Bernard De Baets; Hans De Meyer; Radko Mesiar

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ASYMMETRIC SEMILINEAR COPULAS

Bernard De Baets, Hans De Meyer and Radko Mesiar

We complement the recently introduced classes of lower and upper semilinear copulas by two new classes, called vertical and horizontal semilinear copulas, and characterize the corresponding class of diagonals. The new copulas are in essence asymmetric, with maximum asymmetry given by 1/16. The only symmetric members turn out to be also lower and upper semilinear copulas, namely convex sums of Π and M.

Keywords: asymmetry, copula, diagonal section, semilinear copula, symmetry

AMS Subject Classification: 62H10, 62H20

1. INTRODUCTION

A two-dimensional copula (copula for short) is a $[0,1]^2 \to [0,1]$ function connecting the marginal distribution functions of a random vector (X, Y) with its joint distribution function. For more statistical details, we refer to [5, 8, 12]. A copula $C : [0,1]^2 \to [0,1]$ is characterized by the following properties:

(C1) *Annihilator* 0: $C(x, 0) = C(0, x) = 0$ for any $x \in [0,1]$;

(C2) *Neutral element* 1: $C(x, 1) = C(1, x) = x$ for any $x \in [0,1]$;

(C3) 2*-increasingness*: for any $x, x', y, y' \in [0,1]$ with $x \le x'$ and $y \le y'$ it holds that

$$V_C([x, x'] \times [y, y']) := C(x', y') - C(x, y') - C(x', y) + C(x, y) \ge 0\,.$$

Well-known examples of copulas are $\Pi(x, y) = xy$, expressing the independence of the random variables X and Y; $M(x, y) = \min(x, y)$, expressing the total positive dependence of X and Y; and $W(x, y) = \max(x + y - 1, 0)$, expressing the total negative dependence of X and Y. Although all of these copulas are symmetric, in general a copula need not be symmetric, expressing the possible non-exchangeability of the random variables X and Y.

Sklar's theorem [12] guarantees that for any two continuous random variables X and Y, there exists a unique copula C such that their joint cumulative distribution function $F_{X,Y}$ can be expressed in terms of their marginal cumulative distribution function F_X and F_Y as follows, for any $x, y \in \mathbb{R}$:

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y)) .$$

Moreover, in this case the diagonal section $\delta_C : [0,1] \to [0,1]$ of C, defined by $\delta_C(u) = C(u,u)$, is (the restriction to $[0,1]$ of) the distribution function of the random variable $D = \max(F_X(X), F_Y(Y))$; in particular, if X and Y are uniformly distributed over the unit interval, then $D = \max(X,Y)$.

For the purpose of this paper, it is instructive to recall that the diagonal section δ_C of a copula C has the following properties:

(D1) $\delta_C(0) = 0$, $\delta_C(1) = 1$;

(D2) $\delta_C(u) \leq u$ for any $u \in [0,1]$;

(D3) δ_C is increasing;

(D4) δ_C is 2-Lipschitz, i. e. $|\delta_C(v) - \delta_C(u)| \leq 2|v-u|$ for any $u, v \in [0,1]$.

The set of $[0,1] \to [0,1]$ functions satisfying properties (D1) – (D4) is denoted by $\mathcal{D}$. The members of $\mathcal{D}$ are called *diagonals*.

There exist many ways of constructing a copula with a given diagonal section $\delta \in \mathcal{D}$. For example, the *diagonal copula* C_δ, defined by

$$C_\delta(x,y) = \min(x, y, (\delta(x) + \delta(y))/2) ,$$

is the greatest symmetric copula with a given diagonal section δ [10]. Similarly, the *Bertino copula* B_δ, defined by

$$B_\delta(x,y) = \min(x,y) - \min\{u - \delta(u) \mid u \in [\min(x,y), \max(x,y)]\} ,$$

is the smallest copula with a given diagonal section δ, see [1, 6, 10]. Note that the Bertino copula is symmetric. Several other ways of constructing copulas with a given diagonal section, although mostly applicable only to some special classes of diagonal sections, such as MT-copulas [2] and (upper and lower) semilinear copulas [3], always yield symmetric copulas. For more details on these construction methods, symmetric as well as asymmetric, we refer to [4, 11].

The aim of this paper is to introduce a new method for constructing possibly asymmetric copulas with a given diagonal section. The inspiration for this work can be found in the study of lower and upper semilinear copulas [3]. Such copulas result from a linear interpolation between the values at the lower boundaries (i. e. $C(x,0) = C(0,x) = 0$) or upper boundaries (i. e. $C(x,1) = C(1,x) = x$) of the unit square and the values on the diagonal (i. e. $C(x,x) = \delta(x)$). We will investigate other combinations of boundaries of the unit square.

Our paper is organized as follows. In the next section, we introduce vertical and horizontal semilinear copulas. In Section 3, we characterize the corresponding diagonal sections. We demonstrate in Section 4 that the new classes of copulas are closed under minimum, maximum and convex sums, yet not under log-convex sums. In the last two sections, we identify the symmetric members of our classes and pinpoint the maximum asymmetry (1/16) the new copulas can attain.

2. DEFINITIONS

2.1. Lower and upper semilinear copulas

Semilinear copulas were introduced recently by Durante et al. [3].

Definition 1.

(i) A copula C is called *lower semilinear* if the mappings

$$\begin{aligned} h_1 &: [0,x] \to [0,1], \qquad t \mapsto h_1(t) = C(t,x) \\ v_1 &: [0,x] \to [0,1], \qquad t \mapsto v_1(t) = C(x,t) \end{aligned}$$

are linear for all $x \in [0,1]$.

(ii) A copula C is called *upper semilinear* if the mappings

$$\begin{aligned} h_2 &: [x,1] \to [0,1], \qquad t \mapsto h_2(t) = C(t,x) \\ v_2 &: [x,1] \to [0,1], \qquad t \mapsto v_2(t) = C(x,t) \end{aligned}$$

are linear for all $x \in [0,1]$.

This definition immediately implies that both lower and upper semilinear copulas are symmetric. As the survival copula $\hat{C}(x,y) = x+y-1+C(1-x,1-y)$ of a lower (resp. upper) semilinear copula C is an upper (resp. lower) semilinear copula, we can restrict our attention to lower semilinear copulas only. Lower semilinear copulas are characterized by their diagonal section [3]: a copula C is a lower semilinear copula if and only if it is given by

$$C(x,y) = \begin{cases} y\dfrac{\delta_C(x)}{x}, & \text{if } y \le x\,, \\ x\dfrac{\delta_C(y)}{y}, & \text{otherwise}\,, \end{cases} \tag{1}$$

(with the convention $\frac{0}{0} := 1$, adopted throughout this paper). The diagonals for which the above expression leads to a copula are characterized by

(L1) the function $\varphi_\delta :]0,1] \to [0,1]$ defined by $\varphi_\delta(x) = \frac{\delta(x)}{x}$ is increasing;

(L2) the function $\eta_\delta :]0,1] \to [1,\infty[$ defined by $\eta_\delta(x) = \frac{\delta(x)}{x^2}$ is decreasing.

The class of diagonals satisfying (L1) and (L2) is denoted by $\mathcal{D}_L$. The corresponding conditions (U1) and (U2) for diagonals characterizing upper semilinear copulas are easily derived.

2.2. Vertical and horizontal semilinear copulas

Inspired by the above results, we investigate whether grouping the functions h_1 and h_2, or v_1 and v_2 also lead to meaningful classes of copulas.

Definition 2.

(i) A copula C is called *horizontal semilinear* if the mappings

$$\begin{aligned} h_1 &: [0,x] \to [0,1], \qquad t \mapsto h_1(t) = C(t,x)\,, \\ h_2 &: [x,1] \to [0,1], \qquad t \mapsto h_2(t) = C(t,x) \end{aligned}$$

are linear for all $x \in [0,1]$.

(ii) A copula C is called *vertical semilinear* if the mappings

$$\begin{aligned} v_1 &: [0,x] \to [0,1], \qquad t \mapsto v_1(t) = C(x,t)\,, \\ v_2 &: [x,1] \to [0,1], \qquad t \mapsto v_2(t) = C(x,t) \end{aligned}$$

are linear for all $x \in [0,1]$.

In essence, the above definitions are asymmetric. In Section 5 we will characterize the symmetric members, while in Section 6 we will characterize the maximally asymmetric members. Similarly as for lower and upper semilinear copulas, the members of the above classes can be characterized in terms of their diagonal sections.

Proposition 1. For a copula C, the following statements are equivalent:

(a) C is a vertical semilinear copula;

(b) C is given by

$$C(x,y) = \begin{cases} y\dfrac{\delta_C(x)}{x}, & \text{if } y \le x\,, \\ \dfrac{(y-x)x + (1-y)\delta_C(x)}{1-x}, & \text{otherwise}\,. \end{cases} \tag{2}$$

P r o o f . Suppose that C is a vertical semilinear copula. Eq. (2) easily follows by piecewise linear interpolation between the known values of the copula C: 0 in $(x,0)$, $\delta_C(x)$ in (x,x) and x in $(x,1)$. The converse part is obvious. □

Similarly, we can characterize horizontal semilinear copulas.

Proposition 2. For a copula C, the following statements are equivalent:

(a) C is a horizontal semilinear copula;

(b) C is given by

$$C(x,y) = \begin{cases} x\dfrac{\delta_C(y)}{y}, & \text{if } x \le y\,, \\ \dfrac{(x-y)y + (1-x)\delta_C(y)}{1-y}, & \text{otherwise}\,. \end{cases} \tag{3}$$

Both M (with $\delta_M(x) = x$) and Π (with $\delta_\Pi(x) = x^2$) are vertical as well as horizontal semilinear copulas (as well as lower and upper semilinear copulas). Note that for two vertical (resp. horizontal) semilinear copulas C_1 and C_2 it holds that $C_1 \le C_2$ if and only if $\delta_{C_1} \le \delta_{C_2}$.

The survival copula $\hat{C}$ of a vertical (resp. horizontal) semilinear copula C is again a vertical (resp. horizontal) semilinear copula; the relationship between the corresponding diagonal sections is $\delta_{\hat{C}}(x) = 2x - 1 + \delta_C(1-x)$.

Every horizontal semilinear copula C_H with diagonal section δ is given by $C_H(x,y) = C_V(y,x)$, with C_V the vertical semilinear copula determined by the same diagonal section. Hence, horizontal and vertical semilinear copulas are determined by the same class of diagonals. For that reason, we will restrict our discussion to vertical semilinear copulas.

3. CHARACTERIZATION

Similarly as for lower/upper semilinear copulas, not any diagonal can be the diagonal section of a vertical/horizontal semilinear copula.

Theorem 1. Given a diagonal δ, consider the function $A_\delta : [0,1]^2 \to [0,1]$ defined by

$$A_\delta(x,y) = \begin{cases} y\dfrac{\delta(x)}{x}, & \text{if } y \le x\,, \\ \dfrac{(y-x)x + (1-y)\delta(x)}{1-x}, & \text{otherwise}\,. \end{cases} \tag{4}$$

Then A_δ is a vertical semilinear copula if and only if δ satisfies

(A1) the function $\varphi_\delta :]0,1] \to [0,1]$ defined by $\varphi_\delta(x) = \frac{\delta(x)}{x}$ is increasing;

(A2) the function $\psi_\delta : [0,1[\to [0,1]$ defined by $\psi_\delta(x) = \frac{x-\delta(x)}{1-x}$ is increasing;

(A3) the inequality $\delta(x) \ge x^2$ holds for any $x \in [0,1]$, i.e. $\delta \ge \delta_\Pi$.

Proof. Let A_δ be a vertical semilinear copula.

(i) Consider arbitrary $y > 0$, then the monotonicity of A_δ implies that φ_δ is increasing on $[y,1]$, and hence on $]0,1]$.

(ii) Consider a rectangle $[x,x'] \times [y,y']$ such that $x' \le y$. Since

$$V_{A_\delta}([x,x'] \times [y,y']) = (y'-y)(\psi_\delta(x') - \psi_\delta(x)) \ge 0\,,$$

condition (A2) follows.

(iii) The inequality trivially holds for $x \in \{0,1\}$. Consider $x \in]0,1[$ and a rectangle $[x,x] \times [x',x']$. Since

$$V_{A_\delta}([x,x] \times [x',x']) = (x'-x)(\varphi_\delta(x') - \psi_\delta(x)) \ge 0\,,$$

it follows that $\varphi_\delta(x') \ge \psi_\delta(x)$. Due the continuity of φ_δ and ψ_δ on $]0,1[$, it holds that $\varphi_\delta(x) \ge \psi_\delta(x)$, which is equivalent to (A3).

Conversely, assume that conditions (A1) – (A3) hold. As the boundary conditions are trivially fulfilled, it remains to show that A_δ is 2-increasing. It is sufficient to consider the following three cases.

(i) Consider a rectangle $[x, x'] \times [y, y']$ such that $x \geq y'$. Then

$$V_{A_\delta}([x, x'] \times [y, y']) = (y' - y)(\varphi_\delta(x') - \varphi_\delta(x))$$

is positive due to (A1).

(ii) Consider a rectangle $[x, x'] \times [y, y']$ such that $x' \leq y$. Then

$$V_{A_\delta}([x, x'] \times [y, y']) = (y' - y)(\psi_\delta(x') - \psi_\delta(x))$$

is positive due to (A2).

(iii) Consider $x \in]0, 1[$ and a rectangle $[x, x] \times [x', x']$. Then

$$V_{A_\delta}([x, x] \times [x', x']) = (x' - x)(\varphi_\delta(x') - \psi_\delta(x)) .$$

As condition (A3) states that $\psi_\delta(x) \leq \varphi_\delta(x)$, condition (A2) implies that $\varphi_\delta(x') - \psi_\delta(x) \geq 0$, and hence $V_{A_\delta}([x, x] \times [x', x']) \geq 0$. $\square$

Note that condition (A1) is the same as condition (L1). The class of diagonals satisfying (A1) – (A3) is denoted by $\mathcal{D}_A$, inspired by the fact that the same diagonals lead to horizontal semilinear copulas and that the construction is in essence asymmetric.

Let $\mathcal{D}_{z,a} = \{\delta \in \mathcal{D} \mid \delta(z) = a\}$. Note that $\mathcal{D}_{0,0} = \mathcal{D}_{1,1} = \mathcal{D}$. The following proposition studies the piecewise linear members of $\mathcal{D}_{z,a}$. Its proof is left to the reader.

Proposition 3. Consider the piecewise linear function $\delta_{z,a} : [0, 1] \to [0, 1]$ with graph connecting the points $(0, 0)$, (z, a) and $(1, 1)$, with $z \in]0, 1[$. The following characterizations hold:

(i) $\delta_{z,a} \in \mathcal{D}$ if and only if $a \in [\max(0, 2z - 1), z]$;

(ii) $\delta_{z,a} \in \mathcal{D}_A$ if and only if $a \in [z^2, z]$;

(iii) $\delta_{z,a} \in \mathcal{D}_L$ if and only if $a \in [\frac{z}{2-z}, z]$ [3].

Since

$$\max(0, 2z - 1) < z^2 < \frac{z}{2 - z} ,$$

it is clear that there exist diagonals belonging to $\mathcal{D}_A$, but not to $\mathcal{D}_L$. The following example illustrates that also the converse inclusion does not hold.

Example 1. The diagonal δ defined by

$$\delta(x) = \begin{cases} 2x^2, & \text{if } x \leq 1/2\,, \\ x, & \text{otherwise}\,. \end{cases}$$

belongs to $\mathcal{D}_L$, but not to $\mathcal{D}_A$ as it does not satisfy (A3).

For two continuous random variables whose copula is a vertical/horizontal semilinear copula, various measures of association can be expressed in terms of the corresponding diagonal section. The following proposition is the result of straightforward calculus.

Proposition 4. Let X and Y be continuous random variables whose copula is a vertical/horizontal semilinear copula with diagonal section δ.

(i) The population version of Kendall's tau for X and Y is given by

$$\tau = -2 + 2\int_0^1 \frac{\delta^2(x) + 2x\delta(x) - 4x^2\delta(x) + x^3}{x(1-x)}\,\mathrm{d}x\,.$$

(ii) The population version of Spearman's rho for X and Y is given by

$$\rho = 6\int_0^1 \delta(x)\,\mathrm{d}x\,.$$

(iii) The population version of Gini's gamma for X and Y is given by

$$\gamma = 1 - 4\log 2 + 4\int_{1/2}^1 \frac{\delta(x) + \delta(1-x)}{x}\,\mathrm{d}x\,.$$

4. AGGREGATION OF VERTICAL/HORIZONTAL SEMILINEAR COPULAS

The class of lower (resp. upper) semilinear copulas is closed under a variety of aggregation functions: minimum and maximum, convex sums (weighted arithmetic means) and log-convex sums (weighted geometric means) [3]. These results also partially hold for vertical (resp. horizontal) semilinear copulas.

Proposition 5. The class of vertical (resp. horizontal) semilinear copulas is closed under minimum, maximum and convex sums, and the corresponding diagonal sections are determined by the same aggregation function. More explicitly, if A_{δ_1} and A_{δ_2} are vertical semilinear copulas, then

(i) the copula $A = \min(A_{\delta_1}, A_{\delta_2})$ is a vertical semilinear copula with diagonal section $\delta = \min(\delta_1, \delta_2)$;

(ii) the copula $A = \max(A_{\delta_1}, A_{\delta_2})$ is a vertical semilinear copula with diagonal section $\delta = \max(\delta_1, \delta_2)$;

(iii) the copula $A = \lambda A_{\delta_1} + (1-\lambda)A_{\delta_2}$, with $\lambda \in [0,1]$, is a vertical semilinear copula with diagonal section $\delta = \lambda\delta_1 + (1-\lambda)\delta_2$.

Proof. Since the expressions in (2) are linear w.r.t. the diagonal section, the result obtained by applying one of the mentioned aggregation functions takes the same form, with the corresponding δ obtained by applying the same aggregation function to the individual diagonal sections. Consider for instance $A = \min(A_{\delta_1}, A_{\delta_2})$, then using Eq. (2) it follows that

$$A(x,y) = \begin{cases} y\dfrac{\min(\delta_1(x), \delta_2(x))}{x}, & \text{if } y \le x\,, \\ \dfrac{(y-x)x + (1-y)\min(\delta_1(x), \delta_2(x))}{1-x}, & \text{otherwise}\,, \end{cases}$$

is of the same form with $\delta = \min(\delta_1, \delta_2)$.

This function δ is a diagonal, since $\mathcal{D}$ is closed under the mentioned aggregation functions. Moreover, since also the functions φ and ψ are linear w.r.t. to the diagonal section, the functions φ_δ and ψ_δ can also be obtained by applying the same aggregation function to the corresponding individual functions. Consider for instance again $A = \min(A_{\delta_1}, A_{\delta_2})$, then $\varphi_\delta = \min(\varphi_{\delta_1}, \varphi_{\delta_2})$ and $\psi_\delta = \min(\psi_{\delta_1}, \psi_{\delta_2})$.

It then immediately follows that δ also satisfies properties (A1) – (A3). The claim then follows from Theorem 1. $\square$

In the following example, we show that the class of vertical semilinear copulas is not closed under log-convex sums.

Example 2. One easily verifies that the $[0,1] \to [0,1]$ function δ_1 defined by

$$\delta_1(x) = \max\left(x^2, \frac{3x-1}{2}\right)$$

belongs to $\mathcal{D}_A$. The geometric mean Q of A_{δ_1} and M, i.e. $Q = \sqrt{A_{\delta_1} M}$ is not a copula since

$$V_Q([0.5, 0.75] \times [0.75, 0.81]) < 0\,.$$

Hence, the class of vertical semilinear copulas is not closed under the geometric mean.

Remarkably, the diagonal section ν_1 of Q, i.e. $\nu_1 = \sqrt{\delta_1 \delta_M}$, still is a diagonal. Moreover, a simple computation shows that $\psi_{\nu_1}(1/2) = 1 - 1/\sqrt{2} \simeq 0.29$ and $\psi_{\nu_1}(3/4) = 3 - \sqrt{15/2} \simeq 0.26$. Hence, ν_1 does not belong to $\mathcal{D}_A$.

However, the geometric mean of two diagonals in $\mathcal{D}_A$ is in general not a diagonal. Indeed, the $[0,1] \to [0,1]$ function δ_2 defined by

$$\delta_2(x) = \max\left(x^2, 1.05x - 0.05\right)$$

belongs to $\mathcal{D}_A$, yet the function $\nu_2 = \sqrt{\delta_2 \delta_M}$ is not a diagonal. It suffices to verify that $\nu_2'(0.1) > 2$.

5. SYMMETRIC MEMBERS

Although the definition of vertical and horizontal semilinear copulas is in essence asymmetric, it does not exclude that some of them might be symmetric. If a vertical (resp. horizontal) semilinear copula is symmetric, then it is clearly also a horizontal (resp. vertical) semilinear copula. Conversely, a copula that is both a vertical and a horizontal semilinear copula, is symmetric. Hence, the class of symmetric vertical (resp. horizontal) semilinear copulas coincides with the intersection of the classes of vertical and horizontal semilinear copulas. Moreover, from the definition of the lower and upper semilinear copulas it is immediately clear that the class of symmetric vertical (resp. horizontal) semilinear copulas coincides with the intersection of any two of the classes of lower, upper, vertical or horizontal semilinear copulas. To conclude, the symmetric vertical semilinear copulas are nothing else but the intersection of all classes of semilinear copulas introduced. Next, we characterize this central class.

Proposition 6. A vertical semilinear copula A_δ is symmetric if and only if it has a diagonal section of the form

$$\delta_\lambda(x) = \lambda x^2 + (1-\lambda)x \tag{5}$$

with $\lambda \in [0,1]$.

Proof. Let A_δ be a symmetric vertical semilinear copula. Eq. (5) trivially holds for $x \in \{0,1\}$. Consider $0 < x \le y < 1$, then expressing symmetry leads to

$$\frac{(y-x)x + (1-y)\delta_A(x)}{1-x} = x\frac{\delta_A(y)}{y}\,,$$

or, equivalently,

$$y + (1-y)\frac{\delta_A(x)}{x} = x + (1-x)\frac{\delta_A(y)}{y}\,.$$

Consequently, for any $z > y$ it also holds that

$$z + (1-z)\frac{\delta_A(x)}{x} = x + (1-x)\frac{\delta_A(z)}{z}\,.$$

Combining the above leads to

$$(z-y)\left(1 - \frac{\delta_A(x)}{x}\right) = (1-x)\left(\frac{\delta_A(z)}{z} - \frac{\delta_A(y)}{y}\right)\,,$$

or, equivalently,

$$\frac{1-\frac{\delta_A(x)}{x}}{1-x} = \frac{1}{z-y}\left(\frac{\delta_A(z)}{z} - \frac{\delta_A(y)}{y}\right)\,.$$

Due to (A1) and (A3), the left-hand side takes values in $[0,1]$. Let f be the $]0,1[\to [0,1]$ function defined by this left-hand side, then the above expresses that f is constant on $]0,y]$, for any fixed y. Consequently,

$$\frac{1-\frac{\delta_A(x)}{x}}{1-x} = \lambda\,,$$

for any $x \in]0,1[$, with $\lambda \in [0,1]$. Eq. (5) then follows immediately.

Conversely, one easily verifies that the function δ defined by Eq. (5) is indeed a diagonal and satisfies properties (A1)−(A3). The symmetry of A_δ follows from the above reasoning. □

Proposition 5 then immediately implies the following corollary.

Corollary 1. A vertical semilinear copula is symmetric if and only if it is a convex sum of Π and M.

Example 3. The above results do not imply that the diagonals $\delta_\lambda = \lambda\delta_P + (1-\lambda)\delta_M$ are the only ones belonging to $\mathcal{D}_L \cap \mathcal{D}_A$. However, for any $\delta \in \mathcal{D}_L \cap \mathcal{D}_A$ not of this type, the corresponding lower semilinear copula S_δ and vertical semilinear copula A_δ are different. For instance, the power function $\delta_p : [0,1] \to [0,1]$ defined by $\delta_p(x) = x^p$, with $p \in]0,\infty[$, is a diagonal if and only if $p \in [1,2]$, in which case δ_p also belongs to $\mathcal{D}_L \cap \mathcal{D}_A$. If $p \in]1,2[$, then A_{δ_p} is asymmetric, and thus different from S_{δ_p}. Another example of a diagonal $\delta \in \mathcal{D}_L \cap \mathcal{D}_A$ is the $[0,1] \to [0,1]$ function defined by $\delta(x) = x/(2-x)$.

6. MAXIMALLY ASYMMETRIC MEMBERS

The asymmetry ε_C of a copula C is defined by

$$\varepsilon_C = \sup\{|C(x,y) - C(y,x)| \mid (x,y) \in [0,1]^2\}\,.$$

Klement and Mesiar have shown that the maximal asymmetry of a copula is $1/3$ [7] (see also [9]).

Condition (A3) and the fact that comparable diagonal sections lead to comparable vertical (resp. horizontal) semilinear copulas imply that any vertical (resp. horizontal) semilinear copula A_δ is *positive quadrant dependent*, i.e. $A_\delta \geq \Pi$. We therefore investigate first the asymmetry of these copulas. Remarkably, the maximal asymmetry of such a copula is only about half of the general maximal asymmetry. This suggests that positive quadrant dependence acts in favour of exchangeable models.

Proposition 7. The maximal asymmetry of a positive quadrant dependent copula is $3 - 2\sqrt{2} \simeq 0.172$.

Proof. Let C be a positive quadrant dependent copula. Consider $x < y$ and suppose that $C(x,y) \leq C(y,x)$. If $C(y,x) = \min(x,y) = x$ and $C(x,y) = xy$, then the maximal difference $C(y,x) - C(x,y) = x(1-y)$ is obtained. The monotonicity and 1-Lipschitz property of C imply that

$$C(y,x) = x \leq C(x,x) + y - x \leq C(x,y) + y - x = xy + y - x\,,$$

and hence $x \leq y/(2-y)$. Consequently,

$$C(y,x) - C(x,y) \leq \frac{y(1-y)}{2-y}\,.$$

The $[0,1] \to \mathbb{R}$ function h defined by $h(y) = \frac{y(1-y)}{2-y}$ reaches it maximum value $3-2\sqrt{2}$ in the point $y_0 = 2-\sqrt{2}$, with corresponding $x_0 = y_0/(2-y_0) = \sqrt{2}-1$. Hence, $\epsilon_C \leq 3-2\sqrt{2} \simeq 0.172$.

Moreover, this upper bound is attained, for instance by the bilinear extension [8] of the discrete copula $C_0 : \{0, x_0, y_0, 1\}^2 \to [0,1]$ defined by $C_0(x_0, x_0) = C_0(x_0, y_0) = x_0 y_0$ and $C_0(y_0, y_0) = C_0(y_0, x_0) = x_0$. □

In the next proposition, we study the class of diagonals δ satisfying (A1) – (A3) and $\delta(z) = a$,

Proposition 8. Consider $z \in]0,1[$. The class of diagonals $\mathcal{D}_A \cap \mathcal{D}_{z,a}$ is convex with greatest element δ^*, defined by

$$\delta^*(x) = \begin{cases} \dfrac{xa}{z}, & \text{if } x \in [0,z]\,, \\[2ex] \dfrac{x(1-a)-(z-a)}{1-z}, & \text{otherwise}\,, \end{cases} \tag{6}$$

and smallest element δ_*, defined by

$$\delta_*(x) = \begin{cases} \dfrac{x(1-a)-(z-a)}{1-z}, & \text{if } x \in [\frac{z-a}{1-z}, z]\,, \\[2ex] \dfrac{xa}{z}, & \text{if } x \in [z, \frac{a}{z}]\,, \\[2ex] x^2, & \text{otherwise}\,. \end{cases} \tag{7}$$

Proof. The convexity is an immediate consequence of the convexity of $\mathcal{D}_A$ (Proposition 5) and the convexity of $\mathcal{D}_{z,a}$. Consider $\delta \in \mathcal{D}_A$ such that $\delta(z) = a$. Condition (A1) implies that $\delta(x) \leq (xa)/z$ on $[0,z]$ and $\delta(x) \geq (xa)/z$ on $[z,1]$. Similarly, (A2) implies that $\delta(x) \geq [x(1-a)-(z-a)]/(1-z)$ on $[0,z]$ and $\delta(x) \leq [x(1-a)-(z-a)]/(1-z)$ on $[z,1]$. Combining the upper bounds leads to δ^*. The lower bounds, together with (A3), lead to δ_*. One easily verifies that δ^* and δ_* satisfy (A1) – (A3). □

Setting $a := z$ one easily verifies that $\delta^* = \delta_* = \delta_M$, whence the following corollary.

Corollary 2. If a diagonal $\delta \in \mathcal{D}_A$ has an internal fixpoint, then $\delta = \delta_M$.

Proposition 8 will assist us in identifying the maximal asymmetry of vertical semilinear copulas. As can be expected, this maximal asymmetry is again lower than that for positive quadrant dependent copulas.

Proposition 9. The maximal asymmetry of a vertical semilinear copula is $1/16$. Examples of maximally asymmetric vertical semilinear copulas are:

(i) A_{δ_*} with $\delta_* \in \mathcal{D}_A \cap \mathcal{D}_{1/2,3/8}$ (in the point $(x,y) = (1/4,1/2)$);

(ii) A_{δ^*} with $\delta^* \in \mathcal{D}_A \cap \mathcal{D}_{3/4,9/16}$ (in the point $(x,y) = (1/2,3/4)$).

Proof. Consider a vertical semilinear copula A_δ with $\delta(y) = a$, with $y \in]0,1[$. Consider $0 < x < y$, then

$$|A(x,y) - A(y,x)| = \left|\frac{(y-x)x + (1-y)\delta(x)}{1-x} - \frac{xa}{y}\right|.$$

Due to Proposition 8 it holds that

$$\begin{aligned}|A(x,y) - A(y,x)| \leq \max\Bigg(&\left|\frac{(y-x)x + (1-y)\delta_*(x)}{1-x} - \frac{xa}{y}\right|,\\ &\left|\frac{(y-x)x + (1-y)\delta^*(x)}{1-x} - \frac{xa}{y}\right|\Bigg).\end{aligned}$$

For fixed y and a, the expression

$$\left|\frac{(y-x)x + (1-y)\delta^*(x)}{1-x} - \frac{xa}{y}\right|,$$

attains its maximal value $\frac{(y-a)(1-\sqrt{1-y})^2}{y}$ in $x_0 = 1 - \sqrt{1-y}$. For fixed y, the expression $\frac{(y-a)(1-\sqrt{1-y})^2}{y}$ attains its maximal value $g(y) := (1-y)(1-\sqrt{1-y})^2$ when $a = y^2$. The $[0,1] \to \mathbb{R}$ function g attains its maximal value $1/16$ in $y_0 = 3/4$ (with corresponding $x_0 = 1/2$), thus for the copula A_{δ^*} with $\delta^* \in \mathcal{D}_A \cap \mathcal{D}_{3/4,9/16}$ (in the point $(x_0,y_0) = (1/2,3/4)$).

Similarly, it can be shown that for fixed y and a, the expression

$$\left|\frac{(y-x)x + (1-y)\delta_*(x)}{1-x} - \frac{xa}{y}\right|$$

attains its maximal value $\frac{(a-y^2)(y-a)}{y(1-y)}$ in $x_0 = \frac{y-a}{1-y}$. For fixed y, the expression $\frac{(a-y^2)(y-a)}{y(1-y)}$ attains its maximal value $h(y) := \frac{y(1-y)}{4}$ when $a = \frac{y+y^2}{2}$. The $[0,1] \to \mathbb{R}$ function h attains its maximal value $1/16$ in $y_0 = 1/2$ (with corresponding $x_0 = 1/4$), thus for the copula A_{δ_*} with $\delta_* \in \mathcal{D}_A \cap \mathcal{D}_{1/2,3/8}$ (in the point $(x_0,y_0) = (1/4,1/2)$). □

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Bernard De Baets, Department of Applied Mathematics, Biometrics and Process Control, Ghent University, Coupure links 653, B-9000 Gent. Belgium.
e-mail: bernard.debaets@ugent.be

Hans De Meyer, Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281 S9, B-9000 Gent. Belgium.
e-mail: hans.demeyer@ugent.be

Radko Mesiar, Department of Mathematics and Descriptive Geometry, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia and Institute for Research and Applications of Fuzzy Modelling, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic.
e-mail: mesiar@math.sk