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Non-parametric approximation of non-anticipativity constraints in scenario-based multistage stochastic programming

NON–PARAMETRIC APPROXIMATION OF NON–ANTICIPATIVITY CONSTRAINTS IN SCENARIO–BASED MULTISTAGE STOCHASTIC PROGRAMMING

Jean-Sébastien Roy[1] and Arnaud Lenoir

We propose two methods to solve multistage stochastic programs when only a (large) finite set of scenarios is available. The usual scenario tree construction to represent non-anticipativity constraints is replaced by alternative discretization schemes coming from non-parametric estimation ideas. In the first method, a penalty term is added to the objective so as to enforce the closeness between decision variables and the Nadaraya–Watson estimation of their conditional expectation. A numerical application of this approach on an hydro-power plant management problem is developed. The second method exploits the interpretation of kernel estimators as a sum of basis functions.

Keywords: multistage stochastic programming, scenarios, discrete approximation

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1. INTRODUCTION

A multistage stochastic program deals with the optimization of a system where decisions have to be taken in sequence, at some stage indices $t = 1, \ldots, T$. Decisions are functions (sometimes called *recourse functions*) depending on some alea ω lying in a probability space $(\Omega, \mathcal{F}, \mu)$. The (partial) information available to the decision maker at each stage is modelled by a *filtration* $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_T \subset \mathcal{F}$, where $\mathcal{F}_t$ is the set of distinguishable events of Ω at stage t and where the interlocking property models a progressive acquisition of information without memory loss.

In practice, this information is generated by a sequence of random variables $\{\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_T\}$ where $\boldsymbol{\pi}_t$ represents the *past* of an underlying observation random process (we have $\mathcal{F}_t = \sigma(\boldsymbol{\pi}_t)$ completed with zero-measure sets). Thus, there is an observation space $\mathcal{S} = \mathcal{S}_1 \times \ldots \times \mathcal{S}_T \subset (\mathbb{R}^m)^T$ equipped with its Borel sigma-field

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[1] Jean-Sébastien Roy passed away on the 4th of July 2007 at the age of 33 years.

$\mathcal{B}_{\mathbb{R}^{mT}}$, a random vector:

$$\boldsymbol{S} : (\Omega, \mathcal{F}, \mu) \to (\mathcal{S}, \mathcal{B}_{\mathbb{R}^{mT}})$$
$$\omega \mapsto (\boldsymbol{S}_1(\omega), \dots, \boldsymbol{S}_T(\omega))$$

and for each $t = 1, \dots, T$, the past $\boldsymbol{\pi}_t$ aggregates the first t components of $\boldsymbol{S}$:

$$\boldsymbol{\pi}_t : (\Omega, \mathcal{F}, \mu) \to (\mathcal{S}_1 \times \dots \times \mathcal{S}_t, \mathcal{B}_{\mathbb{R}^{mt}})$$
$$\omega \mapsto (\boldsymbol{S}_1(\omega), \dots, \boldsymbol{S}_t(\omega)).$$

For instance, $\boldsymbol{S}$ can stand for successive observations of the temperature, the number of clients or the price of a portfolio item. We will hereafter call its realizations *scenarios*.

We now denote by $\boldsymbol{U}_t \in L^\infty(\Omega, \mathcal{F}, \mu, \mathbb{R}^n)$ the decision to be taken at stage t. The fact that this decision must be taken only on the basis of information contained in $\mathcal{F}_t$ is achieved by enforcing $\boldsymbol{U}_t$ to be $\mathcal{F}_t$-measurable. These are the so-called *non-anticipativity* constraints: $\boldsymbol{U}_t$ does not anticipate the future from stage $t+1$. In this case, there exists a measurable function φ_t such that $\boldsymbol{U}_t = \varphi_t \circ \boldsymbol{\pi}_t$.

The last thing to be determined is what is going to be optimized. When discovering scenario $s = (s_1, \dots, s_T)$, the cost of making a sequence of decisions $(u_1, \dots, u_T)$ is denoted by $f(u_1, \dots, u_T; s)$ where f is an extended-real valued function. What we want to minimize is the expectation of this cost under the informational constraints i.e. to solve:

$$\text{Minimize}_{(\boldsymbol{U}_1, \dots, \boldsymbol{U}_T)} \quad \mathbb{E}\left(f(\boldsymbol{U}_1, \dots, \boldsymbol{U}_T; \boldsymbol{S})\right) \tag{1a}$$
$$\text{s.t.} \quad \boldsymbol{U}_1, \dots, \boldsymbol{U}_T \in L^\infty(\Omega, \mathcal{F}, \mu, \mathbb{R}^n) \tag{1b}$$
$$\boldsymbol{U}_t \text{ is } \mathcal{F}_t\text{-measurable.} \tag{1c}$$

This problem is infinite-dimensional, hence computationally intractable. That is why one usually deals with a finite-dimensional approximation. Anyway, in many applications, we do not know the law μ but only a set of scenarios $\mathbb{S}_N = \left\{s^{(1)}, \dots, s^{(N)}\right\}$ coming from experiment readings or historical data. In this case, a natural way to approach the problem is to determine a pointwise approximation of $\boldsymbol{U}_t$, $t = 1, \dots, T$, evaluated on the atoms of $\mathbb{S}_N$ i.e. a set of points $\{u_t^{(i)}, i = 1, \dots, N\}$ defining the mapping:

$$\tilde{U}_t : \mathbb{S}_N \to \mathbb{R}^n$$
$$s^{(i)} \mapsto u_t^{(i)}.$$

The expected cost can therefore be approximated by:

$$\mathbb{E}\left(f(\boldsymbol{U}_1, \dots, \boldsymbol{U}_T; \boldsymbol{S})\right) \approx \frac{1}{N} \sum_{i=1}^{N} f\left(u_1^{(i)}, \dots, u_T^{(i)}; s^{(i)}\right). \tag{2}$$

Measurability constraints (1c) have also to be discretized; see [2] for a broad study of the question. A commonly adopted approach consists in constructing a scenario tree

(see [8], [9] for instance), structure of which naturally induces a discrete filtration and is consistent in the epiconvergence sense ([11]).

If only scenarios are available i. e. we do not have a simulation mechanism for the conditional law of the observation process, then an effort must be produced to transform a bundle of scenarios into a tree; see [7]. Moreover, when dealing with multistage stochasic programming, one is usually interested in the first stage decision as well as in the first recourse decisions. Unfortunately, the topology of trees generally does not furnish enough nodes at the first stages to construct feedback approximations (see [3, 4] for a discussion on the subject).

We propose in this paper methods which do not require a modification of sampled data as in tree-based techniques but keep scenarios as they are. Non-anticipativity constraints will be handled by discretization schemes based on ideas from the so-called Nadaraya–Watson non-parametric regression estimator.

In the following section, two discretization methods are developed. Then we give some asymptotic convergence results and a numerical example in the last section.

2. NON–PARAMETRIC DISCRETIZATION OF NON–ANTICIPATIVITY

The informational constraints (1c) can also be written in an analytic way with the help of the conditional expectation:

$$\text{Minimize}_{(\boldsymbol{U}_1,\ldots,\boldsymbol{U}_T)} \quad \mathbb{E}\left(f(\boldsymbol{U}_1,\ldots,\boldsymbol{U}_T;\boldsymbol{S})\right) \tag{3a}$$

$$\text{s.t.} \quad \boldsymbol{U}_1,\ldots,\boldsymbol{U}_T \in L^\infty(\Omega,\mathcal{F},\mu,\mathbb{R}^n) \tag{3b}$$

$$\boldsymbol{U}_t = \mathbb{E}\left(\boldsymbol{U}_t|\boldsymbol{\pi}_t\right). \tag{3c}$$

For a given couple of random vectors $(\boldsymbol{A},\boldsymbol{B})$ with values in $\mathbb{R}^n \times \mathbb{R}^m$, the kernel estimator first proposed by Watson ([13]) and Nadaraya ([10]) arises in non-parametric estimation, as a well known tool to estimate the regression function $g(b) = \mathbb{E}\left(\boldsymbol{A}|\boldsymbol{B} = b\right)$. It is based on the knowledge of a sample of independent and identically distributed random variables $(\boldsymbol{A}^{(1)},\boldsymbol{B}^{(1)}),\ldots,(\boldsymbol{A}^{(N)},\boldsymbol{B}^{(N)})$ with the same law as the couple $(\boldsymbol{A},\boldsymbol{B})$.

If we choose $K : \mathbb{R}^m \to \mathbb{R}^+$ a *kernel* function, and $h_N > 0$ a *kernel bandwidth* depending on N, the estimator is defined as the random function:

$$g_N\left((\boldsymbol{A}^{(1)},\boldsymbol{B}^{(1)}),\ldots,(\boldsymbol{A}^{(N)},\boldsymbol{B}^{(N)});b\right) = \frac{\sum_{i=1}^N \boldsymbol{A}^{(i)} K\left(\frac{\boldsymbol{B}^{(i)}-b}{h_N}\right)}{\sum_{i=1}^N K\left(\frac{\boldsymbol{B}^{(i)}-b}{h_N}\right)} \approx g(b)$$

where $0/0$ is taken to be 0. Convergence properties of this estimator have been broadly studied by Devroye et al. ([5], [6]). We particularly mention the *universal consistency* property proved by Devroye and Wagner [6] and separately by Spiegleman and Sacks [12]:

Theorem 2.1. $B(0,r)$ denotes the closed ball of radius r centered at 0 and $\mathbb{I}_{B(0,r)}$ denotes its characteristic function. Suppose there exists $R_l, R_u, c_l, c_u > 0$ such that

the kernel K satisfies:

$$c_l \mathbb{I}_{B(0,R_l)} \leq K \leq c_u \mathbb{I}_{B(0,R_u)}$$

and that $h \to 0$, $N h^m \to 0$. Then, for all $p \geq 1$ such that $\|\boldsymbol{A}\|_{L^p} < +\infty$:

$$\mathbb{E} \left(\left\| g_N \left((\boldsymbol{A}^{(1)}, \boldsymbol{B}^{(1)}), \ldots, (\boldsymbol{A}^{(N)}, \boldsymbol{B}^{(N)}); \boldsymbol{B} \right) - \mathbb{E} \left(\boldsymbol{A} | \boldsymbol{B} \right) \right\|_{L^p}^p \right) \to 0$$

where the outer expectation is taken with respect to the sample.

From the characterization of measurability with the help of conditional expectation, we obtain the straightforward corollary:

Corollary 2.1. $\boldsymbol{A}$ is $\sigma(\boldsymbol{B})$-measurable if and only if:

$$\mathbb{E} \left(\left\| g_N \left((\boldsymbol{A}^{(1)}, \boldsymbol{B}^{(1)}), \ldots, (\boldsymbol{A}^{(N)}, \boldsymbol{B}^{(N)}); \boldsymbol{B} \right) - \boldsymbol{A} \right\|_{L^p}^p \right) \to 0.$$

P r o o f . This is a direct consequence of the fact that $\boldsymbol{A}$ is $\sigma(\boldsymbol{B})$-measurable if and only if $\boldsymbol{A} = \mathbb{E} \left(\boldsymbol{A} | \boldsymbol{B} \right)$ and of the uniqueness of the limit in Theorem 2.1.   □

This result originates the discretization schemes we propose in the following. Given a sample $(s^{(1)}, \ldots, s^{(N)})$, and the corresponding past samples $(\pi_t^{(1)}, \ldots, \pi_t^{(N)})$ for $t = 1, \ldots, T - 1$, we will approximate the right-hand side of constraint (3c) by:

$$\mathbb{E} \left(\boldsymbol{U}_t | \boldsymbol{\pi}_t = \pi_t \right) \approx g_N \left((u_t^{(1)}, \pi_t^{(1)}), \ldots, (u_t^{(N)}, \pi_t^{(N)}); \pi_t \right), \qquad (4)$$

where the $u_t^{(i)}$'s are the decision variables of the discretized problem constituting the pointwise approximation of $\boldsymbol{U}_t$. Now, so as to approach the whole constraint (3c), the first (naïve) idea would be to add the $N \times T$ following equalities:

$$u_t^{(i)} = g_N \left((u_t^{(1)}, \pi_t^{(1)}), \ldots, (u_t^{(N)}, \pi_t^{(N)}); \pi_t^{(i)} \right) \approx \mathbb{E} \left(\boldsymbol{U}_t | \boldsymbol{\pi}_t = \pi_t^{(i)} \right) \qquad \forall i, \forall t. \quad (5)$$

This naïve discretization is in fact catastrophical because the resulting pointwise approximation is inevitably constant, or in the best case constant on each connected part of $\text{supp}(\boldsymbol{\pi}_t)$ ($\text{supp}(\cdot)$ stands for the support of a function). To see this fact, let us denote:

$$\alpha_t^{(i,j)} = \frac{K \left(\frac{\pi_t^{(i)} - \pi_t^{(j)}}{h_N} \right)}{\sum_{k=1}^{N} K \left(\frac{\pi_t^{(i)} - \pi_t^{(k)}}{h_N} \right)} \quad \text{and} \quad A_t^{(i,j)} = \alpha_t^{(i,j)} I_n$$

with I_n the $n \times n$ identity matrix. Equality (5) rewrites $u_t^{(i)} = \sum_{j=1}^{N} \alpha_t^{(i,j)} u_t^{(j)}$ or $u_t = A_t u_t$ where $A_t = (A_t^{(i,j)})_{i,j}$ is a matrix defined by block. Any two scenarios

i and j are said to be *neighbors* if $\alpha_t^{(i,j)} > 0$. If we gather scenarios $1 \leq i, j \leq N$ connected through a path of neighbors, we then obtain a relation of equivalence $\triangle_t$:

$$i \triangle_t j \iff \begin{cases} \exists\, k_1, k_2, \ldots, k_p \in \{1, \ldots, N\} \\ k_1 = i, k_p = j \\ \forall\, \nu = 1, \ldots, p-1 \quad \alpha_t^{(k_\nu, k_{\nu+1})} > 0. \end{cases}$$

Let's assume for an instant that $n = 1$ and there is only one class, i. e. the matrix $A = (\alpha_t^{(i,j)})_{i,j}$ is irreducible. This occurs for instance if we choose a kernel $K(\cdot)$ of support all $\mathbb{R}$, as a gaussian kernel for example. We denote $\mathcal{E}$ the subspace defined by the system of constraints (5) and we claim that it is equal to the space $\mathcal{E}'$ of vectors with components one equal to each other:

$$\mathcal{E}' = \left\{ u_t / u_t^{(1)} = \ldots = u_t^{(N)} \right\}.$$

Clearly, $\mathcal{E}' \subset \mathcal{E}$ because A computes weighted averages. Now, A is a stochastic matrix, so its spectral radius is 1. The associated eigenspace is consequently 1-dimensional, by the irreducibility of A together with the Perron–Frobenius theorem. This eigenspace coincide with $\mathcal{E}$ by definition and must therefore reduce to $\mathcal{E}'$ since it is itself 1-dimensional. This line easily extends to n-dimensional vectors, reasoning component by component.

The convergence of estimators generally needs each of the scenarios to have many neighbors. Therefore, classes of equivalence will correspond to the connected part of $\mathrm{supp}(\boldsymbol{\pi}_t)$.

3. DISCRETIZATION METHODS

3.1. Penalization of non-anticipativity

We just realised that imposing the estimation of the regression function to be point-wise equal to the decisions $(u_t^{(1)}, \ldots, u_t^{(N)})$ led to a very bad approximation of the space of $\mathcal{F}_t$-measurable functions. The first idea we develop consists in slightly relaxing these T sets of N equalities by penalizing them. If $P(\cdot)$ is a penalization function, for instance:

$$P(x) = \frac{C}{2} \|x\|_2^2 \qquad P(x) = C \|x\|_1 \qquad P(x) = \mathbb{I}_{\{\|x\|_\infty \leq \varepsilon\}} \tag{6}$$

we propose to solve the following finite dimensional program:

$$\text{Minimize}_{\left\{ u_t^{(i)} \right\}} \frac{1}{N} \sum_{i=1}^{N} f(u_1^{(i)}, \ldots, u_T^{(i)}; s^{(i)}) \tag{7a}$$

$$+ \sum_{i,t} P\left(u_t^{(i)} - g_N((u_t^{(1)}, \pi_t^{(1)}), \ldots, (u_t^{(N)}, \pi_t^{(N)}); \pi_t^{(i)}) \right). \tag{7b}$$

The penalty parameters C and ε in (6) should be tuned experimentally with respect to the value of N (see Section 4 for more details).

3.2. Partition of unity method

We now present another point of view about non-parametric estimators which interprets the function g_N in (4) as the sum of N basis functions weighted with coefficients $u_t^{(i)}$. Namely, this basis contains for a given t, the functions $\varphi_t^{(i)}$ defined by:

$$\varphi_t^{(i)}(\cdot) = \frac{K\left(\frac{\pi_t^{(i)}-\cdot}{h_N}\right)}{\sum_{j=1}^{N} K\left(\frac{\pi_t^{(j)}-\cdot}{h_N}\right)} \tag{8}$$

and we look for functions $\varphi_t : \mathbb{R}^{mt} \to \mathbb{R}^n$ in the form $\varphi_t(\cdot) = \sum_{i=1}^{N} \varphi_t^{(i)}(\cdot)u_t^{(i)}$. This family defines a so-called partition of unity in the sense that:

$$\sum_{i=1}^{N} \varphi_t^{(i)} \equiv 1 \quad \text{on} \quad \bigcup_{i=1}^{N} \left\{ \pi_t^{(i)} + h_N \operatorname{supp}(K) \right\}.$$

We will denote by $\Phi_t \cdot u_t$ the element of this space with coefficients $(u_t^{(1)}, \ldots, u_t^{(N)}) = u_t \in (\mathbb{R}^n)^N$. Then:

$$(\Phi_t \cdot u_t)(\pi_t^{(i)}) = g_N\left((u_t^{(1)}, \pi_t^{(1)}), \ldots, (u_t^{(N)}, \pi_t^{(N)}); \pi_t^{(i)}\right) \tag{9a}$$

$$= \sum_{j=1}^{N} \varphi_t^{(j)}(\pi_t^{(i)})u_t^{(j)}. \tag{9b}$$

Instead of trying to relate the pointwise values of $\varphi_t = \Phi_t \cdot u_t$ with the values of its coefficients like in the previous section, we differentiate them by introducing new variables. For $i = 1, \ldots, N$ and $t = 1, \ldots, T$, we set $v_t^{(i)} = (\Phi_t \cdot u_t)(\pi_t^{(i)})$ and using (9b), we propose to solve the following problem:

$$\text{Minimize}_{\left\{u_t^{(j)}, v_t^{(i)}\right\}} \frac{1}{N} \sum_{i=1}^{N} f(v_1^{(i)}, \ldots, v_T^{(i)}; s^{(i)}) \tag{10a}$$

$$v_t^{(i)} = \sum_{j=1}^{N} \varphi_t^{(j)}(\pi_t^{(i)})u_t^{(j)}. \tag{10b}$$

Remark 3.1. Note that variables $v_t^{(i)}$ have been introduced only for sake of clarity but need not be implemented.

We would like to emphasize the fact that this approach is not new. The use of a partition of unity for numerical integration has already been used in [1] to solve partial differential equations. It is worth noting that the idea (see [1, section 2]) comes from numerical integration methods in data fitting. We now present some facts about the convergence of the partition of unity method.

3.3. A convergence result

We hope the convergence results we give in this section will constitute a base for further exploration on approximation of problems (7) and (10). In this section, we forget the stage index t. We set $\Pi = \boldsymbol{\pi}(\Omega)$ and we equip it with the transported measure $\lambda = \mu \circ \boldsymbol{\pi}^{-1}$ and with its Borel σ-field. Then, we consider a sequence $\{\lambda_N\}_N$ of measures on Π of atomic support $\{\pi^{(1)}, \ldots, \pi^{(N)}\}$. The probability assigned to each atom does not matter because we do not study the evaluation of the cost, but only the structure of basis functions $\varphi^{(i)}$ which only depends on the localization of the atoms in the space.

For all N, we choose a kernel bandwidth h_N and we define the set:

$$E_N = \left\{ \varphi = \sum_{i=1}^{N} u^{(i)} \varphi^{(i)}, (u^{(1)}, \ldots, u^{(N)}) \in (\mathbb{R}^n)^N \right\}$$

which is the finite dimensional space of approximations. We are interested in the functions representable by sequences of $\{E_N\}_N$. Convergence in L^p-norm is considered for $1 \le p < \infty$ under the following hypothesis:

H1) $\lim_{N \to \infty} h_N = 0$;

H2) $\lim_{N \to \infty} \lambda \left(\bigcup_{i=1}^{N} \{\pi^{(i)} + h_N \mathrm{supp}(K)\} \right) = 1$;

H3)   there are $c_l, R_l, c_u, R_u > 0$ such that K satisfies hypothesis of Theorem 2.1.

Lemma 3.1.   Let l be defined on Π, continuous with a compact support. Then, under H1, H2, H3, there is a sequence $l_N \in E_N$ converging to l in any norm L^p.

P r o o f. We construct the sequence $\{l_N\}_N$ by defining for every N:

$$l_N(\pi) = \sum_{i=1}^{N} l(\pi^{(i)}) \varphi^i(\pi).$$

Choose $\varepsilon > 0$. If we denote by $A_N = \bigcup_{i=1}^{N} \{\pi^{(i)} + h_N \mathrm{supp}(K)\}$ the involved set in hypothesis H2, then we can choose N large enough so as to obtain:

i) h_N is so small that $\sup_{\|\pi - \pi'\| < R_u h_N} \|l(\pi) - l(\pi')\| < \varepsilon^{\frac{1}{p}}$.

ii) $\lambda(A_N) \ge 1 - \frac{\varepsilon}{(\|l\|_\infty)^p}$.

For every π, we have:

$$\|l_N(\pi) - l(\pi)\| = \left\| \frac{1}{\sum_{i=1}^{N} K\left(\frac{\pi^{(i)} - \pi}{h_N}\right)} \sum_{i=1}^{N} l(\pi^{(i)}) K\left(\frac{\pi^{(i)} - \pi}{h_N}\right) - l(\pi) \right\| \tag{11}$$

the elements of the sum are non-zero for the indices i such that $\frac{\pi^{(i)} - \pi}{h_N} \in \mathrm{supp}(K)$. If there is no such i, it means that $\pi \notin A_N$ and the sum is zero. Else $\pi \in A_N$ and for

all non-zero term, we have $\left\|\pi^{(i)} - \pi\right\| \leq R_u h_N$ by $H3$ so $\left\|l(\pi^i) - l(\pi)\right\| < \varepsilon^{\frac{1}{p}}$ by the choice of h_N. Now the sum in (11) corresponds to a convex combination of points all being in $B(l(\pi), \varepsilon^{\frac{1}{p}})$, the open ball centered in $l(\pi)$ and of radius $\varepsilon^{\frac{1}{p}}$ so the result is also in this ball and we obtain:

$$\|l_N(\pi) - l(\pi)\|^p < \varepsilon.$$

Now, we can integrate this inequality:

$$\int \|l_N(\pi) - l(\pi)\|^p = \int_{A_N} \|l_N(\pi) - l(\pi)\|^p + \int_{\Pi \backslash A_N} \|l_N(\pi) - l(\pi)\|^p$$
$$\leq \lambda(A_N)\varepsilon + \lambda(\Pi \backslash A_N) \|l\|_\infty$$
$$\leq \varepsilon + \varepsilon.$$

As this is true for all ε, we have the convergence. $\square$

We deduce from this result a corollary similar to the universal strong consistency of Theorem 2.1:

Corollary 3.1.   For every $l \in L^p(\Omega, \mathcal{F}, \mu)$. There is a sequence l_N in E_N converging to l in L^p.

P r o o f. This is straightforward from the density of the set of continuous functions with compact support in L^p. $\square$

4. NUMERICAL EXAMPLES

4.1. Optimal control of an hydro-power plant

We consider the problem of managing an hydro-power plant. Two successive production decisions $u_1 \geq 0$ and $u_2 \geq 0$ have to be made. The reservoir initially contains an amount of energy S, so that we require $u_1 + u_2 \leq S$. These decisions have to be taken as feedbacks on successive random selling prices $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. There is a non-anticipativity constraint on the first decision, i.e., u_1 has to be taken prior to any knowledge of the second price, except its conditional law with respect to the first one. Mathematically, we consider the following cost function:

$$f(u_1, u_2, \omega_1, \omega_2) = -u_1\omega_1 - u_2\omega_2 - V(S - u_1 - u_2)$$

where $V(x)$ is the value of the remaining stock at the end of the two steps, and is in our case a quadratic approximation of $\sqrt{\eta + x}$, i.e., $V(x) = \sqrt{\eta} + ax + bx^2$, with $b = \frac{2}{S^2}\left(\sqrt{\eta} - 2\sqrt{\eta + \frac{S}{2}} + \sqrt{\eta + S}\right)$, $a = \frac{\sqrt{\eta+S} - \sqrt{\eta} - b}{S}$ and $\eta = 0.1$. $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ follow independent uniform laws on $[0.4, 2]$, and $S = 1$. Our optimization problem is therefore:

$$J = \min_{u_1(\cdot), u_2(\cdot, \cdot)} \mathrm{E}\left[-\mathrm{u}_1(\boldsymbol{\omega}_1)\boldsymbol{\omega}_1 - \mathrm{u}_2(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)\boldsymbol{\omega}_2 - \mathrm{V}(\mathrm{S} - \mathrm{u}_1(\boldsymbol{\omega}_1) - \mathrm{u}_2(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2))\right]$$

subject to $, \forall (\omega_1, \omega_2) \in [0.4, 2]^2 \ u_1(\omega_1) \in [0, S]$ and $u_2(\omega_1, \omega_2) \in [0, S - u_1(\omega_1)]$.

The problem can be solved exactly by dynamic programming. We compare this exact solution to the approximation obtained by the penalization approach (7). Let $\left(\omega_1^{(j)}, \omega_2^{(j)}\right)_{j=1,\ldots,N}$ be N independent realizations of the prices. We consider the following problem:

$$
\tilde{J}_N \quad = \quad \min_{\left(u_1^{(j)}, u_2^{(j)}\right)_{j=1,\ldots,N}} \frac{1}{N} \sum_{j=1,\ldots,N} f\left(u_1^{(j)}, u_2^{(j)}, \omega_1^{(j)}, \omega_2^{(j)}\right)
$$

$$
+\frac{C}{N} \sum_{j=1}^{N} \left\| u_1^{(j)} - \frac{\sum_{k \neq j} K\left(\frac{\omega_1^{(j)} - \omega_1^{(k)}}{\varepsilon_1}\right) u_1^{(k)}}{\sum_{k \neq j} K\left(\frac{\omega_1^{(j)} - \omega_1^{(k)}}{\varepsilon_1}\right)} \right\|^2
$$

where $K(x) = \mathrm{e}^{-\frac{x^2}{\varepsilon^2}}$. Once this problem is solved, we synthetize the required feedbacks $u_1\left(\cdot\right)$ and $u_2\left(\cdot,\cdot\right)$ using:

$$
u_1\left(\omega_1\right) = \frac{\sum_{j=1}^{N} K\left(\frac{\omega_1^{(j)} - \omega_1}{\varepsilon_1}\right) u_1^{(j)}}{\sum_{j=1}^{N} K\left(\frac{\omega_1^{(j)} - \omega_1}{\varepsilon_1}\right)}
$$

$$
u_2\left(\omega_1, \omega_2\right) = \frac{\sum_{j=1}^{N} K\left(\frac{\left(\omega_1^{(j)}, \omega_2^{(j)}\right) - \left(\omega_1, \omega_2\right)}{\varepsilon_2}\right) u_2^{(j)}}{\sum_{j=1}^{N} K\left(\frac{\left(\omega_1^{(j)}, \omega_2^{(j)}\right) - \left(\omega_1, \omega_2\right)}{\varepsilon_2}\right)}
$$

with ε_2 chosen to provide the best fit for the point set $\left((\omega_1^{(j)}, \omega_2^{(j)}), u_2^{(j)}\right)$. We choose empirically $\varepsilon_2 = \sqrt{\frac{\varepsilon_1}{\pi}}$. The quality of this approximate is then evaluated by a large Quasi-Monte-Carlo simulation.

Figures 1 and 2 present the feedbacks obtained while Table 1 summarizes the quality of these approximations. The results must be compared to the optimum value, $J = -1.7414$, the value obtained without the penalty term, i. e., the anticipative solution assuming the future is known, whose value is -1.786, and the value of the solution obtained by synthetizing from the anticipative solution, feedbacks as detailed above, with $\varepsilon_1 = 0.1$. The value of this last solution is -1.69563. Figure 3 presents this solution, while Figure 4 presents the optimal feedback $u_2\left(\cdot,\cdot\right)$ (the optimal feedback $u_1\left(\cdot\right)$ is drawn on each graph representing an approximation of $u_1(\cdot)$).

Parameters ε_1 and C can be set so as to optimize the value during the simulation. Figures 5 and 6 present the best approximations obtained for $N = 10, 27, 129$ and 999. Table 2 summarize the quality of the approximations, and the parameters chosen. It seems that the optimal value as well as the feedbacks converge to the exact solution as the number of available scenarios grows.
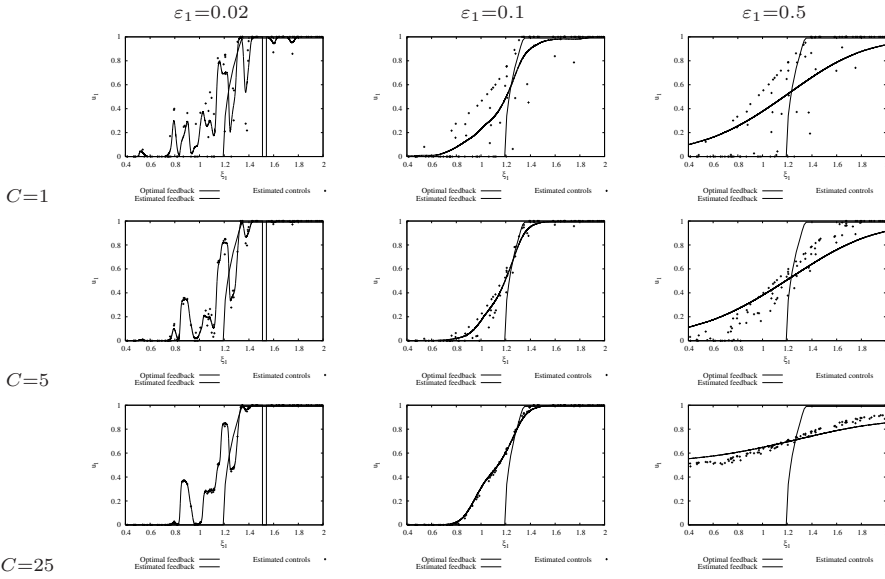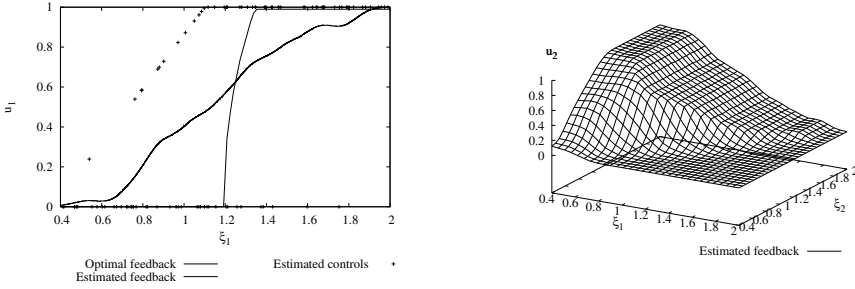
Fig. 1. Approximation of $u_1\left(\cdot\right)$ for $N = 100$ scenarios
as a function of the bandwidth ε_1 and the penalty C.
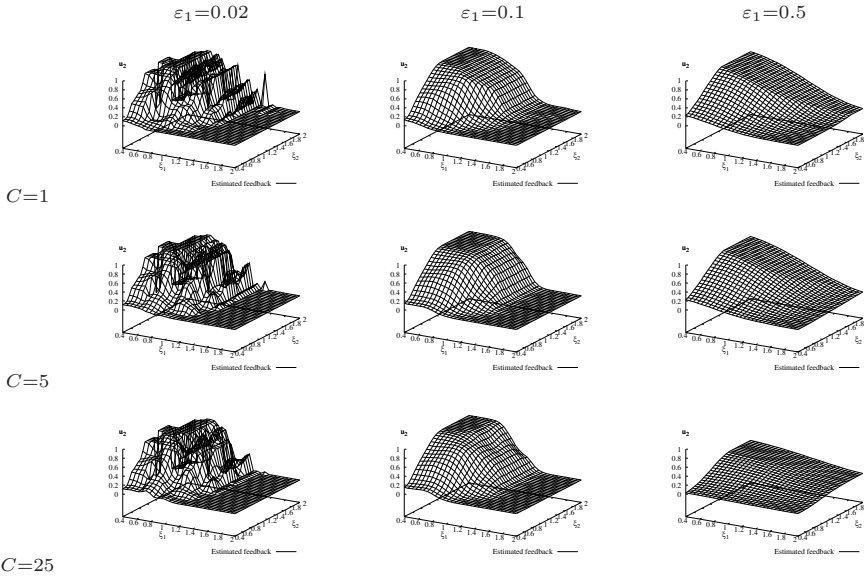
Fig. 2. Approximation of $u_2\left(\cdot,\cdot\right)$ for $N = 100$ scenarios
as a function of the bandwidth ε_1 and the penalty C.

Table 1. Quality of the approximation for $N = 100$ scenarios as a function of the bandwidth ε_1 and the penalty C.

	$\varepsilon_1 = 0.02$	$\varepsilon_1 = 0.1$	$\varepsilon_1 = 0.5$
$C = 1$	-1.6788	-1.72698	-1.67559
$C = 5$	-1.67757	-1.73394	-1.66757
$C = 25$	-1.67638	-1.72863	-1.58707

Fig. 3. Approximations of $u_1 (\cdot)$ and $u_2 (\cdot, \cdot)$, for $N = 100$ scenarios, with bandwidth $\varepsilon_1 = 0.1$ and no penalty, i. e., $C = 0$.
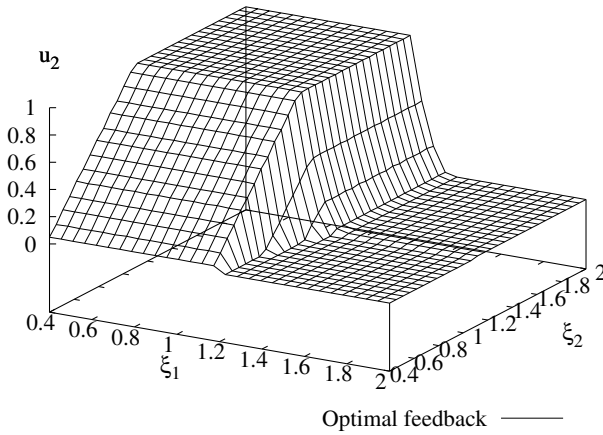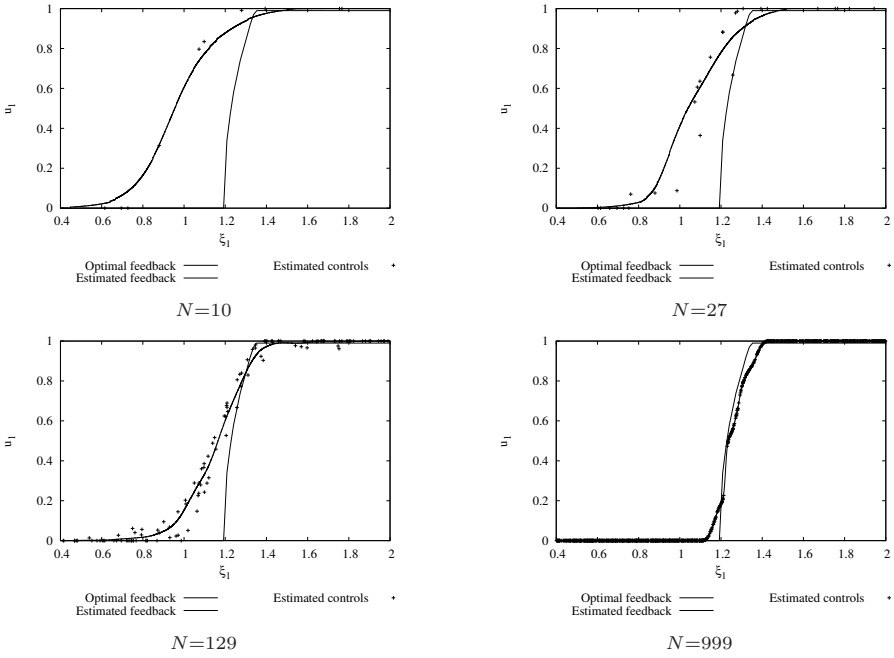

Fig. 4. Optimal feedback $u_2 (\cdot, \cdot)$.

Fig. 5. Best approximation of $u_1\left(\cdot\right)$ for $N = 10$, 27, 129 and 999.
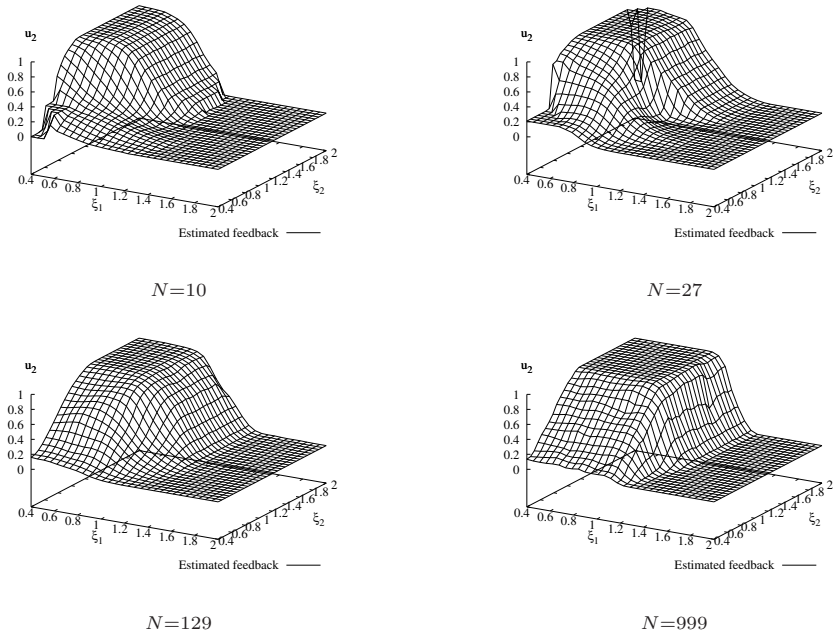

Fig. 6. Best approximation of $u_2\left(\cdot,\cdot\right)$ for $N = 10$, 27, 129 and 999.

Table 2. Quality and values of ε_1 and C
for the best solutions for $N = 10, 27, 129$ and 999.

N	ε_1	C	Value
10	0.215443	5.99484	-1.70561
27	0.129155	2.15443	-1.72187
129	0.0774264	5.99484	-1.73369
999	0.016681	1000	-1.74018

5. CONCLUSION

The work presented in this paper aims at developing methods alternative to the construction of scenario trees, on the one hand to avoid complex pre-processing computation, on the other to get better solution approximation for first stages. However, there is no question that, like the other scenario-based methods, the number of scenarios required to get a good approximation of the problem should be high and necessitate for example to set up decomposition-coordination methods. Moreover, both methods presented here should make profit of the dense computational literature in the field of numerical data fitting.

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