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POLARS AND X-IDEALS IN SEMIGROUPS

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In the first part of the paper the foundations of the theory of polars are generalized from lattice ordered groups (l-groups) to x-ideals in commutative semigroups (see [1]).

In the second part of the paper a characteristic of x-ideals of a finite character is given.

1.

Definition. *Let $(S,.)$ be a semigroup. A mapping x: $2^S \to 2^S$ that fulfils the conditions*

I. $A \subseteq S \Rightarrow A \subseteq A_x$,

II. $A, B \subseteq S,\ A \subseteq B_x \Rightarrow A_x \subseteq B_x$,

III. $A \subseteq S \Rightarrow S \, . \, A_x \subseteq A_x$,

IV. $A, B \subseteq S \Rightarrow A \, . \, B_x \subseteq (A \, . \, B)_x$,

is called an ideal-mapping and a set $A \subseteq S$ with $A_x = A$ is called an x-ideal in S. A system of all x-ideals in S, for the given ideal-mapping x, is called an x-system.

Remark. 1. From I. and II. it follows $A_{xx} = A_x$. 2. If $(S, .)$ is a semigroup, then for any $A, B \subseteq S$ we denote $A : B = \{c \in S : c \, . \, B \subseteq A\}$. With regard to [1], Th. 3 the condition IV. is equivalent to $(A_x : b)_x = A_x : b$, for each $A \subseteq S$, $b \in S$ if we suppose I. and II.

Examples. 1. If $(G, +)$ is a group, $a \circ b = -a - b + a + b$, then $(G, \circ)$ is a semigroup and a mapping x such that it maps every subset $A \subseteq G$ on the normal subgroup A_x in $(G, +)$ generated by A is an ideal-mapping.

2. If $(L, \vee, \wedge)$ is a distributive lattice, $a \, . \, b = a \wedge b$, then $(L, \circ)$ is a semigroup and a mapping x such that it maps every subset $A \subseteq L$ on the lattice-ideal A_x in $(L, \vee, \wedge)$ generated by A is an ideal-mapping.

3. If $(R, +, .)$ is a ring, then a mapping x such that it maps every subset $A \subseteq R$ on the ring-ideal A_x in $(R, +, .)$ generated by A is an ideal-mapping.,

1.1. *Let G be an l-group, A_l be a convex l-subgroup in G generated by $A \subseteq G$, for each $A \subseteq G$. Then l: $2^G \to 2^G$ is an ideal-mapping on the semigroup $(G, \circ)$*

where $a \circ b = |a| \wedge |b|$, *for each* $a, b \in G$. *Further, if* B *is an x-ideal in* $(G, \circ)$, *that is a subgroup in* G, *then* B *is a convex l-subgroup in* G.

Proof. Evidently $A \subseteq A_l$ and $A \subseteq B_l \Rightarrow A_l \subseteq B_l$. For each $a \in A$, $b \in B_l$, $A, B \subseteq G$ there is $a \circ b = |a| \wedge |b| \leq |b|$ and thus $A \,.\, B_l \subseteq B_l$. Now we prove that $A_l : g$ is a convex l-subgroup in G, for each $A \subseteq G$, $g \in G$:

If $a, b \in A_l : g$, $h \in G$, $|h| \leq |a|$, then $h \circ g = |h| \wedge |g| \leq |a| \wedge |g| = a \circ g \in A_l$ and $h \circ g \in A_l$, $h \in A_l : g$, $|-a| \wedge |g| = |a| \wedge |g| = a \circ g \in A_l$, i. e., $-a \in A_l : g$. Further, $(a + b) \circ g = |a + b| \wedge |g| \leq (|a| + |b| + |a|) \wedge |g| \leq (|a| \wedge |g|) + (|b| \wedge |g|) + (|a| \wedge |g|) \in A_l$, $(a + b) \circ g \in A_l$, $a + b \in A_l : g$, $(a \vee 0) \circ g = |a \vee 0| \wedge |g| \leq |a| \wedge |g| \in A_l$, $a^+ \in A_l : g$. Together $(A_l : g)_l = A_l : g$ and according to Remark 2. the mapping l defines an x-system on $(G, \circ)$.

Now, let B be an x-ideal in $(G, \circ)$, B be a subgroup in G. If $b \in B$, $g \in G$, $|g| \leq |b|$, then $|g| = |b| \wedge |g| = b \circ g \in B$, $b^+ = |b \vee 0| = |b| \wedge |b \vee 0| = b \circ (b \vee 0) \in B$.

Remark. We shall suppose that in this paper a semigroup is always commutative.

Definition. *Let* $(S, \,.\,, e)$ *be a commutative semigroup with a zero element, i.e.,* $s \,.\, e = e \,.\, s = e$, *for each* $s \in S$. *Then we define relations* δ^*, δ' *in* S:

$$x\delta^*y \Leftrightarrow x \,.\, y = e, \quad \text{for} \quad x, y \in S$$

$$x\delta'y \Leftrightarrow x \,.\, y = e, \quad \text{for} \quad x, y \in S,\ x \neq y$$

$$x\delta'x \Leftrightarrow x = e, \quad \text{for} \quad x \in S.$$

Further, $K^* = \{s \in S : s\delta^*k$, *for each* $k \in K\}$, $K' = \{s \in S : s\delta'k$, *for each* $k \in K\}$, $K^{**} = (K^*)^*$, $K'' = (K')'$. *A set* $K \subseteq S$ *with the property* $K^{**} = K (K'' = K)$ *is called a* δ^**-polar (a* δ'*-polar).*

Remark. 1. $K \subseteq K^{**}$, $K \subseteq K''$. 2. A zero element e in a semigroup S is contained in every x-ideal in S.

1.2. *Let* $(S, \,.\,, e)$ *be a commutative semigroup with a zero* e. *Then there holds*:

1. $A \subseteq B \subseteq S \Rightarrow A' \supseteq B', A^* \supseteq B^*$,
2. $A \subseteq S \Rightarrow A''' = A', A^{***} = A^*$,
3. $A \subseteq S \Rightarrow A'$ *and* A^* *are subsemigroups in* S,
4. $A \subseteq S \Rightarrow A' \subseteq A^*$,
5. $A \subseteq S \Rightarrow A^* \cap A^{**} \subseteq \{s \in S : s\delta^*s\}$,
6. $A \subseteq S \Rightarrow A' \cap A'' = A' \,.\, A'' = \{e\}$.

1.3. *Let* S *be a commutative semigroup with a zero* e. *For each* $A \subseteq S$ *put* $\pi^*A = A^{**}$. *Then* π^* *is an ideal-mapping.*

Proof. $A \subseteq A^{**}$ and $A \subseteq B^{**} \Rightarrow A^{**} \subseteq B^{****} = B^{**}$. Further, for each $s \in S$, $a \in A^{**}$, $b \in A^*$ there holds $(s \,.\, a) \,.\, b = s \,.\, (a \,.\, b) = s \,.\, e = e$, $s \,.\, a \in A^{**}$, $S \,.\, A^{**} \subseteq A^{**}$. If $s \in S$, $h \in A^*$, $A \subseteq S$, then $A^{**} : s = \{c \in S : c \,.\, s \in A^{**}\}$

and $e = (c \,.\, s) \,.\, h = c \,.\, (s \,.\, h)$, i.e., $c \in (s \,.\, A^*)^*$. If $c \in (s \,.\, A^*)^*$, $h \in A^*$, then $e = c \,.\, (s \,.\, h) = (c \,.\, s) \,.\, h$, i.e., $c \,.\, s \in A^{**}$ and $(A^{**} : s) = (s \,.\, A^*)^*$, $(A^{**} : s)^{**} = A^{**} : s$.

1.4. *If G is an l-group, then for each $a, b \in G$ the following assertions are equivalent:*

1. $|a| \wedge |b| = 0$, 2. $a\delta' b$, 3. $a\delta^* b$.

Definition. *Let $(S, \,.\,, e)$ be a semigroup (commutative) with a zero e. Then $\pi^*(S)$ $(\pi'(S))$ is the system of all δ^*-polars (δ'-polars) in S.*

Remark. If G is an l-group, then $\Gamma(G)$ denotes the system of all polars in G with respect to the relation δ:

$$a\delta b \Leftrightarrow |a| \wedge |b| = 0, \; a, b \in G.$$

1.5. Corollary. *If G is an l-group, then $\pi^*(G) = \pi'(G) = \Gamma(G)$ with respect to a semigroup operation $\circ$ $(a \circ b = |a| \wedge |b|$, $a, b \in G)$* on G.

Remark. Further, let us denote $a^* = \{a\}^*$, $a^{**} = \{a\}^{**}$, $a' = \{a\}'$, $a'' = \{a\}''$.

1.6. *If $(S, \,.\,, e)$ is a semigroup with a zero e, then the following assertions are equivalent:*

1. $a\delta^* b \Leftrightarrow a\delta' b$, *for each pair* $a, b \in S$,
2. $a \,.\, a = e \Rightarrow a = e$, *for each* $a \in S$,
3. $a\delta^* b \Rightarrow a^{**} \cap b^{**} = \{e\}$, *for each pair* $a, b \in S$,
4. $a^* \cap a^{**} = \{e\}$, *for each* $a \in S$.

Definition. *We say that a semigroup $(S, \,.\,, e)$ with a zero e has the property (E) if $a \,.\, a = e \Rightarrow a = e$, for each $a \in S$.*

Remark. A semigroup $(S, \,.\,, e)$ has the property (E) if and only if the relations δ^* and δ' are identical. We shall further suppose in this paper that (E) is valid; a δ^*-polar of S will be called a polar of S.

1.7. Theorem. *Let a commutative semigroup $(S, \,.\,, e)$ have the property (E), x be an ideal mapping on S. Then the following assertions are equivalent:*

1. $\{e\}_x = \{e\}$,
2. $\cap \{A_x : A \subseteq S\} = \{e\}$,
3. *Every polar A in S is the greatest x-ideal B_x in S with respect to $B_x \cap A' = \{e\}$,*
4. *Every polar in S is an x-ideal,*
5. $(A_x)'' = (A'')_x = A''$, $A \subseteq S$,
6. $(A_x)' = (A')_x = A'$, $A \subseteq S$.

Proof. $2 \Rightarrow 1$: From the fact that $e \in A_x$ for every $A \subseteq S$ it follows $\{e\}_x \subseteq \subseteq \cap \{A_x : A \subseteq S\} = \{e\}$.

$1 \Rightarrow 3$: If $p \in A_x$, $c \in A'$, then $c \,.\, p \in A' \,.\, A_x \subseteq (A' \,.\, A)_x = \{e\}_x = \{e\}$, i.e.,

$p \in A'' = A$, $A_x \subseteq A$ and A is an x-ideal in S. Further, if B_x is an x-ideal in $S, B_x \cap A' = \{e\}$, $b \in B_x$, $a \in A$, then $(b \,.\, c) \,.\, a = b \,.\, (c \,.\, a) = b \,.\, e = e$, $b \,.\, c \in A' \cap B_x = \{e\}$. It means that $b \in A''$, $B_x \subseteq A$.
$3 \Rightarrow 4$ evidently. $4 \Rightarrow 2$: $\{e\} \subseteq \cap \{A_x : A \subseteq S\} \subseteq A' \cap A'' = \{e\}$.
$4 \Rightarrow 5$: $(A'')_x = A''$, $(A_x)'' \supseteq A''$ and $A_x \subseteq A'' \Rightarrow (A_x)'' \subseteq A''$. Together $(A_x)'' = (A'')_x = A''$.
$5 \Rightarrow 4$, $6 \Rightarrow 4$ evidently. $4 \Rightarrow 6$: $(A')_x = A'$. Further, from 4 the property 5 follows and thus $(A_x)' = (A_x)''' = [(A_x)'']' = (A'')' = A''' = A'$.

1.8. *If $A, B, A_\lambda (\lambda \in \Lambda)$ are subsets in a commutative semigroup $(S, ., e)$ with the property (E), then* $(\bigcup_{\lambda \in \Lambda} A_\lambda)' = \bigcap_{\lambda \in \Lambda} A'$

Proof. $(\bigcup_{\lambda \in \Lambda} A_\lambda)' \subseteq \bigcap_{\lambda \in \Lambda} A'_\lambda$ (see 1.2., 1) and if $x \in \bigcap_{\lambda \in \Lambda} A'_\lambda$, $y \in \bigcup_{\lambda \in \Lambda} A$:, then $x \,.\, y = e$ and thus $\bigcap_{\lambda \in \Lambda} A'_\lambda \subseteq (\bigcup_{\lambda \in \Lambda} A_\lambda)'$.

1.9. Theorem. *The set $\pi(S)$ of all polars in a commutative semigroup $(S, ., e)$ with the property (E) is a Boolean algebra, where a complement of a polar A in S is A' and the order in $\pi(S)$ is defined by set-inclusion.*

Further, $\bigwedge_{\lambda \in \Lambda} A''_\lambda = \bigcap_{\lambda \in \Lambda} A''_\lambda$, $\bigvee_{\lambda \in \Lambda} A''_\lambda = (\bigcup_{\lambda \in \Lambda} A_\lambda)''$, *for each* $A_\lambda \subseteq S$, $\lambda \in \Lambda$, $A'' \vee B'' = (A'' \cup B'')'' = (A' \cap B')' = (A \cup B)''$, *for each* $A, B \subseteq S$, $A'' \wedge B'' = A'' \cap B'' = (A' \cup B')' = (A \,.\, B)''$, *for each* $A, B \subseteq S$, $A'' \wedge B'' = (A \cap B)''$, *for each x-ideals A, B in S.*

Proof. $S = \{e\}'$ is the greatest element in $\pi(S)$, $\{e\} = S'$ is the smallest element in $\pi(S)$. If $A_\lambda \in \pi(S)$ for $\lambda \in \Lambda$, then $(\bigcap_{\lambda \in \Lambda} A''_\lambda)'' = [\bigcap_{\lambda \in \Lambda} (A'_\lambda)']'' = (\bigcup_{\lambda \in \Lambda} A'_\lambda)' = \bigcap_{\lambda \in \Lambda} A''_\lambda$ and thus $\bigwedge_{\lambda \in \Lambda} A_\lambda = \bigcap_{\lambda \in \Lambda} A_\lambda$. Therefore $\pi(S)$ is a complete lattice and $\bigvee_{\lambda \in \Lambda} A_\lambda = (\bigcup_{\lambda \in \Lambda} A_\lambda)''$ and for each $A \in \pi(S)$ $A \wedge A' = A \cap A' = \{e\}$, $A \vee A' = (A \cup A')'' = (A' \cap A'')' = \{e\}' = S$.

Further, for every $A, B \subseteq S$ there is $A'' \vee B'' = (A'' \cup B'')'' = (A' \cap B')' = (A \cup B)''$, $A'' \wedge B'' = A'' \cap B'' = (A' \cup B')'$ – see 1.8. If $c \in A \,.\, B$, $d \in A'$ are arbitrary elements, then $c = a \,.\, b$ for suitable elements $a \in A$, $b \in B$ and $d \,.\, c = d \,.\, (a \,.\, b) = (d \,.\, a) \,.\, b = e \,.\, b = e$, i.e., $c \in A''$. From this $A \,.\, B \subseteq A''$ and similarly $A \,.\, B \subseteq B''$, thus $A \,.\, B \subseteq A'' \cap B''$, $(A \,.\, B)'' \subseteq A'' \cap B''$. For every $x \in A'' \cap B''$, $y \in (A \,.\, B)'$, $c \in A \,.\, B$ there is $(x \,.\, y) \,.\, c = x \,.\, (y \,.\, c) = x \,.\, e = e$, i.e., $x \,.\, y \in (A \,.\, B)'$ and for each $a \in A$, $b \in B$ we have $e = (x \,.\, y) \,.\, (a \,.\, b) = (x \,.\, y \,.\, a) \,.\, b$, $x \,.\, y \,.\, a \in B' \cap B'' = \{e\}$, $x \,.\, y \in A' \cap A'' = \{e\}$. Finally, $A'' \cap B'' \subseteq (A \,.\, B)''$ and $A'' \cap B'' = (A \,.\, B)''$. If A, B are x-ideals in S, then $(A \,.\, B)'' \subseteq (A \cap B)'' \subseteq (A'' \cap B'')'' = A'' \cap B''$.

Now we prove the distributivity of $\pi(S)$: If $A, B, C \in \pi(S)$, then $(A \vee B) \wedge \cup (A \,.\, C) \cup (A \,.\, B) \cup (B \,.\, C)]'' \subseteq [A \cup (B \cap C)]'' = A \vee (B \wedge C)$ – see the following Remark.

Remark. For every $A \subseteq S, a \in A'', b \in A', s \in S$ there holds that $b \,.\, (a \,.\, s) = (b \,.\, a) \,.\, s = e \,.\, s = e$, i.e., $A'' \,.\, S \subseteq A''$. From this $A'' \,.\, B'' \subseteq A'' \cap B''$.

2.

Definition. *Let x be an ideal mapping on a semigroup S. We say that x defines an x-system of finite character if $A_x = \cup \{N_x : N \subseteq A, \text{ card } N < \aleph_0\}$ for each $A \subseteq S$.*

2.1. *If G is an l-group, $a \,.\, b = |a| \wedge |b|$, for each $a, b \in G$ and $C(G)$ is a set of all convex l-subgroups in G, then $C(G)$ is an x-system of finite character on $(G, .)$.*

Remark. The set of all x-ideals on a semigroup forms a complete lattice with respect to set-inclusion (see [1], Prop. 1).

2.2. Theorem. *If $(S, .)$ is a semigroup, $\mathfrak{S}$ is a lattice of x-ideals, then the following assertions are equivalent:*
1. *$\mathfrak{S}$ is an x-system of finite character.*
2. *$\mathfrak{S}$ is the lattice of all subalgebras of an algebra.*
3. *The join of every upper directed set of x-ideals is an x-ideal.*

Proof. $1 \Rightarrow 2$: We consider an algebra (S, Ω), where Ω is the set of all n-ary operations fulfilling the condition: $\omega \in \Omega$, n-ary, $a_1, \dots, a_n \in S \Rightarrow a_1 \dots a_n\omega = b \in \{a_1, \dots, a_n\}_x$. Hence an x-ideal A_x in S is an algebra in (S, Ω) because for every $\omega \in \Omega$, n-ary, $a_1, \dots, a_n \in A_x$ there holds $a_1 \dots a_n\omega \in \{a_1, \dots, a_n\}_x \subseteq A_x$. Conversely every subalgebra P in (S, Ω) is an x-ideal in S. In fact for every finite set $N \subseteq P$ we have $N_x \subseteq P$ and thus $P_x = \cup \{N_x : N \subseteq P, N \text{ finite}\} \subseteq P, P_x = P$.
$2 \Rightarrow 3$: It follows from [2], Satz 1.
$3 \Rightarrow 1$: If $A \subseteq S$, then $A_x \supseteq \cup \{N_x : N \subseteq A, N \textit{ finite}\}$ and the set $\{N_x : N \subseteq A, N \textit{ finite}\}$ is upper directed, i.e., $A_x = \cup \{N_x : N \subseteq A, N \textit{ finite}\}$.

Definition. *Let A_x be an x-ideal in a semigroup $(S, .)$. The set $\sqrt{A_x} = \{a \in S:$ there exists a positive integer n, $a^n \in A_x\}$ is called a radical of A_x. If $A_x = \sqrt{A_x}$, then A_x is called an x-semiprimeideal. If every x-ideal is an x-semiprimeideal then an x-system is called an x-semiprimesystem.*

2.3. *If a commutative semigroup $(S, . ,e)$ has the property (E), then the set $\pi(S)$ of all polars in S is an x-semiprimesystem.*

Proof. $\pi(S)$ is an x-system (see 1.3) and according to [1], Prop. 11 it is sufficient for every $A \subseteq S$ to prove: $a^2 \in A'' \Rightarrow a \in A''$. If $a^2 \in A''$ for some $a \in S$ and some $A \subseteq S$, then for each $b \in A'$ we have $a^2 \,.\, b = e$ and $(a \,.\, b)^2 = a^2 \,.\, b^2 = (a^2 \,.\, b) \,.\, b = e \,.\, b = e$. From the property (E) it follows $a \,.\, b = e$, $a \in A''$.

Definition. *Let P_x be an x-ideal in a semigroup $(S, .)$. Then P_x is called:*
an irreducible x-ideal, if $P_x = R_x \cap Q_x$, $R, Q \subseteq S$ implies $P_x = R_x$ or $P_x = Q_x$;
a primary x-ideal, if $a, b \in S$, $a \,.\, b \in P_x$, $a \notin P_x$ implies the existence of a positive integer n such that $b^n \in P_x$;
a prime x-ideal, if $a, b \in S$, $a \,.\, b \in P_x$, $a \notin P_x$ implies $b \in P_x$;
a simple x-ideal, if $a, b \in S$, $a \,.\, b = e$, $a \notin P_x$ implies $b \in P_x$, where e is a zero in S.

Remark. The definition of prime, irreducible and primary x-ideals is taken over [1].

2.4. *If $(S, . ,e)$ is a commutative semigroup with the property (E), then every simple x-ideal is a prime x-ideal in S.*

Proof. Let P_x be a simple x-ideal in S. If $P_x = S$, then clearly P_x is a prime x-ideal. If $P_x \neq S$, $a \notin P_x$, $b \notin P_x$, $a \in S$, $b \in S$, $a \,.\, b \in P_x$, then $a' \subseteq P_x$, $b' \subseteq P_x$ and for each $c \in (P_x)'$ there holds $e = (a \,.\, b) \,.\, c = a \,.\, (b \,.\, c) = b \,.\, (a \,.\, c)$, $b \,.\, c \in a' \subseteq P_x$, $c \,.\, a \in b' \subseteq P_x$. It implies that $b \,.\, c$, $c \,.\, a \in (P_x)' \cap P_x = \{e\}$, $c \in (a' \cap b') \cap (P_x)' \subseteq (P_x)' \cap P_x = \{e\}$, i. e., $(P_x)' = \{e\}$, $P_x = S$, which is a contradiction.

2.5. Corollary. *For a commutative semigroup $(S, . ,e)$ with the property (E) and an x-semiprimesystem L in S, $P_x \in L$, the following assertions are equivalent:*

1. *P_x is a prime x-ideal,*
2. *P_x is an irreducible x-ideal,*
3. *P_x is a primary x-ideal,*
4. *P_x is a simple x-ideal.*

Proof. $1 \Leftrightarrow 2$: see [1], Prop. 14, $1 \Rightarrow 3$ is clear, $4 \Rightarrow 1$: see 2.4, $3 \Rightarrow 4$: For $a, b \in S$, $a \,.\, b = e$, $a \notin P_x$ there exists a positive integer n with the property $b^n \in P_x$. If $n = 1$, then $b \in P_x$. Suppose that $n > 1$. Let k be the minimal positive integer with the property $b^k \in P_x$. If $k > 2$, then there exists a positive integer m, $m < k$, $2m > k$. It implies $b^k \,.\, b^{2m-k} \in P_x$, because P_x is an x-ideal in S, i. e., $b^{2m} \in P_x$, $(b^m)^2 \in P_x$. From [1], Prop. 11 there follows $b^m \in P_x$. From this contradiction $k = 2$, $b^2 \in P_x$ follows and $b \in P_x$ again according to [1], Prop. 11.

2.6. The Krull—Stone Theorem ([1], Th. 12). *If $(S, .)$ is a commutative semigroup with an x-system, then for every $A \subseteq S$ there holds that $\sqrt{A_x} = \cap \{P_x : P_x$ is a prime x-ideal in S, $P_x \supseteq A_x\}$.*

Corollaries of the Krull—Stone Theorem:

2.7. *Let G be an l-group. Then there holds:*

1. *The set of all convex l-subgroups in G is an x-semiprimesystem and every convex l-subgroup A_l generated by a set A in G is an intersection of simple l-subgroups in G containing A.*

2. ([3], 2.3, 9) *Every polar A' in G is an intersection of all minimal simple l-subgroups in G not containing A.*

3. There exists an l-group G sich that the set of all l-ideals in G forms no x-system in G.

Proof. 1. It follows from 1.1, 2.5 and the definition of the x-semiprime-system.

2. According to 2.3 for every polar A' in G there is $\sqrt{A'} = A'$ and the rest follows from 2.5 and 2.6.

3. We suppose that the set of all l-ideals in G is an x-system in G. Then it is clearly an x-semiprimesystem and $\{0\}$ is an intersection of simple l-ideals in G. In case that G has no realization, it is impossible.

2.8. *If $(S, \cdot, e)$ is a commutative semigroup with the property (E), then every polar in S is an intersection of maximal polars in G.*

Proof. $\pi(S)$ is an x-semiprimesystem (see 2.3). Every polar in S is an intersection of simple polars in S (see 2.6). Now we prove that a polar P being a simple x-ideal in S is a maximal polar in S (i. e., a dual atom in $\pi(S)$). Namely, if $Q \in \pi(S)$, $Q \supset P$, $S \neq Q \neq P$, then $Q' \subset P \subset Q$ and $Q' = \{0\}$, $Q = G$, which is a contradiction.

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