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Mathematica Slovaca, Vol. 26 (1976), No. 4, 257--269

Persistent URL: http://dml.cz/dmlcz/136125

MATHEMATICA SLOVACA

VOLUME 26 1976 NUMBER 4

ON A CLASS OF GENERALIZED HERMITE POLYNOMIALS

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INTRODUCTION

In this paper we study the polynomials orthonormal on the interval $(-\infty, +\infty)$ with the weight

$$\mathscr{L}(x)=(x^2)^\beta(a+x^2)^\alpha \exp(-x^2+2bx)$$

where α, β, a and b are real numbers satisfying the conditions in Section 2,1.

These polynomials in the present paper are a generalisation of the classical Hermite's polynomials and also of the polynomials introduced by J. Korous in [III]. The methods which we employ are different from those of Mr. Korous.

It is easy to pass from above defined polynomials to the polynomials orthonormal on the interval $(0, +\infty)$ with the weight

$$x^\beta(a+x)^\alpha e^{-x}$$

which are obviously generalized Laguerre's polynomials.

§ 1. Some fundamental properties of orthonormal polynomials

1,1. Notation. In this paper we use of the following notation:

(1,1a) $\mathcal{N}$ is the set of all non-negative integers, $n\in\mathcal{N}$. If $P(x)=\sum_{k=0}^{n} a_k x^k$, $a_n\neq 0$, then the degree of $P(x)$ is n. If $P(x)\equiv 0$, then degree of $P(x)$ is $-\infty$.

(1,1b) $$P(x)=\pi_n \Leftrightarrow \text{the degree of } P(x)\leqq n\,.$$

(1,1c) $$m\in\mathcal{N},\quad n\in\mathcal{N},\quad m\neq n \Rightarrow \delta_{m,n}=0,\quad \delta_{n,n}=1$$

(1,1d) $$\delta>0 \quad\text{and}\quad c_i>0 \quad (i=1, 2, \ldots)$$

are constants independent of n and of x, t, etc. in the interval in question.

(1,1e) $$k_i(x) \quad (i=1, 2\ldots)$$

are functions of x and n such that $|k_i(x)|<c_i$.

The numbering of c_i and $k_i(x)$ is independent in every section.

The integrals in this paper are those of Lebesgue.

1,2. Let $w(x)\geqq 0$ be a function and $\mathcal{J}$ an interval such that for every n

(1,2a) $$0<\int_{\mathcal{J}}|x^n|w(x)\mathrm{d}x<+\infty$$

Then for every n there exists one and only one polynomial.

(1,2b) $$P_n(x)=\sum_{k=0}^{n} a_k^{(n)}x^{n-k}, \quad a_0^{(n)}>0$$

such that

(1,2c) $$m\in\mathcal{N}, \quad n\in\mathcal{N}\Rightarrow\int_{\mathcal{J}}P_m(x)P_n(x)w(x)\mathrm{d}x=\delta_{m,n}$$

$P_n(x)$ is called the orthonormal polynomial with weight $w(x)$ on $\mathcal{J}$.

Proof. [II], p. 74.

1,3. We introduce the following notations: For $m\in\mathcal{N}$, $n\in\mathcal{N}$.

(1,3a) $$P_n(x, t)=\sum_{v=0}^{n}P_v(x)P_v(t)$$

(1,3b) $$p_{m,n}[f(t)]=\int_{\mathcal{J}}f(t)P_m(t)P_n(t)w(t)\mathrm{d}t$$

(1,3c) $$p'_{m,n}[f(t)]=\int_{\mathcal{J}}f(t)P'_m(t)P_n(t)w(t)\mathrm{d}t$$

(1,3d) $$P_{m,n}[x, f(t)]=\int_{\mathcal{J}}f(t)P_m(t)P_n(x, t)w(t)\mathrm{d}t$$

(1,3e) $$P'_{m,n}[x, f(t)]=\int_{\mathcal{J}}f(t)P'_m(t)P_n(x, t)w(t)\mathrm{d}t$$

If $L_n(x)$, $Q_n(x)$ etc are orthonormal polynomials, then $Q_n(x, t)$, $q_{m,n}[f(t)]$ etc are the expressions analogous to those in (1,3a)—(1,3e).

1,4. Put

(1,4a) $$q_0=0, \quad n>0\Rightarrow q_n=\frac{a_0^{(n-1)}}{a_0^{(n)}}$$

Then

(1,4b) $$[x-p_{n,n}(x)]P_n(x)=q_{n+1}P_{n+1}(x)+q_nP_{n-1}(x)$$

and

(1,4c) $$x\neq t\Rightarrow(x-t)P_n(x, t)=q_{n+1}[P_{n+1}(x)P_n(t)-P_n(x)P_{n+1}(t)]\,.$$

(1,4b) is a recurrence formula for the polynomials $P_n(x)$ and (1,4c) is the Christoffel formula.

1,5. Let $P(x)=\sum_{k=0}^{n} a_k x^k$, $a_n \neq 0$. Then

(1,5a) $$P(x)=\sum_{\nu=0}^{n} k_\nu P_\nu(x)$$

where

(1,5b) $$k_\nu = \frac{1}{a_0^{(0)}} p_{\nu,0}[P(t)]$$

Hence

(1,5e) $$P(x)=\frac{1}{a_0^{(0)}} P_{0,n}[x, P(t)]$$

Further

(1,5d) $$p_{0,n}[P(t)]=\frac{a_n}{a_0^{(0)} a_0^{(n)}}$$

and

(1,5e) $$m \in \mathcal{N}, \quad n \in \mathcal{N}, \quad m>n \Rightarrow p_{0,m}[P(t)]=0 .$$

Proof. [II], p. 74.

1,6. Employing the notation introduced in this section we have for $n>0$

(1,6a) $$q_n = p_{n,n-1}(t)$$

and

(1,6b) $$nq_n^{-1} = p'_{n,n-1}(1)$$

Proof. (1,5d).

§ 2. The polynomials $L_n(x)$

2,1. Let $a>0$, α, β be arbitrary real numbers such that

(2,1a) $$b \neq 0 \Rightarrow \beta=0, \quad b=0 \Rightarrow \beta > -\tfrac{1}{2}$$

Denote $\mathscr{E}=(-\infty, +\infty)$ and put

(2,1b) $$\mathscr{L}(x)=(x^2)^\beta (a+x^2)^\alpha \exp(-x^2+2bx)$$

Let

(2,1c) $$\mathscr{L}_n(x)=\sum_{k=0}^{n} l_k^{(n)} x^{n-k}, \; l_0^{(n)}>0$$

be orthonormal polynomial with weight $\mathscr{L}(x)$ on $\mathscr{E}$.

2,2. It follows from the definition that $\mathscr{L}_n(x)$ is an even (odd) function, if n is even (odd) and $b=0$.

2,3. Let $\mathscr{L}_n^{(\beta)}(x)$ for $\beta > -\frac{1}{2}$ be the orthonormal polynomial with weight $x^{2\beta}e^{-x}$ on the interval $(0, +\infty)$, i.e. $\mathscr{L}_n^{(\beta)}(x)$ is the normalized Laguerre polynomial of order 2β. Then for $b=0$ and $a=0$

(2,3a) $$\mathscr{L}_{2n}(x) = \mathscr{L}_n^{(\beta-1/2)}(x^2), \quad \mathscr{L}_{2n+1}(x) = x\mathscr{L}_n^{(\beta+1/2)}(x^2)$$

(See [I], p. 135).

2,4. The polynomials $\mathscr{L}_n(x)$ are a generalization of the polynomials of J. Korous His polynomials have the weight $(a+x^2)^\alpha e^{-x^2+2bx}$. (See [III] and also [IV] and [V]).

§ 3. Properties of the polynomials $\mathscr{L}_n(x)$

3,1. With the notation introduced in Section 2,2 we employ the following notation:

(3,1a) $$\lambda_0 = 0, \quad n>0 \Rightarrow \lambda_n = \frac{l_0^{(n-1)}}{l_0^{(n)}}$$

Taking into account the notation introduced in Section 1,3 we may write

(3,1b) $$x \neq t \Rightarrow (x-t)\mathscr{L}_n(x,t) = \lambda_{n+1}[\mathscr{L}_{n+1}(x)\mathscr{L}_n(t) - \mathscr{L}_n(x)\mathscr{L}_{n+1}(t)]$$

3,2. Making use of the symbol (1,3b) we obtain

(3,2a) $$l_{n,n}(t) = b + al_{n,n}[t(a+t^2)^{-1}] .$$

Proof. 1. Let $b=0$. Then $t(a+t^2)^{-1}\mathscr{L}_n^2(t)\mathscr{L}(t)$ is an odd function of t and (3,2a) is evident.

2. Let $b \neq 0$. By integrating by parts we obtain in virtue of (1,5e)

$$l_{n,n}(t) = -\tfrac{1}{2}\int_t \mathscr{L}_n^2(t)(a+t^2)^\alpha e^{2bt}\, \mathrm{d}(e^{-t^2}) =$$
$$= l'_{n,n}(1) + \tfrac{1}{2}l_{n,n}[2b + 2at(a+t^2)^{-1}] = b + al_{n,n}[t(a+t^2)^{-1}] .$$

3,3. We have

(3,3a) $$\lambda_n = \tfrac{1}{2}\sqrt{n} + \mathfrak{O}(1) \quad \text{for} \quad n \to +\infty .$$

Proof. Making use of (1,6a) we obtain by integrating by parts

(1) $$\lambda_n = l_{n,n-1}(t) =$$
$$= -\tfrac{1}{2}\int_t \mathscr{L}_n(t)\mathscr{L}_{n-1}(t)(t^2)^\beta(a+t^2)^\alpha e^{2bt}\, \mathrm{d}(e^{-t^2}) =$$
$$= \tfrac{1}{2}l'_{n,n-1}(t) + \tfrac{1}{2}l'_{n-1,n}(1) + \beta l_{n,n-1}(t^{-1}) +$$
$$+ al_{n,n-1}[t(a+t^2)^{-1}]$$

It is easy to prove

(2) $$l_{n,n-1}[t(a+t^2)^{-1}] = \mathfrak{O}(1)$$

Taking into account (1,6b) it follows from (1)

(3) $$\lambda_n = \tfrac{1}{2} n\lambda_n^{-1} + \beta\delta_n\lambda_n^{-1} + \mathfrak{O}(1)$$

(3,5a) is a consequence of (3).

3,4. The following formula is true

(3,4a) $$\mathscr{L}_n'(x) = 2\beta\delta_n x^{-1}\mathscr{L}_n(x) - 2\lambda_n\mathscr{L}_{n-1}(x) - \\ - 2a\mathscr{L}_{n,n-1}[x, t(a+t^2)^{-1}] .$$

where $\mathscr{L}_{n,n-1}[x, f(t)]$ is defined by (1,3d).

Proof. Taking into account the remark in Section 2,2 we have

(1) $$\mathscr{L}_n'(x) - 2\beta\delta_n x^{-1}\mathscr{L}_n(x) = \pi_{n-1} = \sum_{\nu=0}^{n-1} k_\nu \mathscr{L}_\nu(x)$$

$$k_\nu = l_{n,\nu}'(1) - 2\beta\delta_n l_{n,\nu}(t^{-1}) .$$

By integration by parts we deduce

$$k_\nu = -l_{n,\nu}\left[\frac{\mathrm{d}}{\mathrm{d}t} lg[\mathscr{L}(t) - 2\beta\delta_n t]^{-1}\right] =$$

$$= 2l_{n,\nu}[t - at(a+t^2)^{-1}] .*$$

Hence

(2) $$k_{\nu-1} = 2\lambda_n - 2al_{n,n-1}[t(a+t^2)^{-1}]$$

and

(3) $$\nu < n-1 \Rightarrow k_\nu = -2al_{n,\nu}[t(a+t^2)^{-1}] .$$

(3,4a) is a consequence of (1), (2) and (3).

3,5. Making use of the notation in the Section 1,3 we have

(3,5a) $$\lambda_n^{-1}(a+x^2)\mathscr{L}_{n,n-1}[x, (a+t^2)^{-1}] = \\ = l_{n,n-1}[(x+t)(a+t^2)^{-1}]\mathscr{L}_n(x) - \\ - l_{n,n}[(x+t)(a+t^2)^{-1}]\mathscr{L}_{n-1}(x) .$$

* n even $\Rightarrow \nu$ odd $\Rightarrow t^{-1}\mathscr{L}_\nu(t) = \pi_{\nu-1}$.

Proof. Since $\mathscr{L}_{n,n-1}(x, t) = \pi_{n-1}$ with respect to the variable t and

$$(a+t^2)^{-1} = (a+x^2)^{-1} - (x-t)(x+t)(a+x^2)^{-1}(a+t^2)^{-1}$$

we deduce (3,5a) by means of (1,4c)

3,6. The following formula similar to (3,5a) is true

(3,6a)
$$\lambda_n^{-1}(a+x^2)\mathscr{L}_{n,n-1}[x, t(a+t^2)^{-1}] =$$
$$= l_{n,n-1}[(tx-a)(a+t^2)^{-1}]\mathscr{L}_n(x) -$$
$$- l_{n,n}[(tx-a)(a+t^2)^{-1}]\mathscr{L}_{n-1}(x)$$

Proof. Since

(3,6b)
$$t(a+t^2)^{-1} - x(a+x^2)^{-1} =$$
$$= (x-t)(tx-a)(a+x^2)^{-1}(a+t^2)^{-1}$$

(3,6a) follows by a similar argument as (3,5a).

3,7. For brevity put

(3,7a)
$$\psi(x, t) = (x+t)(a+t^2)^{-1}[(a+t^2)^{-1} + (a+x^2)^{-1}]$$

Then

(3,7b)
$$\lambda_n^{-1}(a+x^2)\mathscr{L}_{n,n-1}[x, (a+t^2)^{-2}] =$$
$$= l_{n,n-1}[\psi(x, t)]\mathscr{L}_n(x) - l_{n,n}[\psi(x, t)]\mathscr{L}_{n-1}(x)$$

Proof. Since

$$(a+t^2)^{-2} = (a+x^2)^{-2} + (x-t)\psi(x, t) .$$

(3,7b) follows by a similar argument as (3,6a).

3,8. The following formula will be established

(3,8a)
$$\mathscr{L}'_{n,n-1}[x, t(a+t^2)^{-1}] = x(a+x^2)^{-1}\mathscr{L}'_{n-1}(x) +$$
$$+ \lambda_n(a+x^2)^{-1}\{l'_{n,n-1}[(tx-a)(a+t^2)^{-1}]\mathscr{L}_n(x) -$$
$$- [x + l_{n,n}[\psi_1(x,t)]\mathscr{L}_{n-1}(x)\} ,$$

where

(3,8b)
$$\psi_1(x, t) = (a+t^2)^{-1}\{[a - b - \alpha + \tfrac{1}{2} + (\alpha-1)(a+t^2)^{-1}]x -$$
$$- a[t - b - (\alpha-1)t(a+t^2)^{-1}] - \beta x\} .$$

Proof. 1. Making use of (3,6b), (1,5c) and (1,4c) we deduce

(1)
$$\mathscr{L}'_{n,n-1}[x, t(a+t^2)^{-1}] = x(a+x^2)^{-1}\mathscr{L}'_n(x) +$$
$$+ \lambda_n(a+x^2)^{-1}\{l'_{n,n-1}[(tx-a)(a+t^2)^{-1}]\mathscr{L}_n(x) -$$

$$- l'_{n,n}[(tx-a)(a+t^2)^{-1}]\mathscr{L}_{n-1}(x)\}$$

We notice that

(2) $$b=0 \Rightarrow l'_{n,n-1}[t(a+t^2)^{-1}]=0\,,$$
$$l'_{n,n}[(a+t^2)^{-1}]=0\,.$$

2. Integrating by parts we receive

(3) $$l'_{n,n}[t(a+t^2)^{-1}]=\tfrac{1}{2}\int_{\mathscr{I}} t(a+t^2)^{-1}\mathscr{L}(t)\,\mathrm{d}[\mathscr{L}_n^2(t)]=$$
$$=-\tfrac{1}{2}\,l_{n,n}[(a+t^2)^{-1}(-2t^2+2bt+1+2\beta)]-$$
$$-l_{n,n}[(\alpha-1)(a+t^2)^{-2}t^2]=$$
$$=1+\tfrac{1}{2}\,l_{n,n}[(a+t^2)^{-1}(2a-2bt-2\alpha+1-2\beta)]+$$
$$+(\alpha-1)a l_{n,n}[(a+t^2)^{-2}]$$

Similarly we have in virtue of (2)

(4) $$l'_{n,n}[(a+t^2)^{-1}]=-\tfrac{1}{2}\,l_{n,n}[(a+t^2)^{-1}(-2t+2b)]-$$
$$-l_{n,n}[(\alpha-1)(a+t^2)^{-2}t]=$$
$$=l_{n,n}[(a+t^2)^{-1}(t-b)]-(\alpha-1)l_{n,n}[t(a+t^2)^{-1}]$$

as $b\neq 0 \Rightarrow \beta=0$.

(3,8a) is a consequence of (1)—(4).

3,9. For brevity write

(3,9a) $$\mathscr{B}_n(x)=\alpha(a+x^2)^{-1}l_{n,n}[(tx-a)(a+t^2)^{-1}]$$

(3,9b) $$\mathscr{D}_n(x)=\beta\delta_n\lambda_n^{-1}x^{-1}-\alpha(a+x^2)l_{n,n-1}[(tx-a)(a+t^2)^{-1}]\,.$$

Then

(3,9c) $$\tfrac{1}{2}\,\lambda_n^{-1}\mathscr{L}'_n(x)=[1+\mathscr{B}_n(x)]\mathscr{L}_{n-1}(x)+\mathscr{D}_n(x)\mathscr{L}_n(x)$$

If we employ the notation for $1+\mathscr{B}_n(x)\neq 0$

(3,9d) $$d_n(x)=-\frac{\mathscr{B}_n(x)}{1+\mathscr{B}_n(x)}$$

(3,9e) $$e_n(x)=-\frac{\mathscr{D}_n(x)}{1+\mathscr{B}_n(x)}$$

then

(3,9f) $$1+\mathscr{B}_n(x)\neq 0 \Rightarrow 2\lambda_n\mathscr{L}_{n-1}(x)=$$
$$=d_n(x)\mathscr{L}'_n(x)+e_n(x)\mathscr{L}_n(x)$$

The proof follows from (3,4a) and (1,4c)

3,10. Put for brevity

(3,10a) $$f_n(x) = -l_{n,n}[\psi_1(t)] + \frac{\lambda_n^{-1}}{2} x d_n(x)$$

(3,10b) $$h_n(x) = l'_{n,n-1}[(tx-a)(a+t^2)^{-1}] - xe_n(x)$$

where $\psi_1(t)$ is defined by (3,8b)

Then

(3,10c) $$(a+x^2)\mathcal{L}'_{n,n-1}[x, t(a+t^2)^{-1}] =$$

$$= \tfrac{1}{2} x\mathcal{L}'_n(x) - f_n(x)\lambda_n\mathcal{L}_{n-1}(x) + h_n(x)\lambda_n\mathcal{L}_n(x)$$

The proof follows from (3,8a) and (3,9f)

§ 4. Some inequalities

4,1. Let $f(x)$ be a function such that

(4,1a) $$s = \sup_{x\in\mathscr{E}} |f(x)| < +\infty$$

Then

(4,1b) $$m \in \mathcal{N}, \quad n \in \mathcal{N} \Rightarrow l_{m,n}[|f(t)|] \leqq s$$

Proof. By means of Schwarz—Buňakovski's inequality we obtain

$$l_{m,n}[|f(t)|] \leqq s l_{m,m}^{1/2}(1) l_{n,n}^{1/2}(1) = s$$

4,2. Put

(4,2a) $$i_n = \max\{|\alpha| l_{n,n}[(1+t^2)^{-(1/2)}] |\alpha| l_{n,n-1}[(1+t^2)^{-(1/2)}]\}$$

Making use of the notation introduced in Section 1,1 we may write

(4,2b) $$i_n = c_1|\alpha|$$

Proof. (1,4a)

4,3. The following formulas are valid for $x \in \mathscr{E}$

(4,3a) $$\mathcal{L}_{n,n-1}[x, (a+t^2)^{-1}] =$$

$$= \sqrt{n}(1+x^2)^{-(1/2)} i_n[k_1(x)\mathcal{L}_n(x) + k_2(x)\mathcal{L}_{n-1}(x)]$$

(4,3b) $$\mathcal{L}_{n,n-1}[x, t(a+t^2)^{-1}] =$$

$$= \sqrt{n}(1+x^2)^{-(1/2)} i_n[k_3(x)\mathcal{L}_n(x) + k_4(x)\mathcal{L}_{n-1}(x)]$$

(4,3c) $$\mathcal{L}_{n,n-1}[x, (a+t^2)^{-1}] =$$

$$= \sqrt{n}(1+x^2)^{-1} i_n [k_5(x)\mathcal{L}_n(x) + k_6(x)\mathcal{L}_{n-1}(x)]$$

Proof. (3,3a), (3,5a), (3,6a) and (3,7b) in connection with (4,1a).

4,4. The following equations are valid for $x \in \mathscr{E}$ for the expressions defined in Section 3,9.

(4,4a) $$\mathscr{B}_n(x) = \alpha(1+x^2)^{-(1/2)} i_n k_1(x)$$

(4,4b) $$\mathscr{D}_n(x) - \beta\delta_n\lambda_n x^{-1} = \alpha(a+x^2)^{-(1/2)} i_n k_2(x)$$

Further there exists $x_0 \geqq 0$ such that for $|x| > x_0$

(4,4c) $$d_n(x) = \alpha(a+x^2)^{-(1/2)} i_n k_3(x)$$

and

(4,4d) $$e_n(x) = \alpha(a+x^2)^{-(1/2)} i_n l_4(x)$$

Proof. (3,9a), (3,9b), (3,9d) and (3,9e).

4,5. Let n be odd and

$$\beta \bar{\in} \left(\frac{1}{12}, \frac{3}{12}\right) \cup \left(\frac{7}{16}, \frac{9}{16}\right) = \mathscr{I}_0$$

Then

(4,5b) $$l_{n,n}(t^{-2}) = \mathfrak{O}(n) \quad \text{for} \quad n \to +\infty$$

Proof. Clearly

(1) $$l_{n,n}(t^{-2}) = 2\int_0^\infty \mathcal{L}_n^2(t) t^{2\beta-2}(a+t^2)e^{-t^2}\,dt$$

Since

$$t^{-1}\mathcal{L}_n(t) = \mathfrak{O}(1) \quad \text{for} \quad t \to 0-$$

we obtain from (1) by integration by parts for $\beta \bar{\in} \mathscr{I}_0$

(2) $$l_{n,n}(t^{-2}) = \frac{2}{1-2\beta}\{2l'_{n,n}(t^{-1}) + l_{n,n}[\alpha(a+t^2)^{-1} - 2]\}$$

If we employ formula (3,9c) i.e.

$$\mathcal{L}_n'(t) \equiv 2\lambda_n[1+\mathscr{B}_n(t)]\mathcal{L}_{n-1}(t) + 2\lambda\mathscr{D}_n(t)\mathcal{L}_n(t)$$

and the inequalities (4,4a) and (4,4b) i.e.

$$\mathscr{B}_n(t) = k_1(t), \quad \mathscr{D}_n(t) = \beta\lambda_n^{-1}t^{-1} + tk_2(t)$$

we obtain from (2)

(3) $$l_{n,n}(t^{-2})=\frac{4\beta}{1-2\beta}\,l_{n,n}(t^{-2})+$$

$$+\frac{4\lambda_n}{1-2\beta}\,\{l_{n,n}[t^{-1}k_3(t)+l_{n,n-1}[k_4(t)]\}\ .$$

Using Schwarz—Buňakovski's inequality we deduce by (3) and (3,3a)

(4) $$\left(\frac{6\beta-1}{2\beta-1}\right)^2 l_{n,n}(t^{-2})<[c_1l_{n,n}(t^{-2})+c_2]n$$

4,6. Let β satisfy (4,5a) and b be arbitrary. Then

(4,6a) $$\int_{\ell}\mathscr{L}_n'^2(t)\mathscr{L}(t)\,\mathrm{d}t=\mathfrak{O}(n)\quad\text{for}\quad n\to+\infty$$

Proof. Employing (3,9c) and (4,5b) we obtain

$$\int_{\ell}\mathscr{L}_n'^2(t)\mathscr{L}(t)\,\mathrm{d}t<c_1\lambda_n^2[l_{n-1,n-1}(1)+l_{n,n}(1)]+$$

$$+|\beta|\delta_n l_{n,n}(t^{-2})<c_3n$$

4,7. By using Schwarz—Buňakovski's inequality we deduce by (4,5a) for $\beta\in J_0$ and b arbitrary

(4,7a) $$l'_{n,n-1}[(1+t^2)^{-(1/2)}]=\mathfrak{O}(n\sqrt{i_n})\quad\text{for}\quad n\to+\infty$$

4,8. Let $f_n(x)$ and $h_n(x)$ be defined by (3,10a) and (3,10b) respectively. Then for β satisfying (4,5a), $x>|x_0|$, which is defined in Section 4,4, we have

(4,8a) $$f_n(x)=(1+x^2)^{-(1/2)}\iota_nk_1(x)$$

$$h_n(x)=(1+x^2)^{-(1/2)}\sqrt{i_n}k_2(x)$$

Proof. (4,3a), (4,3b), (4,3c), (4,4c), (4,4d) and (4,7a).

§ 5. Differential equations connected with the polynomials $\mathscr{L}_n(x)$

5,1. Notation:

(5,1a) $$\varphi(x,t)=(a+x^2)^{-1}[(2a+2\alpha+2\beta-1)(x+t)+$$

$$+2b(tx-a)+4a(1-\alpha)\psi(x,t)]$$

where $\psi(x,t)$ is defined by (3,7a).

Further in the notation of (3,10a) and (3,10b) put

(5,1b) $$(a+x^2)a_n^*(x)=l_{n,n-1}[\varphi(x,t)]-2h_n(x)$$

(5,1c) $$(a+x^2)j_n(x)=2f_n(x)+l_{n,n}[\varphi(x,t)]$$

and

(5,1d) $$a_n(x)=a_n^*(x)-\tfrac{1}{4}j_n(x)e_n(x)$$

and

(5,1e) $$(a+x^2)b_n(x)=\tfrac{1}{2}d_n(x)j_n(x)$$

Assertion. Let $\mathscr{B}_n(x)$ be defined by (3,9a) and $1+\mathscr{B}_n(x)\neq 0$. Then

(5,1f) $$\mathscr{L}^{-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}[\mathscr{L}_n'(x)\mathscr{L}(x)+$$
$$+[2n+2\beta\delta_n x^{-1}-2\alpha a_n^*(x)]\mathscr{L}(x)=2\alpha j_n(x)\mathscr{L}_{n-1}(x)$$

(5,1g) $$\mathscr{L}^{-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}[\mathscr{L}_n'(x)\mathscr{L}(x)-2\alpha b_n(x)\mathscr{L}_n'(x)+$$
$$+[2n+2\beta\delta_n x^{-2}-2\alpha a_n(x)]\mathscr{L}_n(x)=0\ .$$

Proof. 1. We see at once that

(1) $$\mathscr{A}_n(x)=\mathscr{L}^{-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}[\mathscr{L}_n'(x)\mathscr{L}(x)-2\alpha x(a+x^2)^{-1}\mathscr{L}_n'(x)+$$
$$+2(n-\beta\delta_n x^{-2})\mathscr{L}_n(x)=\pi_{n-1}$$

Therefore

(2) $$\mathscr{A}_n(x)=\sum_{\nu=0}^{n-1}\kappa_\nu\mathscr{L}_\nu(x)$$

where

(3) $$\kappa_\nu+2\alpha l_{n,\nu}'[t(a+t^2)^{-1}]+2\beta\delta_n l_{n,\nu}(t^{-2})=$$
$$=\textstyle\int_{\mathscr{C}}\mathscr{L}_\nu(t)\,\mathrm{d}[\mathscr{L}_n'(t)\mathscr{L}(t)]=\int_{\mathscr{C}}\mathscr{L}_n(t)\,\mathrm{d}[\mathscr{L}_\nu'(t)\mathscr{L}(t)]=$$
$$=\textstyle\int_{\mathscr{C}}\mathscr{L}_n(t)\mathscr{A}_\nu(t)\mathscr{L}(t)\,\mathrm{d}t+2\alpha l_{\nu,n}'[t(a+t^2)^{-1}]+2\beta\delta_n l_{n,\nu}(t^{-2})$$

Since $\mathscr{A}_\nu(x)\lesssim\pi_{n-1}$, it follows from (3)

(4) $$\kappa_\nu=2\alpha\{l_{\nu,n}'[t(a+t^2)^{-1}]-l_{n,\nu}'[t(a+t^2)^{-1}]=$$
$$=2\alpha\int_{\mathscr{C}}t(a+t^2)^{-1}\mathscr{L}_n^2(t)\mathscr{L}(t)\,\mathrm{d}\left[\frac{\mathscr{L}_\nu(t)}{\mathscr{L}_n(t)}\right]$$

2. Integrating by parts we deduce from (4)

(5)
$$\kappa_v + 4\alpha l'_{n,v}[t(a+t^2)^{-1}] = 2\alpha_{\iota n,} [(a+t^2)^{-1}(2t^2) -$$
$$-2bt - 2\beta - 1 - 2(\alpha-1)(a+t^2)^{-1}t^2] =$$
$$= 2\alpha(1-2a-2\alpha-2\beta) l_{n,v}[(a+t^2)^{-1}] -$$
$$-4\alpha b l_{n,v}[t(a+t^2)^{-1}] + 8a\alpha(\alpha-1) l_{n,v}[(a+t^2)^{-1}] .$$

3. It is easily seen from (5) that

(6)
$$\mathscr{A}_n(x) - -4\alpha\mathscr{L}'_{n,n-1}[x, t(a+t^2)^{-1} +$$
$$+ 2\alpha\mathscr{L}_{n,n-1}[x, \varphi_1(x, t)]$$

where

$$\varphi_1(x, t) -$$
$$= (a+t^2)^{-1}[1-2\alpha-2\beta-2a-2bt+4a(\alpha-1)(a+t^2)^{-1}]$$

Using (3,5a), (3,6a), (3,7b) and (3,8d) in connection with (3,9f) and (5,1a) we obtain in case $1 + \mathscr{B}_n(x) \neq 0$ the equation

$$(a+x^2)\mathscr{A}(x) = -2\alpha x\mathscr{L}'_n(x) + 2\alpha\{2f_n(x) +$$
$$+ l_{n,n}[\varphi(x, t)]\lambda_n\mathscr{L}_{\ 1}(x)\} +$$
$$+ \alpha\{2h_n(\quad '_n \ _- [\varphi(x, t)]\}\lambda_n\mathscr{L}_n(x)$$

If we substitute here from 3,9f) i e

$$\lambda_n\mathscr{L}_{n-1}(x) -_2 d_n(x)\mathscr{L}'(x) + \tfrac{1}{2} e_n(x)\mathscr{L}_n(x)$$

we obtain (5,1f) and (5,1g)

5,2. Let x_0 be the number defined in the Section 4,4. Then for $|x| > x_0$ and β satisfying (4,5a) the following equations are true:

(5,2a)
$$a_n^*(x) = (a + \quad ^{-1} n\sqrt{\iota_n} k_1(x)$$

(5,2b)
$$a_{;}\ :) -(1 \quad x^2)^{\ (1/} n\sqrt{i_n} k_2(x)$$

(5,2c)
$$b_n(x) = (^1 + x\ )^{\ (\ -} \iota_n k\ (x)$$

(5,2d)
$$b'_n(x) = (1+x\ )^{\ (1/2)} i_n k_4(x)$$

(5,2e)
$$j_n(x) = (1 \quad x^2)^{-(1/2)} i_n k_5(x)$$

Proof. (4,3a), (4,3b), (4,3c), (4,4c), (4,4d), (4,7c), (4,8), (4,8b).

REFERENCES

[1] SZEGÖ, G.: Ortogolaľnyje mnogočleny, Moscow 1962.
[2] KOROUS, J.: Vybrané stati z matematiky. Ortogonální funkce a ortogonální polynomy, Praha 1958.
[3] KOROUS, J.: O jistém zobecnění Hermiteových polynomu, Praha, 1959 (Sborník VŠD).
[4] KOROUS, J.: Über Reihenentwicklungen nach verallgemeinerten Laguerreschen Polynomen mit drei Parametern, Věstník Král. čes. společnosti nauk, Praha 1937.
[5] KOROUS, J.: Über Entwicklungen der Funktion einer reellen Veränderlichen in Reihe einer gewissen Klasse orthogonaler Polynome im unendlichen Intervalle, Praha 1937 (Věstník Král. čes. společnosti nauk).

Received November 16. 1971.

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О КЛАССЕ ОБОБЩЕННЫХ ЭРМИТОВЫХ МНОГОЧЛЕНОВ

Франтишек Пуховски

Резюме

Исследуются многочлены $\mathscr{L}_n(x)$, $n=0, 1, 2, \ldots$, ортонормальные в интервале $(-\infty, +\infty)$, учитывая функцию

$$\mathscr{L}(x)=(a+x^2)^\alpha(x^2)^\beta e^{-x^2+2bx};$$

где $a<0$, b, α, β-вещественные константы такие, что

$$b\neq 0 \Rightarrow \beta=0, \quad b=0 \Rightarrow \beta>-\tfrac{1}{2}.$$

Быводится отношение (3, 4a) для $\mathscr{L}_n'(x)$ и два линейных дифференциальных уравнения (5,1f) и (5,1g) для многочленов $\mathscr{L}_n(x)$. Кроме того в § 4 выводятся разные неравенства для многочленов.