_{\mathrm{I}}^r(TX)$ has the Baire property, i.e. a countable intersection of open and dense sets is dense.

Let $\tau_X\colon T(X)\to X$ be the natural projection. A vectorfield $\zeta \in \Gamma_{\mathrm{I}}^r(TX)$ is called a second order ordinary differential equation on X if $D\tau_X \circ \zeta = 1_{T(X)}$, where $D\tau_X$ denotes the differential of the mapping τ_X and $1_{T(X)}$ is the identical mapping of $T(X)$ onto $T(X)$. Denote the set of second order ordinary differential equations on X by $\Gamma_{\mathrm{II}}^r(TX)$.

Denote by $H_{\mathrm{I}}^r(A, TX)$ the set of parametrized C^r vectorfields on $T(X)$ with the parameter set A (cf. [1, §21]). Similarly to the case of the set $\Gamma_{\mathrm{I}}^r(TX)$, we can endow the set $H_{\mathrm{I}}^r(A, TX)$ with the Whitney C^r topology. Then the set $H_{\mathrm{I}}^r(A, TX)$ has the Baire property.

A parametrized vectorfield $\xi \in H_{\mathrm{I}}^r(A, TX)$ is called a C^r parametrized second order ordinary differential equation on X if $\xi_a \in \Gamma_{\mathrm{II}}^r(TX)$ for all $a \in A$, where $\xi_a(x)=\xi(a, x)$ for $x \in T(X)$. Denote the set of C^r parametrized second order ordinary differential equations by $H^r(A, X)$. This set is a closed subspace of

$H_1^r(A, TX)$ and we can endow it with the topology induced by the topology on $H_1^r(A, TX)$. Then the set $H^r(A, X)$ has the Baire property.

A property P of a parametrized second order ordinary differential equation is called generic in $H^r(A, X)$ if the set $\{\xi \in H^r(A, X) \mid P\}$ contains a residual set, i.e. a set which is a countable intersection of open and dense sets in $H^r(A, X)$.

We shall suppose that $\dim A = 1$ and $\dim X = n$. Let $\xi \in H^r(A, X)$ and let (U, α), (V, β) be charts on A and X, respectively. Then from the property $D\tau_x \circ \xi_a = id_{T(X)}$ for every $a \in A$ it follows that the local representative ξ' of ξ with respect to these charts has the form

$$(1) \qquad \xi'(\mu, x, v) = (x, v, v, \xi_{\alpha\beta}(\mu, x, v)) \,,$$

where

$$\mu \in \alpha(U),\ (x, v) \in \beta(V) \times R^n,\ \xi_{\alpha\beta}: \alpha(U) \times \beta(V) \times R^n \to R^n \ \text{ is } C^r \,.$$

2. The case of a zero eigenvalue. Let $(TX)_0$ denote the image of the zero section in $T(X)$, i.e. $(TX)_0 = \{0_x \in T(X) \mid x \in X\}$, where 0_x denotes the zero of T_xX. The set $(TX)_0$ is a closed submanifold of $T(X)$, which is diffeomorphic to X. Let $T(TX)_0$ be the tangent bundle of $(TX)_0$ and let $(T^2X)_0 = \{0[x] \in T(TX)_0 \mid x \in (TX)_0\}$, whee $0[x]$ denotes the zero of $T_x(TX)_0$. Since $(TX)_0$ is a closed submanifold of $T(X)$ of dimension n, $(T^2X)_0$ is a closed submanifold of $T^2(X) = T(T(X))$ of dimension n. Since X is compact, $(TX)_0$ and $(T^2X)_0$ are compact too.

Let $\tau_X: T(X) \to X$, $\tau_{T(X)}: T^2(X) \to T(X)$ be the natural projections. Denote by $Y(T^2X)$ the set of $z \in T^2(X)$ with the following properties

$$(1) \qquad \tau_{T(X)}(z) \in (TX)_0$$

$$(2) \qquad D\tau_X(z) \in (TX)_0$$

This set is well defined and the definition is independent of coordinates. It is easy to see that if (U, α) is a chart on X and $(T_\alpha^2, T_\alpha, \tau_X^{-1}(U))$ is a natural C^r vector bundle chart on $T^2(X)$ associated with the chart (U, α), then for $z \in \tau_X^{-1}(U)$, $T_\alpha^2(z) = (x, 0, 0, y)$, where $x \in R^n$, $y \in R^n$. Now, it is clear that the set $Y(T^2X)$ is a C^r submanifold of $T^2(X)$ isomorphic to $T(X)$. Therefore we can identify them. Since $(TX)_0$ is isomorphic to X, we can identify them too. Therefore if $\xi \in H^r(A, X)$, we can consider the mapping $r(\xi) = \xi / A \times (TX)_0$: $A \times (TX)_0 \to Y(T^2X)$ as a mapping $r(\xi): A \times X \to T(X)$.

Now, define the set $H_0^r(A, X) = \{\xi \in H^r(A, X) \mid r(\xi) \bar{\cap} (TX)_0\}$, where $r(\xi) \bar{\cap} (TX)_0$ means that the mapping $r(\xi)$ transversally intersects the submanifold $(TX)_0$ in $T(X)$ (cf. [1, § 17]).

Lemma 1. *The set $H_0^r(A, X)$ is open and dense in $H^r(A, X)$.*

Proof. Define the mapping ϱ: $H^r(A, X) \to C^r(A \times X, T(X))$, $\varrho(\xi) = r(\xi)$ for $\xi \in H^r(A, X)$. This mapping is a C^r representation (For the definition of C^r representation see [1, § 18]). Since $A \times X$ is a compact manifold and $(TX)_0$ is a closed submanifold of $T(X)$, then by [1, Theorem 18.2], the set $H_0^r(A, X)$ is open in $H^r(A, X)$. The density follows from [1, Theorem 19.1]. The assumptions of this theorem can be verified similarly to the proof of [6, Lemma 1].

Denote $C(\xi) = \{(a, x) \in A \times T(X) \mid \xi(a, x) \in (T^2X)_0$. From (1) it follows that $C(\xi) \subset A \times (TX)_0$.

Proposition 1. *If $\xi \in H_0^r(A, X)$, then $C(\xi)$ is a compact 1-dimensional C^r submanifold of $A \times T(X)$.*

Proof. If $\xi \in H_0^r(A, X)$, then $r(\xi) \bar{\cap} (TX)_0$ and by [1, Corollary 17.2] $C(\xi) = [r(\xi)]^{-1}(TX)_0$ is a closed 1-dimensional C^r submanifold of $A \times (TX)_0$ and since $A \times (TX)_0$ is compact, the set $C(\xi)$ is compact too.

Let h_0: $X \to T(X)$ be the zero section. This mapping is a diffeomorphism of X onto $(TX)_0$. Denote $K(\xi) = R(C(\xi))$, where $R = id_A \times h_0^{-1}$, id_A is the identical mapping of A onto A. By Proposition 1, the set $K(\xi)$ is a compact 1-dimensional submanifold of $A \times X$ (We have identified $(TX)_0$ and X).

Since the mapping $r(\xi)$: $A \times X \to T(X)$ for $\xi \in H^r(A, X)$ is a parametrized vectorfield, then if $(a, x) \in K(\xi)$, we can define the Hessian $\dot{r}(\xi)_a(x) : T_xX \to T_xX$ at x of the vectorfield $r(\xi)_a$, where $r(\xi)_a(y) = r(\xi)\,(a, y)$ for $y \in X$ (cf. [1, § 22]).

Denote $X_1(\xi) = \{(a, x) \in K(\xi) \mid \dot{r}(x)_a(x)$ is not surjective$\}$. Let $Z_1(\xi) = R^{-1}(X_1(\xi)) \subset A \times T(X)$. By almost the same procedure used in [6], it is possible to prove the following proposition.

Proposition 2. *There exists an open, dense subset $H_{01}^r(A, X)$ in $H_0^r(A, X)$ such that for every $\xi \in H_{01}^r(A, X)$*

(1) *$Z_1(\xi)$ is finite*

(2) *If $(a_0, x_0) \in Z_1(\xi)$, then there exists a chart (W, h) on $A \times T(X)$ at (a_0, x_0) such that*
$h(C(\xi)) = \{(\mu, y_1, \ldots, y_n, 0, \ldots, 0) \in R^{2n+1} \mid \mu = \varphi_0(y_n),\ y_i = \varphi_i(y_n),\ i = 1, 2, \ldots, n-1,\ y_n \in J\}$,
where $\varphi_i \in C^r$ on J for $i = 0, 1, \ldots, n-1$, J is an open interval, $0 \in J$, $\dfrac{d^2\,\varphi_0(0)}{dy_n^2} \neq 0$.

(3) *The principal part ξ_h of the local representative of ξ has the form*
(*) $\xi_h(\mu, x_1, y, v) = (v, \alpha\mu + \beta x_1^2 + \omega(\mu, x_1, y, v), By + \chi(\mu, x_1, y, v))$,
where B is a regular $(n-1) \times (n-1)$ matrix, $y = (x_2, x_3, \ldots, x_n)$, $\omega, \chi \in C^r$, $\chi(0, 0, \ldots, 0) = 0$, $\omega(\mu, x_1, 0, 0)$ contains only μ^2, μx_1 and terms of higher order than 2, $\alpha \neq 0$.

Lemma 2. *Let $C, D \in A(n, n)$, $C=(c_{ij})$, $D=(d_{ij})$. Let $c_{i1}=0$ for $i=1, 2, \ldots, n$ and*

$$\det\begin{bmatrix} c_{12}, \ldots, c_{1n}, d_{11} \\ \ldots\ldots\ldots\ldots \\ c_{n2}, \ldots, c_{nn}, d_{n1}\end{bmatrix} \neq 0 .$$

Then the matrix

$$H=\begin{bmatrix} 0_n & E_n \\ C & D\end{bmatrix}$$

has one eigenvalue $\lambda=0$ of multiplicite 1. (0_n is the zero matrix in $A(n, n)$ and E_n is the unit matrix in $A(n, n)$, $A(i, j)$ denotes the set of all $i\times j$ matrices).

Proof. From the form of the matrix H it follows that $\lambda=0$ is the eigenvalue of H. Denote by $P_Q(\lambda)$ the characteristic polynomial of a matrix Q. Then $P_H(\lambda)=\lambda P_{H_1}(\lambda)$, where

$$H_1=\begin{bmatrix} 0_{n,n-1} & E_{n-1} \\ C_1 & D\end{bmatrix}, \qquad C_1=\begin{bmatrix} c_{12}, \ldots, c_{1n} \\ \ldots\ldots\ldots \\ c_{n2}, \ldots, c_{nn}\end{bmatrix},$$

$0_{n,n-1}$ is the zero matrix in $A(n, n-1)$. Since

$$P_{H_1}(0)=\det\begin{bmatrix} c_{12}, \ldots, c_{1n}, d_{11} \\ \ldots\ldots\ldots\ldots \\ c_{n2}, \ldots, c_{nn}, d_{n1}\end{bmatrix} \neq 0 ,$$

then $P_{H_1}(\lambda)$ has no eigenvalue equal to zero and therefore $\lambda=0$ is an eigenvalue of H of multiplicity 1.

Let $\xi \in H^r_{01}(A', X)$, $(a_0, x_0)\in Z_1(\xi)$ and let $(U\times V, h_1\times h_2)$ be a chart at (a_0, x_0) such that $\xi_h(\varphi_0(x_1), x_1, \varphi_2(x_1), ., \varphi_n(x_1), 0, \ldots, 0)=0$ for $x_1\in J$, $\varphi_i \in C^r$ on J, where $\xi_h(\mu, x_1, y, v)$ has the form (*). Then $H(x_1)=D_2\xi_h(\varphi_0(x_1), x_1, \varphi_2(x_1), \ldots, \varphi_n(x_1), 0, \ldots, 0)=\begin{bmatrix} 0_n & E_n \\ C(x_1) & D(x_1)\end{bmatrix}$, where $D_2\xi_h$ denotes the derivative in (x_1, y, v) and

$$C(x_1)=\begin{bmatrix} 2\beta x_1+o(x_1), & 0, \ldots, 0 \\ 0 & \\ 0 & B\end{bmatrix}, \qquad D(x_1)=\frac{\partial \xi_h}{\partial v}(\varphi_0(x_1, \ldots, 0) .$$

Denote by $H_{02}(A, X)$ the set of all $\xi\in H_{01}(A, X)$ such that $d_{11}(0)\neq 0$, where $D(x_1)=(d_{ij}(x_1))$. It is easy to prove that this set is open and dense in $H^r_{01}(A, X)$. If $\xi \in H^r_{02}(A, X)$, then by Lemma 2 the matrix $H(0)$ has the eigenvalue $\lambda=0$ of multiplicity 1. From the form of $H(x_1)$ it follows that $\det H(x_1)=$

$(2\beta x_1 + o(x_1))$ det B. From this and from the continuous dependence of eigenvalues of $H(x_1)$ on x_1 it follows that the eigenvalues of $H(x_1)$ do not change the sign of its real parts in J for J sufficiently small except of one eigenvalue.

We have proved the following theorem.

Theorem 1. *Assume $r \geqq 3$. Then there is an open, dense subset $H^r_{02}(A, X)$ in $H^r(A, X)$ with the following properties:*

(1) *For $\xi \in H^r_{02}(A, X)$, $C(\xi)$ is a compact 1-dimensional C^r submanifold of $A \times T(X)$.*

(2) *For a fixed $a \in A$, the set $\{x \in T(X) \mid (a, x) \in C(\xi)\}$ consists of isolated points.*

(3) *The set $Z_1(\xi)$ is finite.*

(4) *For every $(a_0, x_0) \in C(\xi) - Z_1(\xi)$ there is a chart (W, h) on $A \times T(X)$ at (a_0, x_0), $h(W) = U \times V$, $h(a_0, x_0) = (0, 0)$ and a C^r mapping $\varphi: U \to V$ such that $h(C(\xi) \cap W) = \{(\mu, z) \mid z = \varphi(\mu),\ \mu \in U\}$.*

(5) *For every $(a_0, x_0) \in Z_1(\xi)$, there is a chart $(U \times V, h_1 \times h_2)$ on $A \times T(X)$ at (a_0, x_0), $h(a_0, x_0) = (0, 0)$ such that*

(a) *$(h_1 \times h_2)(C(\xi) \cap W) = \{(\mu, y_1, y_2, \dots, y_n, 0, \dots, 0)/\mu = \varphi_0(y_1),\ y_i = \varphi_i(y_1),\ i = 2, 3, \dots, n,\ \mu \in J\}$, where J is an open interval, $0 \in J$,*

$$\varphi_0(0) = 0, \qquad \frac{d\varphi_0(0)}{dy_1} = 0, \qquad \frac{d^2\varphi_0(0)}{dy_1^2} \neq 0 .$$

(b) *For μ from one side of 0 there are no critical points of $\xi_{h_1^{-1}(\mu)}$ in V and for μ from the other side of 0 there are exactly two critical points of $\xi_{h_1^{-1}(\mu)}$ in V and the following is true: The point $(0, 0)$ divides the set $C(\xi) \cap (U \times V)$ into two components K_1, K_2 and the number of eigenvalues of the mapping $\dot{\xi}_a(y)$ $(a = h_1^{-1}(\mu),\ y = h_2^{-1}(x))$* with the real part greater than 0 *is constant in the components K_1, K_2 and differs by one.*

(6) *if $(a, x) \in Z_1(\xi)$, then the mapping $\dot{\xi}_a(x)$ has exactly one eigenvalue equal to 0.*

Example. Let us consider the following second order orginary differential equation on R:

$$\dot{x} = v$$
$$\dot{v} = -x^2 + v + \mu, \quad x \in R, \quad \mu \in R ,$$

or in the form of the equation:

$$\ddot{x} - \dot{x} + x^2 - \mu = 0 .$$

The set of critical points is a parabola in the (μ, x)-plane. For $\mu < 0$, there are no critical points and for $\mu > 0$ there are exactly two critical points. The derivative of the right-hand side of the equation at the point $(\mu, x) \in C(\xi) = \{(\mu, x, 0) \mid \mu = x^2\}$ has the form $H(x) = \begin{bmatrix} 0 & 1 \\ -2x & 1 \end{bmatrix}$. The characteristic polynomial of this matrix is

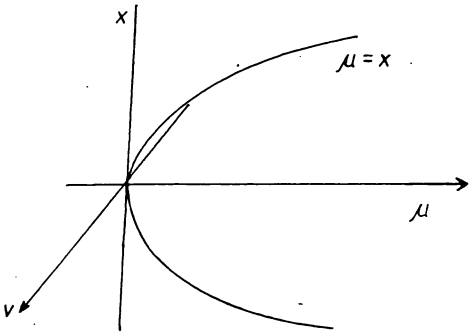

Fig. 1

$P(\lambda)=\lambda^2-\lambda+2x$, which has the roots $\lambda_1=\frac{1+\sqrt{1-8x}}{2}$, $\lambda_2=\frac{1-\sqrt{1-8x}}{2}$. Therefore

$$\begin{aligned} &\lambda_1>0,\ \lambda_2>0 \quad \text{for} \quad x>0, \\ &\lambda_1>0,\ \lambda_2=0 \quad \text{for} \quad x=0 \quad \text{and} \\ &\lambda_1>0,\ \lambda_2<0 \quad \text{for} \quad x<0. \end{aligned}$$

We have the following pictures of trajectories:

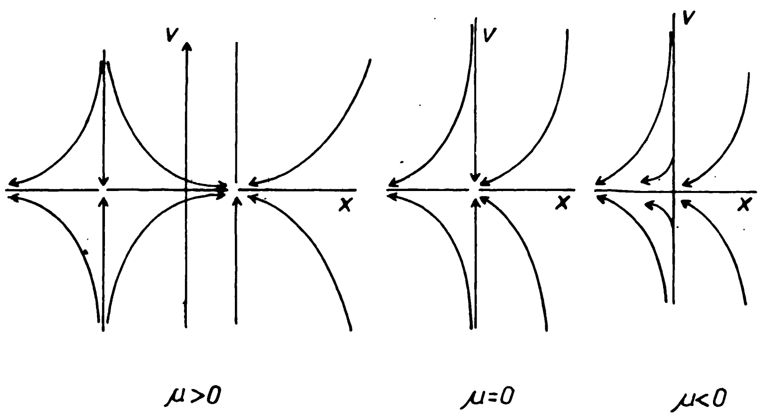

Fig. 2

3. The case of a pair of pure imaginary eigenvalues. First we shall give an example which describes well the generic situation of the case of 1-parametric dynamical systems (cf. [3], [6]):

$$\begin{aligned} \dot{x} &= -\omega y+\mu x+cx(x^2+y^2) \\ \dot{y} &= \omega x+\mu y+cy(x^2+y^2) \end{aligned}$$

$$\dot{u} = -u$$
$$\dot{v} = v\,,$$

$$\dim x = \dim y = 1, \quad \dim u = R^{n_-}, \quad \dim v = R^{n_+}, \quad n_- + n_+ + 2 = n\,.$$

For the study of the topological structure of trajectories of this system in neighbourhoods of invariant manifolds it is enough to consider this system on the submanifold $u=0$, $v=0$. This system has a stable focus at the point (0, 0) which changes to unstable focus if μ cross the zero and there arises a closed orbit in a neighbourhood of 0. We shall show that this is the same in the case of the second order differential equations, too.

Let $\eta \in \dot{\Gamma}_{11}(TX)$ and let $x \in T(X)$ be a critical point of η. We say x is a nonelementary critical point of multiplicity k, if the mapping $\dot{\eta}(x)$ has a pure imaginary eigenvalue of multiplicity $k(\dot{\eta}(x)$ denotes the Hessian of the vectorfield η at x, (cf. [1, §22]) and has no other pure imaginary eigenvalue.

Denote by $H_{11}(A, X)$ the set of all $\xi \in H^r(A, X)$ such that if for $a \in A$ the vectorfield ξ_a has a nonelementary critical point, then it has multiplicity 1. Denote by $Z_2(\xi)$ the set of points $(a, x) \in C(\xi)$ for which x is a nonelementary critical point of ξ_a.

Lemma 3. *The set $H_{11}(A, X)$ $(r \geqq 1)$ is open and dense in $H^r(A, X)$.*

Denote by $\hat{A}(2n, 2n)$ the set of $C \in A(2n, 2n)$ of the form $C = \begin{bmatrix} 0_n & E_n \\ A & B \end{bmatrix}$, where $A, B \in A(n, n)$, 0_n is the zero in $A(n, n)$, E_n is the unit matrix in $A(n, n)$. The set $\hat{A}(2n, 2n)$ is a C^r manifold of dimension $2n^2$.

Let $A_1 = \{(C, \lambda_1, \lambda_2) \in \hat{A}(2n, 2n) \times R^2 \mid \lambda_1 = 0,\ P_1(\lambda_1, \lambda_2) = P_1'(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = P_2'(\lambda_1, \lambda_2) = 0$, where $P(\lambda) = P_1(\operatorname{Re}\lambda, \operatorname{Im}\lambda) + iP_2(\operatorname{Re}\lambda, \operatorname{Im}\lambda)$ is the characteristic polynomial of C and $P_1' + iP_2' = \frac{\partial P}{\partial \lambda}$. It is possible to prove analogously to [4, §2)] that $A_1 = \bigcup_{j=1}^{r_1} A_{1j}$, $j = 1, 2, \ldots, r_1$ are disjoint submanifolds of $\hat{A}(2n, 2n) \times R^2$ of a strictly decreasing dimension and $\bigcup_{j=\varrho_0}^{r_1} A_{1j}$ is closed for $0 < \varrho_0 \leqq r_1$, codim $A_{1j} \geqq 4$ for $j = 1, 2, \ldots, r_1$.

Proof of Lemma 3. Let $\xi, \eta \in H^r(A, X)$, $(a_1, x_1), (a_2, x_2) \in A \times T(X)$ and let (W, h) be a chart on $T(X)$. Let ξ_1, η_1 be the principal part of the local representative of ξ_{a_1}, ζ_{a_2}, respectively, with respect to (W, h). We say that (ξ, a_1, x_1) is k-equivalent to (η, a_2, x_2) if and only if $a_1 = a_2$, $x_1 = x_2$ and $(\xi_1(h(x_1)), D\xi_1(h(x_1)), \ldots, D^k\xi_1(h(x_1)) = (\eta_1(h(x_2)), D\eta_1(h(x_2)), \ldots, D^k\eta_1(h(x_2))$. Obviously, the k-equivalence is an equivalence. Let $J^k\xi(a, x)$ denote the class of triples equivalent to (ξ, a, x). Denote by $\hat{J}^k(A, X)$ the set of all classes $J^k\xi(a, x)$. The mapping $\pi^1: \hat{J}^1(A, X) \to A \times T(X)$, $\pi^1(J^1\xi(a, x)) = (a, x)$ is a C^{r-1} vector bundle over $A \times T(X)$. For $\xi \in H^r(A, X)$ define the mapping $\varrho_\xi: A \times T(X) \to \hat{J}^1(A, X)$, $\varrho_\xi(a, x) = J^1\xi(a, x)$ for $(a, x) \in A \times T(X)$. Define the mapping $\tilde{\varrho}_\xi$: $A \times T(X) \times R^2 \to \hat{J}^1(A, X) \times R^2$, $\tilde{\varrho}_\xi = \varrho_\xi \times id$, where id is the identical mapping of R^2 onto R^2. The mapping ϱ: $H^r(A, X) \to C^{r-1}(A \times T(X) \times R^2, \hat{J}^1(A, X) \times R^2)$, $\varrho(\xi) = \tilde{\varrho}_\xi$ for $\xi \in H^r(A, X)$ is a C^{r-1} representation. Let $(\alpha, \alpha_0 \times \beta_0, U \times V)$ be a natural chart on π^1 and let $W \subset \hat{J}^1(A, X) \times R^2$ be the set of $(p, \lambda_1, \lambda_2) \in \hat{J}^1(A, X) \times R^2$ such that $(\alpha(p), \lambda_1, \lambda_2) = (\mu, y, 0, 0, C, \lambda_1, \lambda_2)$, $\mu \in R$, $y \in R^n$, 0 is the zero in R^n, $(C, \lambda_1, \lambda_2) \in A_1$. It is easy to prove that this definition is independent of coordinates. Since $A_1 = \bigcup_{j=1}^{r_1} A_{1j}$, then $W = \bigcup_{j=1}^{r_1} W_j$, where W_j are disjoint submanifolds of $\hat{J}^1(A, X) \times R^2$ of strictly decreasing dimensions, $\bigcup_{i=\varrho_0}^{r_1} W_i$ is closed for $0 < \varrho_0 \leqq r_1$ and codim $W_j \geqq 2n+4$ for every j. Let ev_ϱ: $H^r(A, X) \times A \times T(X) \times R^2 \to C^{r-1}(A \times T(X) \times R^2, \hat{J}^1(A, X) \times R^2)$, $ev_\varrho(\xi, a, x, \lambda_1, \lambda_2) = \tilde{\varrho}_\xi(a, x, \lambda_1, \lambda_2)$. It is easy to prove that $ev_\varrho \,\bar{\pitchfork}\, N$ for every submanifold N of $\hat{J}^1(A, X) \times R^2$ and so $ev_\varrho \,\bar{\pitchfork}\, W$. Let $\xi \in H^r_{11}(A, X)$, and let $(\beta, \alpha_0 \times \beta_0, U \times V)$ be a natural chart on π^1 as in the definition of W and $\beta(J^1\xi(a, x)) = (\alpha_0(a), \beta_0(x), \xi'_a(x), D\xi'_a(x))$. Since $(TX)_0$ is a compact subset of $T(X)$, there is a neighbourhood $N(\xi)$ of ξ in $H^r(A, X)$ and a number $q > 0$ such that for every $\eta \in N(\xi)$, $(a, x) \in A \times (TX)_0$, every eigenvalue $\lambda(\eta, a, x)$ of $D\eta'_a(x)$ is such that $|\lambda(\eta, a, x)| < q$, where $\beta(J^1\xi(a, x) = (\alpha_0(a), \beta_0(x), \eta'_a(x), D\eta'_a(x))$. Therefore for $\eta \in N(\xi)$, $\varrho(\eta) \,\bar{\pitchfork}\, W$ if and olny if $\varrho(\eta) \,\bar{\pitchfork}\, W$ on the set $A \times (TX)_0 \times [-q, q]$. Denote $\Psi_i = \{\eta \in N(\xi) \,|\, \varrho(\eta) \,\bar{\pitchfork} \bigcup_{j=r_1-i+1}^{r_1} W_i$ on $A \times (TX)_0 \times [-q, q]\}$ for $i = 1, 2, \ldots, r_1$. From [1, Theorem 18.2] it follows that the sets Ψ_i, $i = 1, 2, \ldots, r_1$ are open in $N(\xi)$. Science codim $W_j \geqq 2n+4$ for all j, then $\varrho(\eta) \,\bar{\pitchfork}\, W$ on $A \times (TX)_0 \times [-q, q]$ means that $\varrho(\eta)(A \times T(X) \times [-q, q]) \cap W = \emptyset$ and so the set $H^r_{11}(A, X)$ is open in $H^r(A, X)$. The density follows from [1, Theorem 19.1] analogously to the proof of [6, Lemma 6].

Let $A_2 = \{(C, \lambda_1, \lambda) \in \hat{A}(2n, 2n) \times R^2 \,|\, P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = \lambda_1 = 0\}$, where $C = \begin{bmatrix} 0_n & E_n \\ A & B \end{bmatrix}$. Similarly to [4, § 2] it is possible to prove that $A_2 = \bigcup_{j=1}^{r_2} A_{2j}$, where A_{2j}, $j = 1, 2, \ldots, r_2$ are disjoint submanifolds of $\hat{A}(2n, 2n) \times R^2$ of strictly decreasing dimensions and the set $\bigcup_{j=\varrho_0}^{r_2} A_{2j}$ is closed for $0 < \varrho_0 \leqq r_2$, codim $A_{21} = 3$.

Let $\pi^1: \hat{J}^1(A, X) \to A \times T(X)$ be the mapping as above and let $(\alpha, \alpha_0 \times \beta_0, U \times V)$ be a natural chart on π^1. Let $W' \subset \hat{J}^1(A, X) \times R^2$ be the set of $(p, \lambda_1, \lambda_2) \in \hat{J}^1(A, X) \times R^2$ such that $(\alpha(p), \lambda_1, \lambda_2) = (\mu, y, 0, 0, C, \lambda_1, \lambda_2)$, $\mu \in R$, $y \in R^n$, 0 is the zero in R^n, $(c, \lambda_1, \lambda_2) \in A_2$. Since $A_2 = \bigcup_{j=1}^{r_2} A_{2j}$, so $W' = \bigcup_{j=1}^{r_2} W'_j$, where

W'_j are disjoint submanifolds of strictly decreasing dimensions, $\bigcup_{j=\varrho_0}^{r_2} W'_j$ is closed for $0<\varrho_0\leqq r_2$ and codim $W'_j\geqq 2n+4$ for $j>1$ and codim $W'_1=2n+3$. Let $\varrho\colon H^r(A, X)\to C^{r-1}(A\times T(X)\times R^2,\ \hat{J}^1(A, X)\times R^2)$ be the mapping from the proof of Lemma 3. Let $H^r_{12}(A, X)=\{\xi\in H^r_1(A, X)\,|\,\varrho(\xi)\,\bar{\cap}\, W'\}$. Similarly to the proof of Lemma 3, the following lemma can be proved.

Lemma 4. *The set $H^r_{12}(A, X)$ is open and dense in $H^r(A, X)$.*

Denote $H^r_{13}(A, X)=H^r_{02}(A, X)\cap H^r_{12}(A, X)$. Let $\xi\in H^r_{13}(A, X)$, $(a_0, x_0)\in C(\xi)$ and let (V, β) be a chart on $A\times T(X)$ at (a_0, x_0). Let ξ_β be the principal part of the local representative of ξ. Denote by $F(t)=D_y\xi_\beta(t)$ for $t\in I=\beta(V\cap C(\xi))$, where $D_y\xi_\beta$ is the derivative of $\xi_\beta(\mu, y)$ $(y\in R^{2n})$ with respect to y. Let $T=\{(s_1, s_2)\in R^2\,|\,s_1=0\}$.

If λ_0 is a simple eigenvalue of $F(t_0)$ for $t_0\in I$, then by [4, Lemma 6] there is a neighbourhood N of t_0 in I and an unique C^r function $\lambda\colon N\to C$ such that $\lambda(t_0)=\lambda_0$ and $\lambda(t)$ is an eigenvalue of $F(t)$ for $t\in N$. Further, there is a nonsingular C^r matrix $C(t)$ on N such that $C^{-1}(t)F(t)C(t)=B(t)$ for $t\in N$, where the first column of B is transpose of $(\lambda(t), 0, \ldots, 0)$. Let $\lambda(t)==\lambda_1(t)+i\lambda_2(t)$, $\hat{\lambda}\colon N\to R^2$, $\hat{\lambda}(t)=(\lambda_1(t), \lambda_2(t))$. Similarly to [4, Proposition 3] it is possible to prove that $\hat{\lambda}\,\bar{\cap}\, T$ if $\xi\in H^r_{13}(A, X)$. Therefore if $\xi\in H^r_{13}(A, X)$, then the set $Z_2(\xi)$ is finite.

Lemma 5. *There is an open and dense set $H^r_1(A, X)$ $(r\geqq 1)$ in $H^r(A, X)$, which has the following properties*

(1) $H^r_1(A, X)\subset H^r_{13}(A, X)$

(2) *If $(a, x)\in Z_2(\xi)$, then the mapping $\xi_a(x)$ has exactly one pair of conjugate pure imaginary eigenvalues.*

The proof of this lemma is the same as the proof of [6, Lemma 10].

Let $\xi\in H^r_1(A, X)$, $(a_0, x_0)\in Z_2(\xi)$ and let $(U\times V, \alpha'\times\beta')$ be a natural chart on $A\times T(X)$ at (a_0, x_0) such that $\alpha'(a_0)=0$, $\beta'(x_0)=0$. Let ξ' be the principal part of the local representative of ξ with respect to this chart and let $(\alpha'\times\beta')(a, x)==(\mu, y, v)\in\epsilon(\alpha\times\beta)(U\times V)=U'\times V'\times R^n$, where A, B are C^r $2n\times 2n$ matrices on U', $\omega(\mu, y, v)=o(|y|+|v|)$. We have the following system of differential equations

$$\dot{y}=v$$
$$\dot{v}=A(\mu)y+B(\mu)v+\omega(\mu, y, v)\,.$$

Since $\xi\in H^r_1(A, X)$, we can transform this system by a regular transformation $Y=(x_1, x_2, w, z)^T=C(\mu)\,(v, y)^T(C(\mu)\in A(2n, 2n)$ is a regular C^r matrix on U', u^T means the transpose of u) to the form

$$\dot{Y}=\hat{A}(\mu)Y+\hat{\omega}(\mu, Y)\,,$$

where $\hat{A}(\mu)=C(\mu)\begin{bmatrix}0_n & E_n\\ A & B\end{bmatrix}C^{-1}(\mu)=\operatorname{diag}(A_1(\mu),\quad H_1(\mu),\quad H_2(\mu)),\quad A_1(\mu)=$

$\begin{bmatrix} \alpha(\mu) & \beta(\mu) \\ -\beta(\mu) & \alpha(\mu) \end{bmatrix}$ for all μ, $\alpha(0)=0$, $\beta(0)\neq 0$, all eigenvalues of $H_1(\mu)$ have negative real parts, all eigenvalues of $H_2(\mu)$ have positive real parts, i.e. we have the following system of differential equations

$$(*)\qquad \begin{aligned} \dot{x}_1 &= \alpha(\mu)x_1+\beta(\mu)x_2+Y_1(\mu, x_1, x_2, w, z) \\ \dot{x}_2 &= -\beta(\mu)x_1+\alpha(\mu)x_2+Y_2(\mu, x_1, x_2, w, z) \\ \dot{w} &= H_1(\mu)w+Y_3(\mu, x_1, x_2, w, z) \\ \dot{z} &= H_2(\mu)z+Y_4(\mu, x_1, x_2, w, z)\,, \end{aligned}$$

$Y=(Y_1,\ Y_2,\ Y_3,\ Y_4)=C(\mu)\ (0,\ \omega(\mu,\ C^{-1}(\mu)\ (x_1,\ x_2,\ w,\ z)^T))^T$. If $C(\mu)=\begin{bmatrix} C_1(\mu) & C_2(\mu) \\ C_3(\mu) & C_4(\mu) \end{bmatrix}$, where $C_i(\mu)\in A(2, n)$, $i=1, 2$, $C_j(\mu)\in(2n-2, n)$, $j=3, 4$, then $Y(\mu, x_1, x_2, w, z))=(C_2(\mu)\omega^*(\mu,\ x_1, x_2, w, z), C_4(\mu)\omega^*(\mu, x_1, x_2, w, z)$, where $\omega^*(\mu, x_1, x_2, w, z)=\omega(\mu,\ C^{-1}(\mu)\ (x_1, x_2, w, z)^T)$.

By [1, Appendix C] there exists a center manifold $M_\mu=\{(x_1, x_2, w, z)\mid w=u(\mu, x_1, x_2),\ z=v(\mu, x_1, x_2)\}$ for μ sufficiently small, where $u, v\in C^{r-1}$, $u(0,0,0)=v(0,0,0)=du(0,0,0)=dv(0,0,0)=0$. The mappings u and v are given by the following system of equations

$$(1)\quad u(\mu, x_1, x_2)=\int_{+\infty}^{0} e^{-H_1(\mu)\sigma}Y_3(\mu,\ \eta_1,\ \eta_2,\ u(\mu,\ \eta_1,\ \eta_2,\ v(\mu,\ \eta_1,\ \eta_2))d\sigma$$

$$(2)\quad \begin{aligned} \dot{\eta}_1 &= \alpha(\mu)\eta_1+\beta(\mu)\eta_2+Y_1(\mu,\ \eta_1,\ \eta_2,\ u(\mu,\ \eta_1, \eta_2),\ v(\mu,\ \eta_1,\ \eta_2)) \\ \dot{\eta}_2 &= -\beta(\mu)\eta_1+\alpha(\mu)\eta_2+Y_2(\mu,\ \eta_1,\ \eta_2,\ u(\mu,\ \eta_1,\ \eta_2),\ v(\mu,\ \eta_1,\ \eta_2)) \end{aligned}$$

$$(3)\quad v(\mu, x_1; x_2)=\int_{+\infty}^{0} e^{-H_2(\mu)\sigma}Y_4(\mu,\ \eta_1,\ \eta_2,\ u(\mu,\ \eta_1,\ \eta_2,\ v(\mu,\ \eta_1,\ \eta_2))d\sigma,$$

where $\eta=(\eta_1,\ \eta_2)=(\eta_1(t,\ \mu,\ x_1,\ x_2),\ \eta_2(t,\ \mu,\ x_1,\ x_2))$ is the solution of the system (2) with the initial condition $\eta(0,\ \mu,\ x_1,\ x_2)=(x_1,\ x_2)$.

If we introduce the change of variables

$$\begin{aligned} p &= w-u(\mu, x_1, x_2) \\ q &= z-v(\mu, x_1, x_2)\,, \end{aligned}$$

then in these new coordinates the system (*) has the form

$$(**)\qquad \begin{aligned} x_1 &= \alpha(\mu)x_1+\beta(\mu)x_2+Y_1(\mu, x_1, x_2, p+u(\mu, x_1, x_2), q+v(\mu, x_1, x_2)) \\ \dot{x}_2 &= -\beta(\mu)x_1+\alpha(\mu)x_2+Y_2(\mu, x_1, x_2, p+u(\mu, x_1, x_2), q+v(\mu, x_1, x_2)) \\ \dot{p} &= H_1(\mu)p+X(\mu, x_1, x_2, p, q) \\ \dot{q} &= H_2(\mu)q+Z(\mu, x_1, x_2, p, q), \end{aligned}$$

where $X, Z\in C^{r-1}$, $X(\mu, x_1, x_2, 0, q)\equiv 0$, $Z(\mu, x_1, x_2, p, 0)\equiv 0$.

Let $\varphi=(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ be the parametrized solution of the system (**) in some neighbourhood V'' of 0. If $\bar{p}\neq 0$, $\bar{q}\neq 0$, then $\varphi(\mu, \bar{x}_1, \bar{x}_2, \bar{p}, \bar{q}, t)\notin V''$ for a sufficiently large t. If $\dim q=0$ and $\bar{p}\neq 0$, then $\varphi(\mu, x_1, x_2, p, t)\notin V''$ for a sufficiently large $-t$, $t<0$. Therefore, if for $\mu \in U'$, there is an invariant set of the system (**) in V'', then it must be a part of the submanifold $p=0$, $q=0$. Now it suffices to consider the restriction of this system to the submanifold $p=0$, $q=0$, i.e. the system

$$\dot{x}_1 = \alpha(\mu)x_1 + \beta(\mu)x_2 + \Phi_1(\mu, x_1, x_2)\,,$$
$$\dot{x}_2 = -\beta(\mu)x_1 + \alpha(\mu)x_2 + \Phi_2(\mu, x_1, x_2)\,,$$

where $\Phi_k(\mu, y_1, x_2) = \sum_{j=1}^{n} \beta_{kj}\omega_j^*(\mu, x_1, x_2, u(\mu, x_1, x_2), v(\mu, x_1, x_2))$, $\omega^* = (\omega_1^*, \omega_2^*, \dots, \omega_n^*)$, $C_2(\mu) = (\beta_{kj}(\mu))$.

Proposition 3.

$$\sum_{k=1}^{2}\sum_{j=1}^{n} |\beta_{kj}(\mu)| \neq 0 \textit{ for all } \mu\,.$$

Proof. Suppose that $\sum_{k=1}^{2}\sum_{j=1}^{n} |\beta_{kj}(\mu)| = 0$. Since

$$C\begin{bmatrix} 0_n & E_n \\ A & B \end{bmatrix} = \hat{A}C, \quad \text{so} \quad \begin{bmatrix} C_2A & C_1 + C_2B \\ C_3A & C_3 + C_4B \end{bmatrix} = \begin{bmatrix} A_1C_1 & A_1C_2 \\ A_2C_3 & A_2C_4 \end{bmatrix},$$

where $A_2 = \operatorname{diag}(H_1, H_2)$. Therefore $C_2A = A_1C_1$ and since by the assumption $C_2(\mu) = 0$, then $A_1(\mu)C_1(\mu) = 0$. The matrix $A_1(\mu)$ is regular and so $C_1(\mu) = 0$. But this is impossible, because the matrix $C(\mu)$ regular and this proves Proposition 3.

The properties of ω_j^* imply that

$$\omega_j^*(\mu, x_1, x_2, w, z) = R_{2j}(\mu, x_1, x_2) + R_{3j}(\mu, x_1, x_2) + R_{4j}(\mu, x_1, x_2, w, z) + $$
$$+ R_{5j}(\mu, x_1, x_2, w, z) + R_j(\mu, x_1, x_2, w, z), \qquad j = 1, 2, \dots, n,$$

where

$$R_{2j}(\mu, x_1, x_2) = r_{20}^j(\mu)x_1^2 + r_{11}^j(\mu)x_1x_2 + r_{02}^j(\mu)x_2^2\,,$$

$$R_{3j}(\mu, x_1, x_2) = r_{30}^j(\mu)x_1^3 + r_{12}^j(\mu)x_1^2x_2 + r_{21}^j(\mu)x_1x_2^2 + r_{03}^j(\mu)x_2^3\,,$$

$$r_{ik}^j \in C^{r-3)} \text{ on } U'\,,$$

$$R_{4j}(\mu, x_1, x_2, w, z) = \sum_{i=1}^{p^*} (c_{20}^j w_i^2 + c_{01}^j w_i x_1 + c_{02}^j w_1 x_2) +$$
$$+ \sum_{i=1}^{q^*} (d_{20}^j z_i^2 + d_{01}^j z_i x_1 + d_{02}^j z_i x_2)\,,$$

$$R_{5j}(\mu, x_1, x_2, w, z) = \sum_{i=1}^{p^*} (c_{30}^j w_i^3 + c_{21}^j w_i^2 x_1 + c_{22}^j w_i^2 x_2 + c_{12}^j w_i x_1^2 +$$
$$+ c_{13}^j w_i x_2^2) + \sum_{i=1}^{q^*} (d_{30}^j z_i^3 + d_{21}^j z_i^2 x_1 + d_{22}^j z_i^2 x_2 + d_{12}^j z_i x_1^2 + d_{13}^j z_i x_2^2)\,,$$

where w_i, z_i are components of w, z respectively, dim $w = p^*$, dim $z = q^*$, $c_{ik} = c_{ik}(\mu)$, $d_{ik} = d_{ik}(\mu)$ are C^r functions on U', $R_j(\mu, x_1, x_2, w, z)$ contains only terms of orders higher than 3.

Lemma 6. *Let* $u(\mu, x_1, x_2)$, $v(\mu, x_1, x_2)$ *be the mappings defined by equations* (1), (2), (3) *and let* $u=(u_1, u_2, \ldots, u_{p^*})$, $v=(v_1, v_2, .., v_{q^*})$. *Then*

$$u_i(\mu, x_1, x_2)=u^i_{20}x_1^2+u^i_{11}x_1x_2+u^i_{02}x_2^2+u_i^*(\mu, x_1, x_2)\ ,$$

$$v_j(\mu, x_1, x_2)=v^j_{20}x_1^2+v^j_{11}x_1x_2+v^j_{02}x_2^2+v_j^*(\mu, x_1, x_2)\ ,$$

$i=1, 2, \ldots, p^*$; $j=1, 2, \ldots, q^*$, $u_{kl}=u_{kl}(\mu)$, $v_{kl}=v_{kl}(\mu)$ are C^{r-2} on U' *and these coefficients depend only on the elements of* H_1, H_2, α, β *and on* $D^2\omega_j^*(\mu, 0, 0)$, $j=1, 2, \ldots, n$, *but these do not depend on* $d^m\omega_j^*(\mu, 0, 0)$, $j=1, 2, \ldots, n$, $m>2$; $u_i^*(\mu, x_1, x_2)$, $v_j^*(\mu, x_1, x_2)$ *contain only terms of orders higher* than 2.

Proof. We shall prove the lemma for u^i_{20} only, because for the remaining coefficients the proof is similar.

$$u^i_{20}=\frac{\partial u_i(\mu, 0, 0)}{\partial x_1^2}\ .$$

The formula (1) implies that

$$\frac{\partial u_i(\mu, x_1, x_2)}{\partial x_1}=\int_{+\infty}^{0} e^{-H_1\sigma}\left[\frac{\partial Y_3}{\partial x_1}\frac{\partial \eta_1}{\partial x_1}+\frac{\partial Y_3}{\partial x_2}\frac{\partial \eta_2}{\partial x_1}+\right.$$

$$\left.+\frac{\partial Y_3}{\partial w}\left(\frac{\partial u}{\partial x_1}\frac{\partial \eta}{\partial x_1}+\frac{\partial u}{\partial x_2}\frac{\partial \eta_2}{\partial x_1}\right)+\frac{\partial Y_3}{\partial z}\left(\frac{\partial v}{\partial x_1}\frac{\partial \eta_1}{\partial x_1}+\frac{\partial v}{\partial x_2}\frac{\partial \eta_1}{\partial x_1}\right)\right]d\sigma\ .$$

It is obvious that $\frac{\partial u_i(\mu, 0, 0)}{\partial x_1}=0.$

$$\frac{\partial^2 u_i(\mu, x_1, x_2)}{\partial x_1^2}=\int_{+\infty}^{0} e^{-H_1}\left[\frac{\partial}{\partial x_1}\left(\frac{\partial Y_3}{\partial x_1}\right)\frac{\partial \eta_1}{\partial x_1}+\frac{\partial Y_3}{\partial x_1}\frac{\partial^2 \eta_1}{\partial x_1^2}+\right.$$

$$+\frac{\partial}{\partial x_1}\left(\frac{\partial Y_3}{\partial x_2}\right)\frac{\partial \eta_2}{\partial x_1}+\frac{\partial Y_3}{\partial x_2}\frac{\partial^2 \eta_2}{\partial x_1^2}+\frac{\partial}{\partial x_1}\left(\frac{\partial Y_3}{\partial w}\right)\left(\frac{\partial u}{\partial x_1}\frac{\partial \eta_1}{\partial x_1}+\frac{\partial u}{\partial x_2}\frac{\partial \eta_2}{\partial x_1}\right)+$$

$$+\frac{\partial Y_3}{\partial w}\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\frac{\partial \eta_1}{\partial x_1}+\frac{\partial u}{\partial x_2}\frac{\partial \eta_2}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\frac{\partial Y_3}{\partial z}\right)\left(\frac{\partial v}{\partial x_1}\frac{\partial \eta_1}{\partial x_1}+\frac{\partial v}{\partial x_2}\frac{\partial \eta_1}{\partial x_1}\right)+$$

$$\left.+\frac{\partial Y_3}{\partial z}\frac{\partial}{\partial x_1}\left(\frac{\partial v}{\partial x_1}\frac{\partial \eta_1}{\partial x_1}+\frac{\partial v}{\partial x_2}\frac{\partial \eta_1}{\partial x_1}\right)\right]d\sigma.$$

Since $dY_3(\mu, 0, 0, 0, 0)=0$, it is obvious that $\frac{\partial^2 u_i(\mu, 0, 0)}{\partial x_1^2}$ dependes on $\eta_1, \eta_2, \frac{\partial \eta_1}{\partial x_1}, \frac{\partial \eta_2}{\partial x_1}, d^2Y_3(\mu, 0, 0, 0, 0)$ only and does not depend on derivatives of η_1, η_2 of orders higher than 1. By [1,22.3] $\frac{\partial \eta_1(\mu, 0, 0)}{\partial x_1}$, $\frac{\partial \eta_2(\mu, 0, 0)}{\partial x_1}$ depend on the elements of H_1, H_2, α, β only and therefore $u^i_{20}(\mu)=\dfrac{\partial^2 u_i(\mu, 0, 0)}{\partial x_1^2}$ depends on the elements of H_1, H_2, α, β and it is a polynomial of the coefficients of $d^2\omega^*_i(\mu, 0, 0, 0, 0)$ and it does not depend on $d^m\omega^*_i(\mu, 0, 0, 0, 0)$, $m>2$. The proof is complete.

For the simplicity of computations, we shall suppose that $\dim w=p=1$, $\dim z=0$. In a general case the procedure is the same. Let $u(\mu, x_1, x_2)=u_{20}x_1^2+u_{11}x_1x_2+u_{02}x_2^2+u^*(\mu, x_1, x_2)$, where $u_{ik}=u_{ik}(\mu)\in C^r$, $u^*(\mu, x_1, x_2)$ contains only terms of orders higher than 2. Then

$$\omega^*_i(\mu, x_1, x_2, u(\mu, x_1, x_2))=R_{2i}(\mu, x_1, x_2)+R_{3i}(\mu, x_1, x_2)+$$
$$+R_{4i}(\mu, x_1, x_2, u(\mu, x_1, x_2))+$$
$$+R_{5i}(\mu, x_1, x_2, u(\mu, x_1, x_2))+R_i(\mu, x_1, x_2, u(\mu, x_1, x_2)),$$

where

$$R_{4i}(\mu, x_1, x_2, u(\mu, x_1, x_2))=c^i_{20}u^2(\mu, x_1, x_2)+c^i_{01}u(\mu, x_1, x_2)x_1+$$
$$+c^i_{02}u(\mu, x_1, x_2)x_2=c^i_{01}(u_{20}x_1^2+u_{11}x_1x_2+u_{02}x_2^2)x_1+c^i_{02}(u_{20}x_1^2+u_{11}x_1x_2+u_{02}x_2^2)x_2+$$
$$+\text{term of orders higher than 3, i.e.}$$

$$R_{4i}(\mu, x_1, x_2, u(\mu, x_1, x_2))=c^i_{01}u_{20}x_1^3+(c^i_{01}u_{11}+c^i_{02}u_{20})x_1^2x_2+(c^i_{01}u_{02}u_{11})x_1x_2^2+$$
$$+c^i_{02}u_{02}x_1^3+$$
$$+\text{term of orders higher than 3.}$$

$R_{5i}(\mu, x_1, x_2, u(\mu, x_1, x_2))$ contains only terms of orders higher than 4. Therefore

$$\omega^*_i(\mu, x_1, x_2, u(\mu, x_1, x_2))=R^*_{2i}(\mu, x_1, x_2)+R^*_{3i}(\mu, x_1, x_2)+R^*_i(\mu, x_1, x_2),$$

where $R^*_{2i}(\mu, x_1, x_2)=R_{2i}(\mu, x_1, x_2)$,
$R_{3i}(\mu, x_1, x_2)=s^i_{30}x_1^3+s^i_{21}x_1^2+s^i_{12}x_1x_2^2+s^i_{03}x_2^2$, $s^i_{30}=r^i_{30}+c^i_{01}u_{20}$,
$s^i_{21}=r^i_{21}+c^i_{01}u_{11}+c^i_{02}u_{20}$, $s^i_{12}=r^i_{12}+c^i_{01}u_{02}+c^i_{02}u_{11}$, $s^i_{03}=r^i_{03}+c^i_{02}u_{02}$
and $R^*_i(\mu, x_1, x_2)$ contains only terms of orders higher than 5. Then

$$\Phi_1(\mu, x_1, x_2)=P_2(\mu, x_1, x_2)+P_3(\mu, x_1, x_2)+P(\mu, x_1, x_2),$$
$$\Phi_2(\mu, x_1, x_2)=Q_2(\mu, x_1, x_2)+Q_3(\mu, x_1, x_2)+Q(\mu, x_1, x_2),$$

where

$$P_2(\mu, x_1, x_2)=a_{20}x_1^2+a_{11}x_1x_2+a_{02}x_2^2,$$
$$P_3(\mu, x_1, x_2)=a_{30}x_1^3+a_{21}x_1^2x_2+a_{12}x_1x_2^2+a_{03}x_2^3,$$
$$Q_2(\mu, x_1, x_2)=b_{20}x_1^2+b_{11}x_1x_2+b_{02}x_2^2,$$
$$Q_3(\mu, x_1, x_2)=b_{30}x_1^3+b_{21}x_1^2x_2+b_{12}x_1x_2^2+b_{03}x_2^3,$$

$a_{ik}=a_{ik}(\mu)=\sum_{j=1}^{n}\beta_{1j}r^j_{ik}$, $b_{ik}=b_{ik}(\mu)=\sum_{j=1}^{n}\beta_{2j}s^j_{ik}$ for $(i, k)=(2, 0)$, $(1, 1)$, $(0, 2)$ and

$a_{ik}=a_{ik}(\mu)=\sum_{j=1}^{n}\beta_{1j}s^j_{ik}$, $b_{ik}=b_{ik}(\mu)=\sum_{j=1}^{n}\beta_{2j}s^j_{ik}$ for $(i, k)=(3, 0), (2, 1), (1, 2), (0, 3)$.

We have proved that only a_{30}, b_{30} depend on r_{30} and only a_{03}, b_{03} depend on r_{03}.

If r_0 is a sufficiently small positive number, we can define the function d: $[0, r_0)\to R^1$ in the following way: For $0\leqq\bar{x}_1<r_0$, $d(\bar{x}_1)=\bar{y}_1$, where the point $(\bar{y}_1, 0)$ is the point of the first intersection of the trajectory of the system

$$(\Phi)\qquad \begin{aligned}\dot{x}_1&=\Phi_1(0, x_1, x_2),\\ \dot{x}_2&=\Phi_2(0, x_1, x_2)\end{aligned}$$

through the point $(\bar{x}_1, 0)$ with the x_1-axis. This trajectory intersects the x_1-axis at least at one point different from $(\bar{x}_1, 0)$ because the point $(0, 0)$ is a focus of the system (Φ). By [2, IX] $d'''(0)=3!\alpha_3$, where

$$\alpha_3=\frac{\pi}{4\beta}[3(a_{30}+b_{03})+a_{12}+b_{21}]$$
$$-\frac{\pi}{4\beta^2}[2(a_{20}b_{20}-a_{02}b_{02})-a_{11}(a_{02}+a_{20})+b_{11}(b_{02}+b_{20})]$$

(cf. [2, IX]).

Now we shall prove the following lemma.

Lemma 7. *Let $H_{03}^r(A, X)$ be the set of $\xi\in H_1^r(A, X)$ such that if $(a_0, x_0)\in Z_2(\xi)$, then $\alpha_3\neq 0$. Then this set is open and dense in $H_1^r(A, X)$.*

Proof. We can consider α_3 as a polynomial function of the variables r_{ik}^j and c_{ik}^j.

$$\alpha_3=\frac{\pi}{4\beta^2}\sum_{j=1}^{n}3[\beta_{1j}'s_{30}^j+\beta_{2j}'s_{03}^j]-\gamma=\frac{\pi}{4\beta^2}\sum_{j=1}^{n}3\beta_{1j}'(r_{30}^j+c_{01}^ju_{20})+\beta_{2j}'(r_{03}^j+c_{02}^ju_{02})]-\gamma,$$

where $\beta_{ij}'=\beta_{ij}(0)$, γ is a polynomial of the variables r_{ik}^j, $(i, k)=(2, 0)$, $(1, 1)$, $(0, 2)$, $(2, 1)$, $(1, 2)$, c_{ik}^j, $(i, k)=(2, 0)$, $(0, 1)$, $(0, 2)$, but it does not depend on r_{30}^j, r_{03}^j. Now the opennes is obvious, because α_3 depends continuously on r_{ik}^j, c_{ik}^j.

Density. Suppose that the set $H_{03}^r(A, X)$ is not dense. Then there is a $\xi\in H_1^r(A, X)$ such that $\alpha_3=\alpha_3(r_{30}^1, \dots, r_{03}^n, \dots)\equiv 0$ on some open set in the corresponding euclidean space. Therefore α_3 has all coefficients equal to zero. The formula for α_3 and the above computations show that in the expression of α_3 there is only one term of the form $K\beta_{1j}'r_{03}^j$, $j=1, 2, \dots, n$ $\left(K=\frac{3\pi}{4\beta^2}\right)$ and only one term of the form $K\beta_{2j}'r_{03}^j$, $j=1, 2, \dots, n$. The other terms do not contain the variables r_{03}^j and r_{30}^j. This implies that $\beta_{1j}(0)=\beta_{2j}(0)=0$ for all $j=1, 2, \dots, n$, but this contradicts Proposition 3.

From Lemmas 3—7 and from [2, p. 274] we obtain the following theorem.

Theorem 2. *There is an open and dense set $H_2^r(A, X)$ in $H^r(A, X)$ $(r\geqq 3)$ such that for every $\xi\in H_2^r(A, X)$*

(A) (1) *the set $Z_2(\xi)$ is finite.*

(2) *If* $(a_0, x_0) \in Z_2(\xi)$, *then the mapping* $\dot{\xi}_{a_0}(x_0)$ *has exactly one pair of conjugate pure imaginary eigenvalues.*

(B) *There is a neighbourhood* $U \times V$ *of* (a_0, x_0) *such that the point* (a_0, x_0) *divides the set* $C(\xi) \cap (U \times V)$ *into two components* K_1 *and* K_2, *where*

(1) *for* $(a, x) \in K_1$ *there is no closed orbit of* ξ_a *in* V,

(2) *for* $(a, x) \in K_2$ *there exists exactly one closed orbit of* ξ_a *in* V. *Moreover, if* din $X = 1$ *and* $\alpha_3 < 0$ $(\alpha_3 > 0)$, then this orbit is stable (unstable).

Example. Let us consider the following second order ordinary differential equation on R^1:

(S)
$$\dot{x} = v$$
$$\dot{v} = -x + \mu v + v(x^2 + v^2), \quad \mu \in R^1 ,$$

or in the form of the equation

$$\ddot{x} - \mu\dot{x} + x + \dot{x}[x^2 + (\dot{x})^2] = 0 .$$

Denote $\varrho = \frac{1}{2}(x^2 + v^2)$. The form of the system (S) implies that for ϱ we have the following differential equation:

$$\dot{\varrho} = v^2(\varrho^2 + \mu) .$$

This implies that ϱ is constant on the parabola $\varrho^2 + \mu = 0$ and this means that for $\mu < 0$ the circle γ: $x^2 + v^2 = -\mu$ is a closed orbit of the system (S). For $\mu < 0$ all eigenvalues of the matrix of the first derivatives of the right-hand side of (S) at (0, 0) have negative real parts and therefore the critical point (0, 0) of the system (S) is a stable focus and the closed orbit γ is unstable. For $\mu > 0$ the system (S) has no closed orbit and the point (0, 0) is an unstable focus, because all eigenvalues of the matrix of the first derivatives of the vectorfield (S) have positive real parts. Therefore we have the following pictures of trajectories:

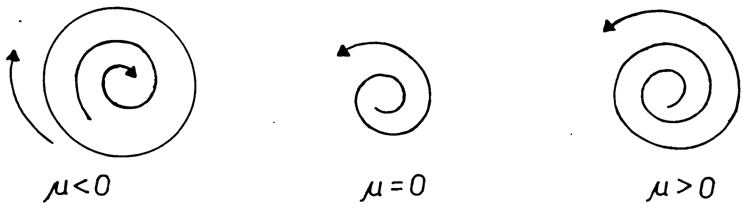

Fig. 3

It is easy to compute that for the equation (S) $\alpha_3 = \pi$ and therefore this case is generic.

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ТИПИЧНЫЕ СВОЙСТВА ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА НА ДИФФЕРЕНЦИРУЕМЫХ МНОГООБРАЗИЯХ

Милан Медведь

Резюме

В этой статье рассматриваются типичные бифуркации траекторий однопараметрических обыкновенных дифференциальных уравнений второго порядка в окрестности критических точек. Доказывается, что возможны два типичных случая: Матрица первых производных векторного поля имеет

1. одно собственное число равно 0
2. пару чисто мнимых собственных чисел.

Изучаются соответственные к случаям 1 и 2 биаурккации.