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Jiří Zikmund

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COMPOSITE CONTROL OF THE n–LINK CHAINED MECHANICAL SYSTEMS

Jiří Zikmund

In this paper, a new control concept for a class of underactuated mechanical system is introduced. Namely, the class of n-link chains, composed of rigid links, non actuated at the pivot point is considered. Underactuated mechanical systems are those having less actuators than degrees of freedom and thereby requiring more sophisticated nonlinear control methods. This class of systems includes among others frequently used for the modeling of walking planar structures. This paper presents the stabilization of the underactuated n-link chain systems with a wide basin of attraction. The equilibrium point to be stabilized is the upright inverted and unstable position.

The basic methodology of the proposed approach consists of various types of partial exact linearization of the model. Based on a suitable exact linearization combined with the so-called "composite principle", the asymptotic stabilization of several underactuated systems is achieved, including a general n-link. The composite principle used herein is a novel idea combining certain fast and slow feedbacks in different coordinate systems to compensate the above mentioned lack of actuation.

Numerous experimental simulation results have been achieved confirming the success of the above design strategy. A proof of stability supports the presented approach.

Keywords: nonlinear systems, exact linearization, underactuated mechanical systems

AMS Subject Classification: 70E60, 70K42, 70F10, 37C75

1. INTRODUCTION

Effective control of underactuated systems is still an open problem. This paper deals with a new control strategy that can be used for a class of underactuated systems. The classification of simple underactuated systems can be found in [12]. The system under interest, a mechanical chain actuated in every joint except the pivot, consists of n-links connected by rotary or prismatic joints.

The control problem for this class of systems is very popular within the control community for its complex non-linear behavior and numerous control strategies that have been applied. Typically, two different problems are solved. The first problem investigates how to swing up such a given system (e. g. the Acrobot, or $(3, 4, \dots)$-link inverted pendulum) from an arbitrary initial position to their inverted position and stabilize them [1, 7]. The second main area concerns the control of their motion along its unstable equilibrium manifold [2, 5, 6, 11]. These approaches usually take advantage of different kind of switching schemes of the controllers thereby trying to extend their domain of attraction. The so-called composite control introduced here is the effective way how to stabilize such a system in its unstable inverted equilibrium and track the system around it. Practically the same set of linearizing coordinates as in [9] is used. Compared to [9], our approach does not require an invertibility of the coordinates transformation and generalizes the stability conditions and controller design.

The contribution of this paper is as follows. The stabilization of the inverted unstable equilibrium of the n-link system from a wide range of initial conditions without any switching scheme is developed. The control algorithm based on maximum and partial exact linearization of the n-link system dynamics has been derived in opposition to [9] where approximate linearization is chosen. That kind of control is called "composite" since the control strategy is a combination of two controllers computed from two partially exactly linearized coordinate systems. Both control terms act simultaneously. These two controllers are shown to provide local asymptotic stability and are easy to tune. In contrast to the standard control methods available for the control and stabilization of this class of underactuated systems, the stabilization law presented herein provides a large domain of attraction and does not include any switching scheme.

The paper is organized as follows. The modeling of the systems under interest is briefly presented in Section 2. Section 3 is devoted to the well-known partial exact linearization method. Section 4 presents the principle of the composite control and includes all necessary proofs. Throughout the paper, the theory is tested on the special case of the 2-link (Acrobot) example. In Section 5, some numerical results are given for the 2-link and 4-link systems.

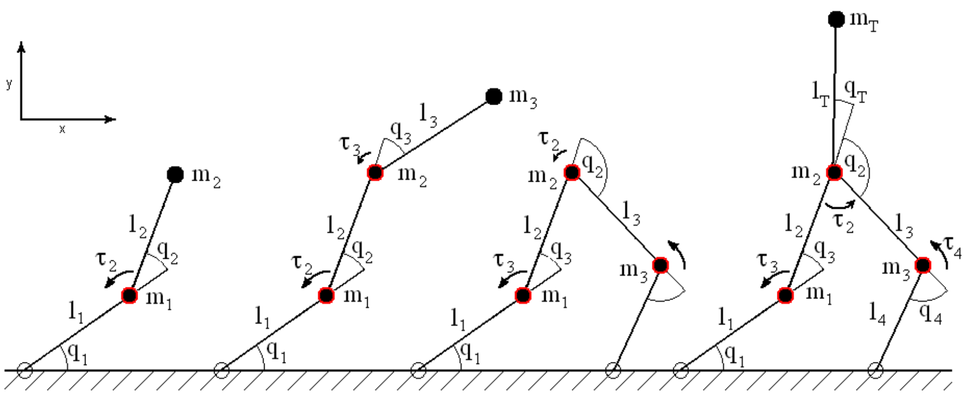

Fig. 1. n-link mechanical underactuated systems: from 2-link inverted pendulum to 5-link planar walking robot.

2. DYNAMIC MODEL OF n–LINK CHAINED SYSTEM

Simple n-link mechanical systems are studied, which consist of a chain of connected rigid, massless links. First of all, a set of generalized coordinates has to be chosen which completely describes the system. Let q_1 be the absolute coordinate defining the orientation of the systems in the Cartesian frame and relative positions are described by the configuration variables q_i.

The system is attached to a fixed pivot reference by the angle q_1. The second angular position q_2 corresponds to chosen centre-joint of the system. Its properties and function will be shown in the sequel, for this moment it is supposed to be the joint possessing the most power controlling a movement of the center of mass. For instance, for a walking robot, q_2 will represent the relative position of the hips. All joints (rotary or prismatic) between the links are actuated except the pivot point and can be numerated from pivot point, except the q_2 corresponding to centre-joint. All relative-angle position is described by the configuration variables q_i and all connections are supposed to be frictionless. Example of such a system is presented in Figure 2.

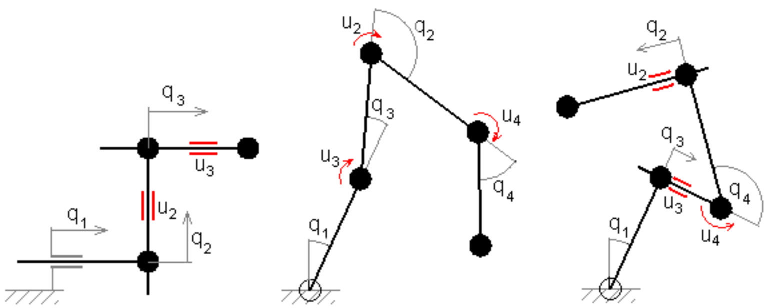

Fig. 2. Systems with prismatic and rotary joints, and notation.

To model these systems, the Lagrangian approach is being used. To do so, the Lagrangian equation is the difference between the kinetic and the potential energy:

$$\mathcal{L}(q,\dot{q}) = K - V = \frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q), \tag{1}$$

where q denotes the n-dimensional configuration vector and $D(q)$ is the inertia matrix; K is the kinetic energy and V is the potential energy of the system.

For independent external forces $u = (0, \tau_2, \tau_3, \dots, \tau_n)^T$ applied to the system, the Euler–Lagrange equations are the following

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} \\ \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \frac{\partial \mathcal{L}}{\partial q_n} \end{bmatrix} = u = \begin{bmatrix} 0 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix}. \tag{2}$$

Systems under our interest have $n-1$ nonzero parts of input vector, that means they have fewer actuators then degrees of freedom. Systems having this property are called underactuated [8]. Equations (2) lead to a dynamic equation in the form

$$D(q_2,\dots,q_n)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u \tag{3}$$

where

$$D(q) = \begin{pmatrix} d_{11}(q_2,\dots,q_n) & \cdots & d_{1n}(q_2,\dots,q_n) \\ \vdots & \ddots & \vdots \\ d_{n1}(q_2,\dots,q_n) & \cdots & d_{nn}(q_2,\dots,q_n) \end{pmatrix} \in \mathbb{R}^{n\times n} \quad \text{is the inertia matrix,}$$

$$C(q,\dot{q}) = \begin{pmatrix} c_{11}(q,\dot{q}) & \cdots & c_{1n}(q,\dot{q}) \\ \vdots & \ddots & \vdots \\ c_{n1}(q,\dot{q}) & \cdots & c_{nn}(q,\dot{q}) \end{pmatrix} \in \mathbb{R}^{n\times n} \quad \text{contains Coriolis and centrifugal terms,}$$

and $\quad G(q) = (g_1(q),\dots,g_n(q))^T \in \mathbb{R}^n \quad$ contains gravity terms.

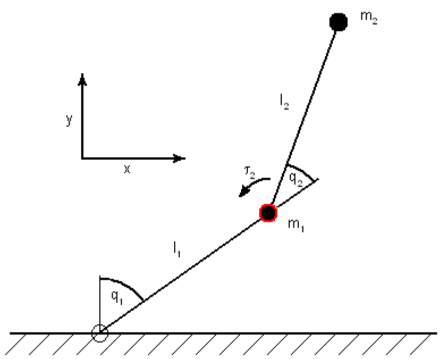

Fig. 3. Acrobot.

Example 1. 2-link system with rotary joints (Acrobot): the dynamic model in the form (3) is easily obtained using the method of Lagrange

$$D(q) = \begin{pmatrix} \theta_1 + \theta_2 + 2\theta_3\cos q_2 & \theta_2 + \theta_3\cos q_2 \\ \theta_2 + \theta_3\cos q_2 & \theta_2 \end{pmatrix}$$

$$C(q,\dot{q}) = \begin{pmatrix} -\theta_3\sin q_2\dot{q}_2 & -\theta_3\sin q_2\dot{q}_2 - \theta_3\sin q_2\dot{q}_1 \\ \theta_3\sin q_2\dot{q}_1 & 0 \end{pmatrix}$$

$$G(q) = \begin{pmatrix} -\theta_4 g \sin q_1 - \theta_5 g \sin(q_1 + q_2) \\ -\theta_5 g \sin(q_1 + q_2) \end{pmatrix}$$

where θ_i are constant coefficients

$$\begin{aligned} \theta_1 &= m_1 l_1^2 + m_2 l_1^2 \\ \theta_2 &= m_2 l_2^2 \\ \theta_3 &= m_2 l_1 l_2 \\ \theta_4 &= m_1 l_1 + m_2 l_1 \\ \theta_5 &= m_2 l_2 \end{aligned}$$

The system has a second-order nonholonomic constraint and a kinetic symmetry (that means the inertia matrix does not depend on variable q_1).

3. PARTIAL EXACT LINEARIZATION OF NONLINEAR SYSTEMS

The partial linearization method is based on the state transformation into a new system of coordinates that display linear dependence between some auxiliary outputs and new inputs introduced via a suitable feedback transformation. A detailed exposition on this topic may be found in [4].

From the theoretical point of view, a mechanical system dynamics is described by a $2n$-dimensional state space $(q, \dot{q})$, (3). Static state feedback linearization of a suitable output function with relative degree r yields a linear subsystem of dimension r. The relative degree[1] of the output function is defined as the number, how many times we have to differentiate this function before the any components τ_i of input vector u appears explicitly. In other words, maximum feedback linearization requires a so-called linearizing output function with maximal relative degree; then the problem is how to find it.

In the sequel, two methods of the partial exact linearization are presented for the class of system under interest.

First, it was shown in [8] that 2-body underactuated systems with input τ_2 can be partially linearized by the feedback

$$\tau_2 = \left(d_{11} - \frac{d_{12} d_{21}}{d_{11}}\right) v + \left(f_2 - \frac{d_{12} f_1}{d_{11}}\right) \tag{4}$$

into the normal form

$$\begin{aligned} \ddot{q}_1 &= J(q) v + R(q, \dot{q}) \\ \ddot{q}_2 &= v, \end{aligned} \tag{5}$$

where $J(q) = -d_{11}/d_{12}$ and $R(q, \dot{q}) = -f_2/d_{12}$ are composed of the corresponding parts in matrices $D(q)$ and $F(q, \dot{q}) = C(q, \dot{q})\dot{q} + G(q)$ in (3). This so-called *collocated* linearization (4) refers to a control that linearizes the actuated variable and its *non-collocated* version [8] refers to linearizing the passive degree of freedom.

[1] Definition of the relative degree is consistent with linear systems theory.

Notice that the feedback (4) is globally defined since ($d_{ii} > 0$) on the configuration manifold Q. It can easily be generalized to the n-link chained systems which are nonactuated at the pivot point.

$$\begin{aligned}
\ddot{q}_1 &= \sum_{k=2}^{n} J_k(q_2, \ldots, q_n) v_k + R(q, \dot{q}) \\
\ddot{q}_2 &= v_2 \\
&\vdots \\
\ddot{q}_n &= v_n
\end{aligned} \tag{6}$$

where $J_k = (d_{1,k}/d_{1,1})$, $d_{1,k}$ are the entries in the first row of inertia matrix D.

Later on, in [3, 10] was shown that if the generalized momentum conjugate to the cyclic variable q_1 is not conserved (as it is the case of systems under consideration), then there exists a set of outputs that defines a one-dimensional exponentially stable zero dynamics.

That means it is possible to find a set of functions $y(q, \dot{q})$ with relative degree 3 that transforms the original system (3) by a local transformation $z = T(q, \dot{q})$

$$\begin{aligned}
\xi_1 &= t_1(q, \dot{q}) \\
\xi_2 &= t_2(q, \dot{q}) = y \\
\xi_3 &= t_3(q, \dot{q}) = \dot{y} \\
\xi_4 &= t_4(q, \dot{q}) = \ddot{y} \\
\xi_5 &= q_3 \\
\xi_6 &= \dot{q}_3 \\
&\vdots \\
\xi_{2n-1} &= q_n \\
\xi_{2n} &= \dot{q}_n
\end{aligned} \tag{7}$$

into the new partially input/output linear system (8) with unobservable nonlinear dynamics of order 1. Some transformations $t_1(q, \dot{q})$ have to be arbitrarily chosen independently from $t_2(q, \dot{q})$, $t_3(q, \dot{q})$ and $t_4(q, \dot{q})$, $\dim(t_1(q, \dot{q})) = 1$ (in case of n-DOF systems $\dim(t_1(q, \dot{q})) = 1$).

$$\begin{aligned}
\dot{\xi}_1 &= \psi_1(\xi) + \psi_2(\xi) u \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= \alpha(\xi) u + \beta(\xi) = w
\end{aligned}$$

$$\begin{bmatrix} \dot{\xi}_5 \\ \dot{\xi}_6 \\ \vdots \\ \dot{\xi}_{2n-1} \\ \dot{\xi}_{2n} \end{bmatrix} = \begin{bmatrix} \dot{\xi}_6 \\ 0 \\ \vdots \\ \dot{\xi}_{2n} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_3 \\ \vdots \\ 0 \\ v_n \end{bmatrix}.$$

More generally, for n-link mechanical systems, nonactuated at the pivot, there are two independent functions having relative degree 3. The first candidate is

$$\sigma = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \sum_{k=1}^{n} d_{1,k}(q_2, \dots, q_n)\dot{q}_k \tag{8}$$

while its associated normalized 1-form

$$\mathrm{d}\omega = \mathrm{d}q_1 + \sum_{k=2}^{n} \frac{d_{1,k}(q_2, \dots, q_n)}{d_{1,1}(q_2, \dots, q_n)} \, \mathrm{d}q_k \tag{9}$$

and leads to the second function $p_2(q)$ having relative degree 3 with respect to v_2 and relative degree 2 with respect to $v_k; k = 3, \dots, n$. See [3] for details. This function $p_2(q)$ can be computed as follows

$$p_2 = q_1 + \int \frac{d_{1,2}(q_2, \dots, q_n)}{d_{1,1}(q_2, \dots, q_n)} \mathrm{d}q_2 \tag{10}$$

which leads to

$$\dot{p}_2 = \frac{\sigma}{d_{1,1}(q_2, \dots, q_n)} + \sum_{k=3}^{n} \beta_k(q_2, \dots, q_n)\dot{q}_k, \tag{11}$$

where

$$\beta_k(q_2, \dots, q_n) = \int \frac{\partial}{\partial q_k} \frac{d_{1,2}(\tau, q_3, \dots, q_n)}{d_{1,1}(\tau, q_3, \dots, q_n)} \, \mathrm{d}\tau - \frac{d_{1,k}(q_2, \dots, q_n)}{d_{1,1}(q_2, \dots, q_n)}. \tag{12}$$

Example 2. In case of the Acrobot, there are two independent functions with relative degree 3

$$\begin{aligned}
\sigma &= \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = (\theta_1 + \theta_2 + 2\theta_3 \cos q_2)\dot{q}_1 + (\theta_2 + \theta_3 \cos q_2)\dot{q}_2, \\
p &= q_1 - \frac{q_2}{2} - \frac{2\theta_2 - \theta_1 - \theta_2}{\sqrt{(\theta_1 + \theta_2)^2 - 4\theta_3}} \arctan\left(\sqrt{\frac{\theta_1 + \theta_2 - 2\theta_3}{\theta_1 + \theta_2 + 2\theta_3}} \tan \frac{q_2}{2} \right)
\end{aligned}$$

that transform the system into the form (8). The zero dynamics is used to investigate the internal stability when the corresponding output is forced to zero. For the special cases $y = Kp$ or $y = K\sigma$ the resulting zero dynamics is only critically stable. However, considering the output function $y = K_1 p(q) + K_2 \sigma(q, \dot{q})$, where K_1, K_2 are the real coefficients, one gets the following zero dynamics

$$\dot{p} + \frac{K_1}{K_2 d_{11}} p = 0 \tag{13}$$

which is asymptotically stable whenever K_1/K_2 is positive.

The maximum linearizations defined above are either only locally defined with many complex singular points, or yield a critically stable zero dynamics with tractable

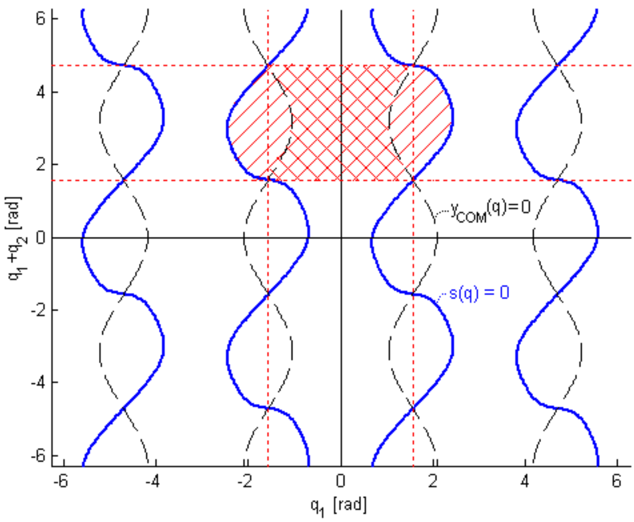

Fig. 4. Singularities $s(q) = 0$ and possible regular set of the coordinate change (14) of the Acrobot. Function $y_{COM}(q)$ is the vertical position of the center of gravity.

singular points. Our main result takes advantage of the linearization of the output $y = \sigma$. The corresponding linearizing transformation will be shown to have quite limited singularities. The set of these singular points is displayed in Figure 4. The singularities occur when the Acrobot's center of gravity passes through $x_g = 0$ which can obviously be excluded in all reasonable physical situations. To be more specific, using the set of functions with maximal relative degree, the following transformation

$$T: \quad \xi_1 = p, \quad \xi_2 = \sigma, \quad \xi_3 = \dot{\sigma}, \quad \xi_4 = \ddot{\sigma} \tag{14}$$

can be defined. Notice, that the following relation

$$\dot{p} \quad = \quad d_{11}(q_2)^{-1}\sigma \tag{15}$$

holds, $d_{11}(q_2)$ being the corresponding element of the inertia matrix D in (3). Applying (14), (15) to (3) we obtain the Acrobot's dynamics in the partially linearized form

$$\begin{aligned} \dot{\xi}_1 &= d_{11}(q_2)^{-1}\xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= \xi_4 \\ \dot{\xi}_4 &= \alpha(q,\dot{q})\tau_2 + \beta(q,\dot{q}) = w \end{aligned} \tag{16}$$

with the new coordinates ξ and the input w being well defined wherever $\alpha(q, \dot{q}) \neq 0$. To determine the region where such a transformation can be applied, let us express it in an explicit way. Namely, straightforward computations show that

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = T(q_1, q_2, \dot{q}_1, \dot{q}_2) := \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}, \tag{17}$$

$$\begin{bmatrix} T_1 \\ T_3 \\ T_2 \\ T_4 \end{bmatrix} = \begin{bmatrix} p(q_1, q_2) \\ \theta_4 g \sin q_1 + \theta_5 g \sin(q_1 + q_2) \\ \Phi_2(q_1, q_2) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \end{bmatrix}, \tag{18}$$

where p is given by (10) and Φ_2 by (23) later on. Further, denote the inverse transformations

$$\phi = \begin{bmatrix} \phi_1(\xi_1, \xi_3) \\ \phi_2(\xi_1, \xi_3) \end{bmatrix}, \text{ such that} \tag{19}$$

$$\begin{aligned} T_1(\phi_1(\xi_1, \xi_3), \phi_2(\xi_1, \xi_3)) &= \xi_1 \\ T_3(\phi_1(\xi_1, \xi_3), \phi_2(\xi_1, \xi_3)) &= \xi_3. \end{aligned} \tag{20}$$

It holds by (17,18) that

$$\frac{\partial[\xi_1, \xi_3, \xi_2, \xi_4]^T}{\partial[q^T, \dot{q}^T]^T} = \begin{bmatrix} \Phi_1(q_1, q_2) & 0 \\ \Phi_3(q, \dot{q}) & \Phi_2(q_1, q_2) \end{bmatrix}, \tag{21}$$

where $q := [q_1, q_2]^T$, $\Phi_3(q, \dot{q})$ is a certain (2×2) matrix of smooth functions whereas

$$\Phi_1(q_1, q_2) = \begin{bmatrix} 1 & \frac{\theta_2 + \theta_3 \cos q_2}{\theta_1 + \theta_2 + 2\theta_3 \cos q_2} \\ \begin{array}{l} \theta_4 g \cos q_1 + \\ \theta_5 g \cos(q_1 + q_2) \end{array} & \theta_5 g \cos(q_1 + q_2) \end{bmatrix}, \tag{22}$$

$$\Phi_2(q_1, q_2) = \begin{bmatrix} \theta_1 + \theta_2 + 2\theta_3 \cos q_2 & \theta_2 + \theta_3 \cos q_2 \\ \begin{array}{l} \theta_4 g \cos q_1 + \\ \theta_5 g \cos(q_1 + q_2) \end{array} & \theta_5 g \cos(q_1 + q_2) \end{bmatrix}. \tag{23}$$

Further, it obviously holds for (19), (20) that

$$\frac{\partial \phi(\xi_1, \xi_3)}{\partial[\xi_1, \xi_3]^T} = \Phi_1^{-1}(q_1, q_2) = \frac{1}{s(q)} \begin{bmatrix} \theta_5 g \cos(q_1 + q_2) & -\frac{\theta_2 + \theta_3 \cos q_2}{\theta_1 + \theta_2 + 2\theta_3 \cos q_2} \\ \begin{array}{l} -\theta_4 g \cos q_1 - \\ \theta_5 g \cos(q_1 + q_2) \end{array} & 1 \end{bmatrix}, \tag{24}$$

where

$$s(q) := \det\Phi_1 = \frac{\det\Phi_2}{d_{11}(q)} \tag{25}$$

$$= g d_{11}^{-1}(q)\big((\theta_1 + \theta_3 \cos q_2)\theta_5 \cos(q_1 + q_2) - (\theta_2 + \theta_3 \cos q_2)\theta_4 \cos q_1\big).$$

Moreover, the coordinate change (17) is locally invertible at each point where

$$s(q) \neq 0. \tag{26}$$

For any n-link systems unactuated in the pivot, define $\sigma = \partial\mathcal{L}/\partial\dot{q}_1$. From the first equation of (2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_1} - \frac{\partial\mathcal{L}}{\partial q_1} = 0 \tag{27}$$

where the Lagrange function

$$\mathcal{L}(q,\dot{q}) = K - V = \frac{1}{2}\dot{q}^T D(q_2,\ldots,q_n)\dot{q} - V(q) \tag{28}$$

has a kinetic symmetry with respect to q_1, i.e. $\frac{\partial K}{\partial q_1} = 0$. Then, for systems that are underactuated in the pivot point, one can easily see that function σ can be differentiated three times before explicit dependence on any input arises. In the next, σ will be the only function used for the maximum exact linearization, $t_2(q,\dot{q})$ in (8), due to the occurrence of a complex singular set attached to p. Nevertheless, the useful structural properties of function p will be used through the choice of $t_1(q,\dot{q}) = p$ in (8) which completes the transformation set (8).

4. COMPOSITE CONTROL

Maximum linearization is only locally defined and the output function $y = K\sigma$ will be used below since the set of singular points attached to the linearization problem is much less complex than the set of singular points obtained when linearizing output is $y = K_1 p + K_2\sigma$. Thus, the asymptotic stability of the zero dynamics is lost and it requires another control layer for the full stabilization. Nevertheless we gain a large domain of attraction to the equilibrium excluding any singular point.

To be more specific, the two-scale control combines the control of the horizontal position x_g of the center of mass of the whole system with the control of the angular positions of each link q_i. All stable inverted positions correspond to $x_g = 0$, when the center of the gravity is exactly above the pivot point. Moreover, the upright inverted position has all relative angular positions q_i equal to 0 too. Thanks to the relation $\dot{\sigma} = -1/\sum m_i x_g$ the stabilization of the center of gravity is obtained by stabilizing σ. The system in the stable inverted position has the coordinates $(\sigma,\dot{\sigma},\ddot{\sigma})$ equal to zero and $x_g = 0$. After reaching a neighborhood of the inverted position the influence of a second control loop, originally much slower then the first one, starts to have a decisive effect and controls the angular coordinates to the desired positions. During the stabilization of the angular positions, the first control loop ensures the stability of the overall system by keeping the center of gravity at zero. Finally, the center of gravity is at zero and all angular coordinates are at the desired position (in case of upright inverted position these are $q_i = 0$). The existence of unlimited combinations of angular positions q_i corresponding to $x_g = 0$ makes these simultaneous controls possible.

Example 3. In the case of 2 links with rotary joints and unactuated pivot point there is one input $u = (0, \tau_2)$ only. Coordinates transformation (8) defined as $t_1(q,\dot{q}), t_2(q,\dot{q}) = p(q), \sigma(q,\dot{q})$ leads to

$$\begin{aligned} \xi_1 &= p(q) \\ \xi_2 &= \sigma(q,\dot{q}) \\ \xi_3 &= \dot{\sigma}(q) \\ \xi_4 &= \ddot{\sigma}(q,\dot{q}), \end{aligned} \tag{29}$$

and the relation $p = d_{11}^{-1}\sigma$ yields the partially linearized form

$$\begin{aligned} \dot{\xi}_1 &= d_{11}^{-1}(T^{-1}(\xi))\xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= \xi_4 \\ \dot{\xi}_4 &= w. \end{aligned} \tag{30}$$

Recall, that $d_{11}(T^{-1}(\xi))$ is the appropriate entry of the inertia matrix D (3). Denote $d_{11}(T^{-1}(\xi))^{-1} = \mu(t)$, $\mu(t)$ can be considered as a bounded uncertainty with

$$a_{\min} \le \mu(t) \le a_{\max}, \tag{31}$$

where the bounds $a_{\min}, a_{\max}$ are positive reals given by the lower and upper limits of the corresponding part d_{11} of inertia matrix in (3) and can be computed as follows

$$\begin{aligned} a_{\min} &= \frac{1}{m_2(l_1+l_2)^2 + m_1 l_1^2 + I_1 + I_2} \\ a_{\max} &= \frac{1}{m_2(l_1-l_2)^2 + m_1 l_1^2 + I_1 + I_2}. \end{aligned} \tag{32}$$

Therefore, the first equation of (30) can be replaced by $\dot{\xi}_1 = \mu(t)\xi_2$ where $\mu(t)$ satisfies (31). Based on this, the main result of this paper formulated by the following theorem gives a constructive way how to stabilize such a system.

Theorem 1. Consider the system (30). Define the following feedback

$$w = -K_1\xi_1 - \Theta^3 K_2\xi_2 - \Theta^2 K_3\xi_3 - \Theta K_4\xi_4, \tag{33}$$

where any $K_1 > 0$ and $K_{2,3,4}$ such that the polynomial $s^3 + K_4 s^2 + K_3 s + K_2$ is Hurwitz. Then, there exists a sufficiently large $\Theta > 0$ such that the feedback (33) globally stabilizes the system (30).

Before proving Theorem 1, note that when Θ is large enough then $\sigma$and p represent the fast and the slow parts of the system dynamics.

P r o o f . First, let us transform linearly the system (8) into new coordinates $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4)$ defined as follows

$$\bar{\xi}_1 = \frac{K_1}{\Theta^3 K_2}\xi_1 \ \ ; \ \ \bar{\xi}_2 = \xi_2 + \frac{K_1}{\Theta^3 K_2}\xi_1 \ \ ; \ \ \bar{\xi}_3 = \Theta^{-1}\xi_3 \ \ ; \ \ \bar{\xi}_4 = \Theta^{-2}\xi_4 \ \ . \tag{34}$$

Denote $K(t) = \mu(t)\frac{K_1}{\Theta^3 K_2}$. In this new coordinates the system (8) takes the form

$$\dot{\bar{\xi}}_1 \ = \ K(t)\left(\bar{\xi}_1 - \bar{\xi}_2\right) \tag{35}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \bar{\xi}_2 \\ \bar{\xi}_3 \\ \bar{\xi}_4 \end{bmatrix} \ = \ \Theta A \begin{bmatrix} \bar{\xi}_2 \\ \bar{\xi}_3 \\ \bar{\xi}_4 \end{bmatrix} - K(t)\begin{bmatrix} \bar{\xi}_1 - \bar{\xi}_2 \\ 0 \\ 0 \end{bmatrix}. \tag{36}$$

Consider the following Lyapunov function $V(\bar{\xi})$ for the system (35), (36) defined as follows

$$V(\xi) = \frac{1}{2}(\bar{\xi}_1^2) + [\bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4]S[\bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4]^T \tag{37}$$

where the positive definite symmetric matrix S is the solution of the Lyapunov matrix equation

$$A^T S + SA \ = \ -I \tag{38}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_2 & -K_3 & -K_4 \end{bmatrix}; \qquad S := (s_{ij})_{i,j=1,2,3}. \tag{39}$$

Further, denote $\tilde{\xi} = [\bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4]^T$ and compute the full-time derivative of $V(\bar{\xi})$ along the trajectories (35), (36):

$$\begin{aligned} \frac{\mathrm{d}V(\bar{\xi})}{\mathrm{d}t} \ &= \ -K(t)\bar{\xi}_1^2 + K(t)\bar{\xi}_1\bar{\xi}_2 + \dot{\tilde{\xi}}^T S\tilde{\xi} + \tilde{\xi}S\dot{\tilde{\xi}}^T \\ &= \ -K(t)\bar{\xi}_1^2 + K(t)\bar{\xi}_1\bar{\xi}_2 + \left(\Theta A\begin{bmatrix} \bar{\xi}_2 \\ \bar{\xi}_3 \\ \bar{\xi}_4 \end{bmatrix}\right)^T S\tilde{\xi} + [\bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4]S\Theta A\begin{bmatrix} \bar{\xi}_2 \\ \bar{\xi}_3 \\ \bar{\xi}_4 \end{bmatrix} \\ &\quad -K(t)[\bar{\xi}_1 - \bar{\xi}_2, 0, 0]S\tilde{\xi} - K(t)\tilde{\xi}^T S\begin{bmatrix} -\bar{\xi}_1 - \bar{\xi}_2 \\ 0 \\ 0 \end{bmatrix} \\ &= \ -K(t)\bar{\xi}_1^2 - \Theta(\bar{\xi}_2^2 + \bar{\xi}_3^2 + \bar{\xi}_4^2) + K(t)\bar{\xi}_1\bar{\xi}_2 \\ &\quad -2K(t)(s_{11}\bar{\xi}_2 + s_{21}\bar{\xi}_3 + s_{31}\bar{\xi}_4)(\bar{\xi}_1 - \bar{\xi}_2). \end{aligned} \tag{40}$$

In other words, we have shown that

$$\begin{aligned} \frac{\mathrm{d}V(\bar{\xi})}{\mathrm{d}t} \ &= \ -K(t)\bar{\xi}_1^2 - \Theta(\bar{\xi}_2^2 + \bar{\xi}_3^2 + \bar{\xi}_4^2) + K(t)(\bar{\xi}_1\bar{\xi}_2(1 - s_{12}) + \bar{\xi}_1\bar{\xi}_3(-2s_{21}) \\ &\quad +\bar{\xi}_1\bar{\xi}_4(-2s_{31}) + \bar{\xi}_2^2(2s_{11}) + \bar{\xi}_2\bar{\xi}_3(2s_{21}) + \bar{\xi}_2\bar{\xi}_4(s_{31})) \end{aligned} \tag{41}$$

where $K(t)$ is given by

$$K(t) = \mu(t)\frac{K_1}{\Theta^3 K_2}. \tag{42}$$

Thus,

$$\frac{a_{\min}K_1}{\Theta^3 K_2} \leq K(t) \leq \frac{a_{\max}K_1}{\Theta^3 K_2} \tag{43}$$

where $a_{\min}$, $a_{\max}$ are given by (32), in particular $a_{\max} > a_{\min} > 0$. Denote

$$K_{\max} = \frac{a_{\max}K_1}{K_2}; \qquad K_{\min} = \frac{a_{\min}K_1}{K_2} \tag{44}$$

then obviously

$$K_{\max} \geq K_{\min} \geq 0. \tag{45}$$

Therefore, it holds

$$\begin{aligned}\frac{\mathrm{d}V(\bar{\xi})}{\mathrm{d}t} \quad = \quad & -\frac{K_{\min}}{\Theta^3}\bar{\xi}_1^2 - \Theta(\bar{\xi}_2^2 + \bar{\xi}_3^2 + \bar{\xi}_4^2) + \frac{K_{\max}}{\Theta^3}\big|\bar{\xi}_1\bar{\xi}_2(1 - s_{12}) - 2s_{21}\bar{\xi}_1\bar{\xi}_3 \\ & -2s_{31}\bar{\xi}_1\bar{\xi}_4 + 2s_{11}\bar{\xi}_2^2 + 2s_{21}\bar{\xi}_2\bar{\xi}_3 + 2s_{31}\bar{\xi}_2\bar{\xi}_4\big|.\end{aligned} \tag{46}$$

Note, that by construction, $K_{\min}$, $K_{\max}$, s_{11}, s_{21}, s_{31} are fixed and independent on Θ. Therefore

$$\begin{aligned}\frac{\mathrm{d}V(\bar{\xi})}{\mathrm{d}t} \quad \leq \quad & -\left(\frac{K_{\min}}{3\Theta^3}\bar{\xi}_1^2 + \frac{\Theta}{3}\bar{\xi}_2^2 - \frac{|1-s_{11}||\bar{\xi}_1||\bar{\xi}_2|}{\Theta^3}\right) \\ & -\left(\frac{K_{\min}}{3\Theta^3}\bar{\xi}_1^2 + \frac{\Theta}{2}\bar{\xi}_3^2 - \frac{2|s_{21}||\bar{\xi}_1||\bar{\xi}_3|}{\Theta^3}\right) \\ & -\left(\frac{K_{\min}}{3\Theta^3}\bar{\xi}_1^2 + \frac{\Theta}{2}\bar{\xi}_4^2 - \frac{2|s_{31}||\bar{\xi}_1||\bar{\xi}_4|}{\Theta^3}\right) \\ & -\left(\left(\frac{\Theta}{3} - 2|s_{11}|\right)\bar{\xi}_2^2 + \frac{\Theta}{2}\bar{\xi}_3^2 - \frac{2|s_{21}||\bar{\xi}_2||\bar{\xi}_3|}{\Theta^3}\right) \\ & -\left(\frac{\Theta}{3}\bar{\xi}_2^2 + \frac{\Theta}{2}\bar{\xi}_4^2 - \frac{2|s_{31}||\bar{\xi}_2||\bar{\xi}_4|}{\Theta^3}\right)\end{aligned}$$

each of these terms is a negative definite quadratic form with respect to two its arguments for $\Theta \geq 0$ large enough. Indeed, e. g. first of them has the matrix

$$\begin{bmatrix}\bar{\xi}_1 & \bar{\xi}_2\end{bmatrix}\begin{bmatrix} -\frac{K_{\min}}{3\Theta^3} & +\frac{|1-s_{11}|}{2\Theta^3} \\ +\frac{|1-s_{11}|}{2\Theta^3} & -\frac{\Theta}{3}\end{bmatrix}\begin{bmatrix}\bar{\xi}_1 \\ \bar{\xi}_2\end{bmatrix}. \tag{47}$$

This symmetric matrix has always negative trace $\left(-\frac{K_{\min}}{3\Theta^3} - \frac{\Theta}{3}\right)$ and its determinant is $\left(\frac{K_{\min}}{5\Theta^2} - \frac{(1-s_{11})^2}{4\Theta^6}\right)$, i. e. it is obviously positive for $\Theta^4 \geq \frac{3(1-s_{11})^2}{2K_{\min}}$. By analogy, one can treat remaining terms, i. e. the overall expression is a negative definite quadratic form of arguments $(\overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_3, \overline{\xi}_4)$. The standard version of the Lyapunov method for exponential stability, see e. g. Khalil [4], finishes the proof. □

The proof made above for 2-links system can easily be extended to the n-link case as follows. Any system having the mentioned properties (i. e. having $n-1$ actuators and nonactuated at pivot point) can be rewritten using (4), (8) into the following triangular-like form

$$
\begin{aligned}
\dot{\xi}_1 &= -\mathrm{d}_{11}^{-1}\xi_2 + \sum_{k=3}^{n} f_k(\xi_1, \xi_2, \ldots, \xi_n)\xi_{2k} \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= w \\
\dot{\xi}_5 &= \xi_6 \\
\dot{\xi}_6 &= v_3 \\
\vdots &\quad \vdots \\
\dot{\xi}_{2n-1} &= \xi_{2n} \\
\dot{\xi}_{2n} &= v_n
\end{aligned}
\tag{48}
$$

where $(\xi_5, \ldots, \xi_{2n})$ is a linear subsystem of dimension $(2n-4)$ having $(n-2)$ inputs. The right hand side in the first equation results from (12), in particular $f_3, \ldots, f_n$ are suitable smooth functions.

Theorem 2. Consider the n-link system (49). Let $(n-2)$ controllers $(v_3(\xi), \ldots, v_n(\xi))$ be chosen exponentially to stabilize the $(2n-4)$ dimensional linear subsystem with $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 0$. Then for any $K_1 > 0$ and $K_{2,3,4}$ such that the polynomial $s^3 + K_4 s^2 + K_3 s + K_2$ is Hurwitz, the feedback $w = -K_1\xi_1 - \Theta^3 K_2\xi_2 - \Theta^2 K_3\xi_3 - \Theta K_4\xi_4$ with sufficiently large Θ and the above feedbacks $(v_3(\xi), \ldots, v_n(\xi))$ locally stabilize the system (49).

P r o o f . By assumptions of Theorem 2, the linear part $\xi_5, \ldots, \xi_n$ of the triangular-like system

$$
\begin{aligned}
\dot{\xi}_1 &= -\mathrm{d}_{11}^{-1}\xi_2 + f(\xi_5, \ldots, \xi_n) \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= -K_1\xi_1 - \Theta^3 K_2\xi_2 - \Theta^2 K_3\xi_3 - \Theta K_4\xi_4 \\
\dot{\xi}_5 &= \xi_6 \\
\dot{\xi}_6 &= v_3 \\
\vdots &\quad \vdots
\end{aligned}
\tag{49}
$$

$$\begin{aligned} \dot{\xi}_{n-1} &= \xi_b \\ \dot{\xi}_n &= v_n \end{aligned}$$

can easily be stabilized by a set of linear feedbacks $v_i = K_{i1}\xi_{2i-1} + K_{i2}\xi_{2i}$, $i = 3, \dots, n$ where $K_{i1}, K_{i2} < 0$. It was shown in [4] that the asymptotic stability of the origin of the full closed-loop system follows from the asymptotic stability of its zero dynamics associated to $(q_3, \dots, q_n)$

$$\begin{aligned} \dot{\xi}_1 &= -\mathrm{d}_{11}^{-1}\xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= \xi_4 \\ \dot{\xi}_4 &= -K_1\xi_1 - \Theta^3 K_2\xi_2 - \Theta^2 K_3\xi_3 - \Theta K_4\xi_4. \end{aligned} \tag{50}$$

More precisely, the origin of (49) with the above v_i's is asymptotically stable if the origin of (50) is asymptotically stable. The stability of (50) is guaranteed by Theorem 1. □

Note that function $f(\xi_5, \dots, \xi_n)$ in (49) has a form $\sum_{i=1}^{n} f_i(\xi_1, \xi_3, \dots, \xi_{n-1}) \cdot \xi_{2i}$ where the last variable ξ_{2i} represents a multiplication by angular velocities q_i. To eliminate the influences of $f(\xi_5, \dots, \xi_n)$ the velocities of the fully linearized actuated coordinates $(q_3, \dots, q_n)$ have to be limited. Corresponding "slow" controllers are inferior with respect to "fast" control that stabilizes the subsystem $(\xi_1, \dots, \xi_4)$.

5. SIMULATION RESULTS

Examples in this section illustrate how the proposed control law locally stabilizes the systems under consideration. Two systems from our class of underactuated systems are chosen: the Acrobot presented below and the 4-links robot with rotary joints. The role of the center point of n-link system is presented as well.

Properties of the Acrobot are demonstrated below. The Figures 5 and 6 present its stabilization for a wide range of initial positions. Next, Figure 7 shows the influence of tuning a factor Θ in (33), and the last one Figure 8 shows the asymptotic stabilization into the upright position from a set of close initial conditions.

The dynamical model of the 4-link shown in Figure 9 is easily obtained via the Lagrange method but it is much larger than the model of the Acrobot and it is not extensively presented here.

The stabilizing controls w and (v_3, v_4) fulfill the conditions of Theorem 4 to asymptotically stabilize the 4-link system around its equilibrium point and provide the asymptotical stabilization in a neighborhood of its inverted position.

The control w can be expressed as a combination of the linearized inputs v_2, v_3, v_4, where $v_2 = \ddot{q}_2$, as

$$\dot{\xi}_4 = \frac{\mathrm{d}^4\sigma}{\mathrm{d}t^4} = h_1(\xi) + h_2(\xi)v_2 + h_3(\xi)v_3 + h_4(\xi)v_4 = w. \tag{51}$$

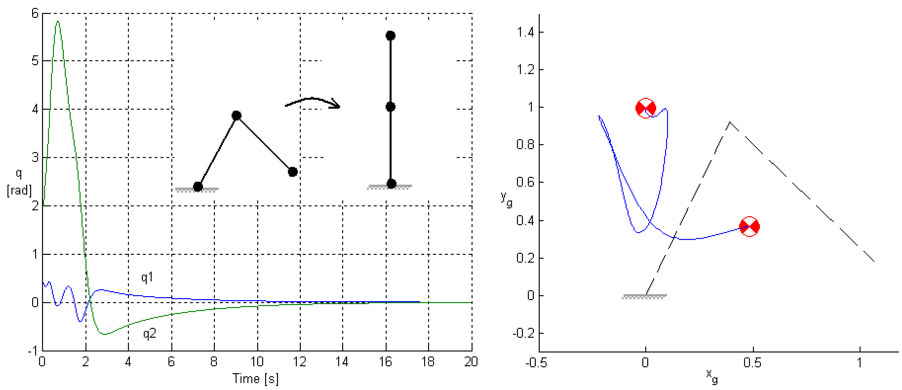

Fig. 5. Stabilization of the 2-link system with rotary joints from $q(0) = (0.4; 2)$, $\dot{q}(0) = (0; 0)$. The controller (33) has been taken as $w = -200\xi_1 - 6\Theta^3\xi_2 - 12\Theta^2\xi_3 - 8\Theta^1\xi_1$, $\Theta = 3$.

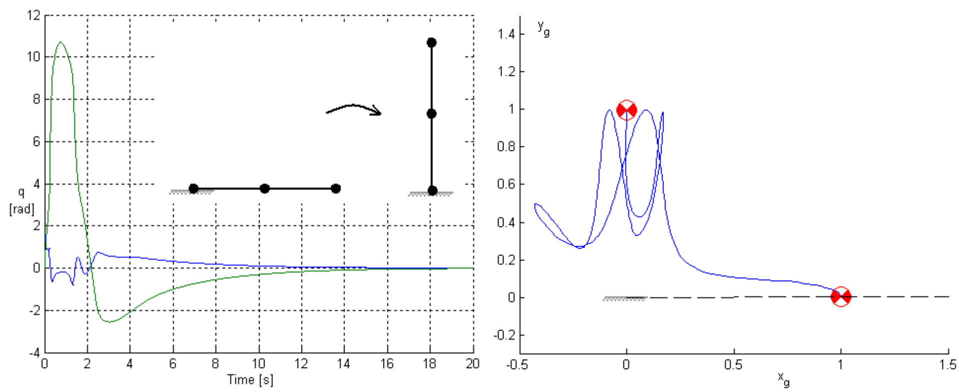

Fig. 6. Stabilization of the 2-link system with rotary joints from $q(0) = (\pi/2; 0)$, $\dot{q}(0) = (0; 0)$. The controller (33) is $w = -200\xi_1 - 6\Theta^3\xi_2 - 12\Theta^2\xi_3 - 8\Theta^1\xi_1$, $\Theta = 3$.

Equation (51) can be solved either in v_2, v_3 or v_4, depending on the selection of the centre joint q_2. For instance,

$$v_2 = \frac{1}{h_2(q,\dot{q})}\left(w - h_1(q,\dot{q}) - h_3(q,\dot{q})v_3 - h_4(q,\dot{q})v_4\right)$$

and acts at the center-joint. In this case $v_3 = -K_{31}\xi_5 - K_{32}\xi_6$, $v_4 = -K_{31}\xi_7 - K_{32}\xi_8$ and w conforms to Theorem 1. Figure 10 shows the stabilization of such a system having the center-joint in the first actuated joint.

For example, a planar walking biped represented as a 4-link mechanical chain unactuated in the pivot point has the centre-joint in its hips, the place where the most important control effects are expected. Stabilization of such a system is presented in the next simulations, Figures 11, 12. In opposition to the previous case, q, the numbering of new angular coordinates $\widetilde{q}$ is different. In this case $\widetilde{q}_2 = q_3$ and $\widetilde{q}_3 = q_2$.

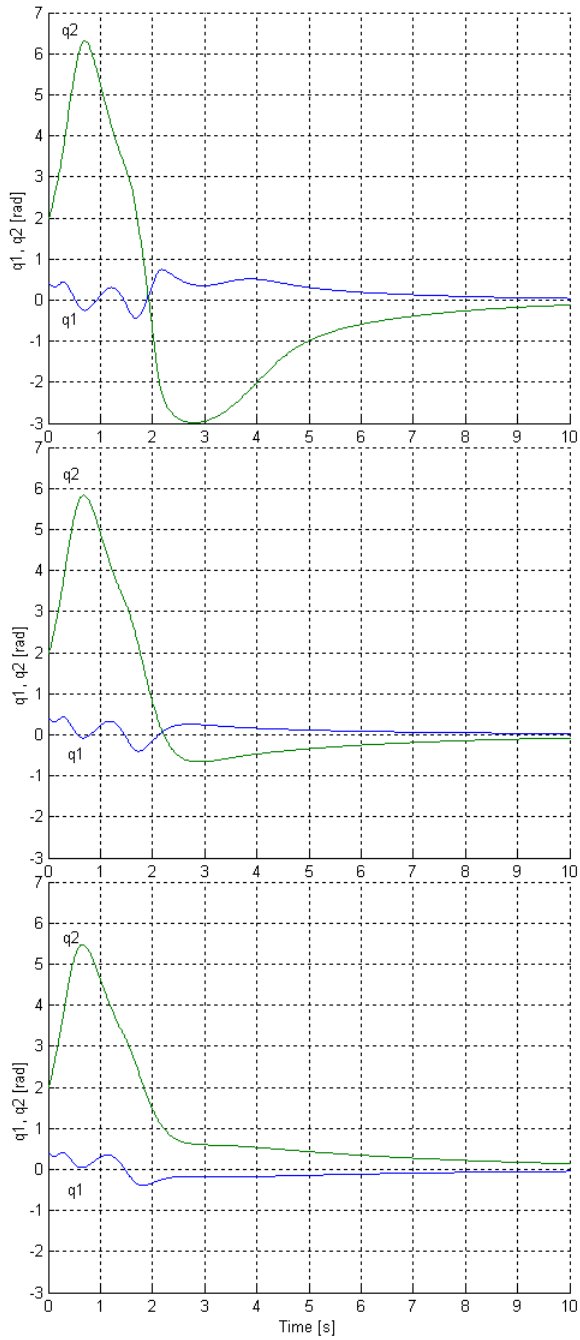

Fig. 7. Stabilization of the 2-link system with rotary joints from $q(0) = (0.4; 2)$, $\dot{q}(0) = (0; 0)$. The controller (33) is $w = -200\xi_1 - 6\Theta^3\xi_2 - 12\Theta^2\xi_3 - 8\Theta^1\xi_1$,
a) $\Theta = 2.8$, b) $\Theta = 3.0$, c) $\Theta = 3.2$.

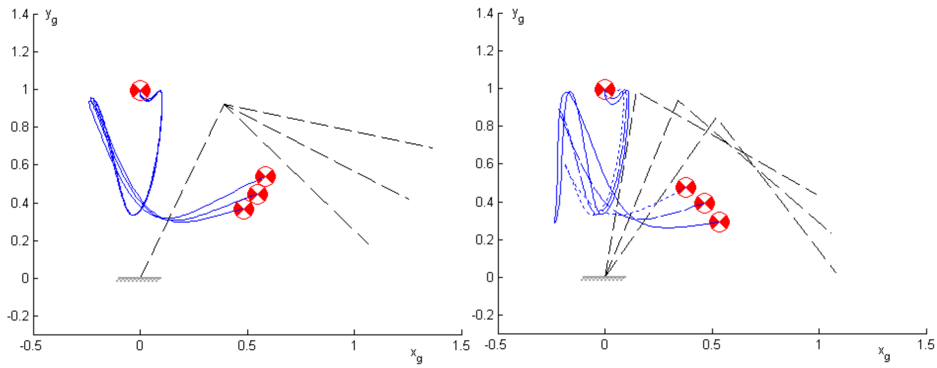

Fig. 8. Stabilization of the 2-link system with rotary joints, trajectories of the centre of gravity for: a) fixed $q_1 = 0.4$ and $q_2 = (1.4; 1.7; 2.0)$; b) $q1 = (0.15; 0.35; 0.55)$ and $q_2 = 2.0$; $\dot{q}(0) = (0; 0)$.

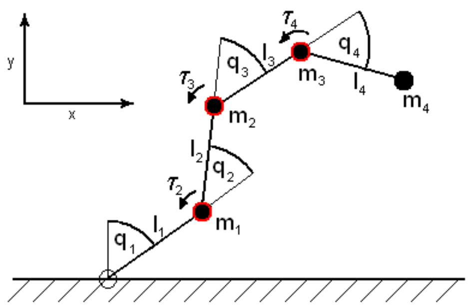

Fig. 9. 4-link system with rotary joints unactuated in the pivot point.

One can easily see that $\widetilde{v}_2 = v_3$ in (51). The function σ and corresponding states ξ_2, ξ_3, ξ_4 are independent of this choice, however the function p is dependent on it. The choice of the center-joint allows to locate the control w into the desired actuated joint and simplifies the notation of dynamic equations.

6. CONCLUSION

The presented work is the part of the complex problem of efficient control of the underactuated mechanical systems. The derived control law stabilizes some of these systems around the inverted position or, more generally, around any other inverted angular configuration.

By a simple tuning of the linear controller the fast convergence with realistic demands on input torque was obtained. Moreover, it does not implement any switching scheme as elsewhere. In the case of stabilization, the controllers are not difficult to tune. Due to singularities it is still not possible to use this control as the global one,

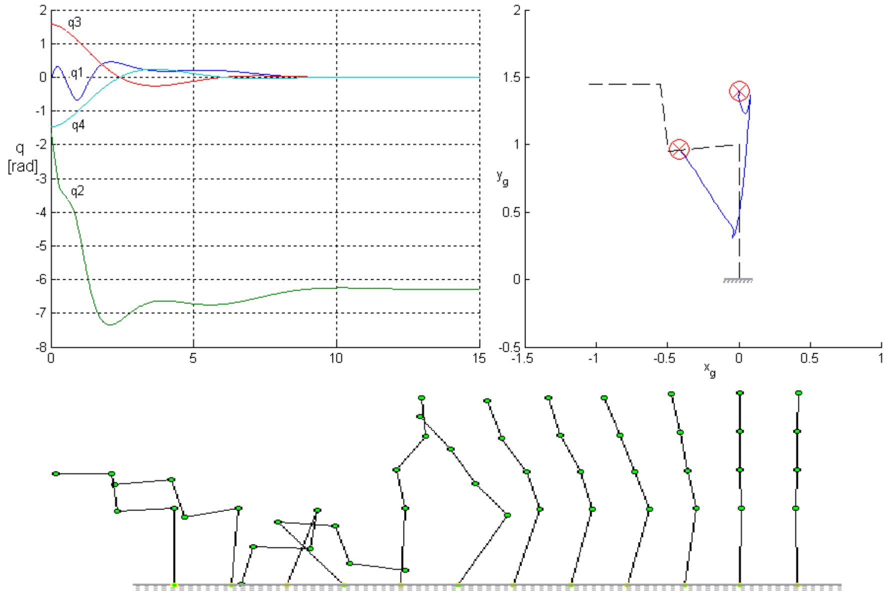

Fig. 10. Stabilization of the 4-link system with centre-joint between the first and the second link.

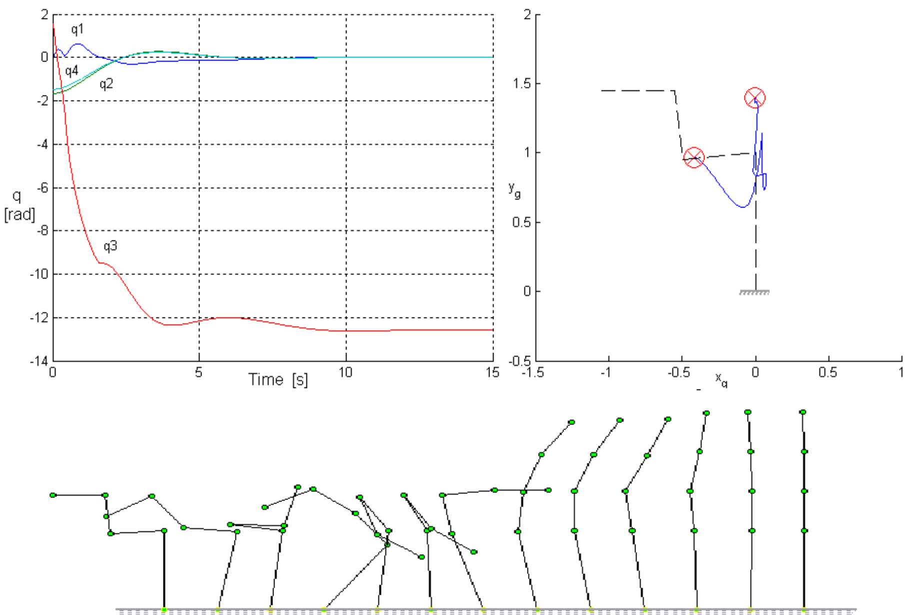

Fig. 11. Stabilization of 4-link system with centre-joint between second and third link.

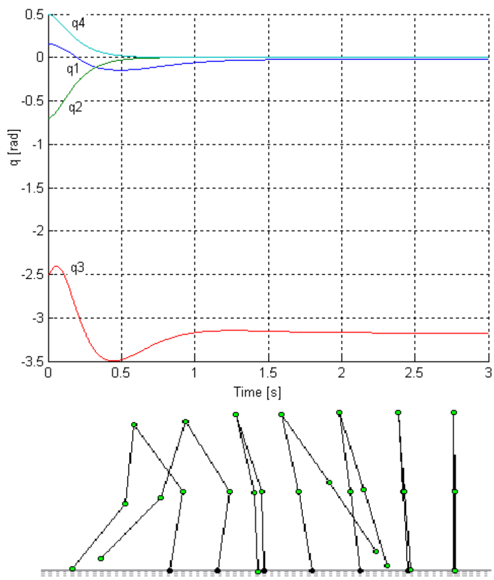

Fig. 12. Step-like stabilization of the 4-link system.

but tractable structural singularities allow to constrain the system inside a suitable domain that excludes any singular points. Therefore only local stability is guaranteed as it is standard for non-linear systems. That is due to the local character of coordinate change. Nevertheless, as was presented on Acrobot's case (Figure 4), the set where transformations and consequently the stability holds include almost all physically reasonable states. The main advantage of the new approach therefore lies in a substantial extension of the domain of attraction.

The theoretical results illustrated here on the simple systems that do represent a walking structure can be extended to a broad range of underactuated mechanical systems. Stabilization and asymptotic tracking around an equilibrium point is the first step to asymptotic trajectory tracking around more general walking like reference trajectories, that will be the subject of further research.

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Jiří Zikmund, Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8. Czech Republic.
e-mail: zikmund@utia.cas.cz