Kybernetika

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Kybernetika, Vol. 42 (2006), No. 5, 539--556

Persistent URL: http://dml.cz/dmlcz/135734

KYBERNETIKA — VOLUME 42 (2006), NUMBER 5, PAGES 539–556

NONLINEAR FILTERING IN SPATIO–TEMPORAL DOUBLY STOCHASTIC POINT PROCESSES DRIVEN BY OU PROCESSES

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Doubly stochastic point processes driven by non-Gaussian Ornstein–Uhlenbeck type processes are studied. The problem of nonlinear filtering is investigated. For temporal point processes the characteristic form for the differential generator of the driving process is used to obtain a stochastic differential equation for the conditional distribution. The main result in the spatio-temporal case leads to the filtering equation for the conditional mean.

Keywords: Cox process, filtering, Ornstein–Uhlenbeck process

AMS Subject Classification: 60G55, 60K35

1. INTRODUCTION

The filtering in point processes is an old problem which comes from applications in medical diagnostics and optical communications (see [14] and the references therein). The counts registered in time are modelled as a realization of an inhomogeneous point process the intensity of which depends on a random function. This is the doubly stochastic point process situation (particularly if the inhomogeneous point process is conditionally Poisson then the resulting doubly stochastic point process is Cox). The filtering aims to estimate the random function given the observed counts. A nonlinear estimation leads to the conditional expectation evaluation studied by many authors (cf. [3, 10, 11, 15]). It is of interest to consider the case when the random function has a well-defined representation as a solution of a stochastic differential equation. In this case the result is again in a form of a stochastic differential equation which can be solved numerically. Various special cases – when the random function is a Gauss–Markov diffusion process or a Poisson driven Markov process, were solved in [15].

In [2] we extended the solution for temporal Cox processes to the case when the random intensity is a function of an Ornstein–Uhlenbeck (OU) type process ([1]) derived from a Lévy process. This leads to a wider class of intensity processes with prescribed marginal distribution, typically nonnegative. Therefore the diffusion component of the Lévy process was omitted (also because for this component the problem was solved). We can surely obtain nonnegativity by a suitable transformation but in that way we loose the linear character of innovations given by the stochastic differential equation for OU processes.

Further generalization which is of great interest is that from temporal to spatio-temporal point processes. Here still the time dynamics is dominant but a random spatial coordinate is added to each count, representing e. g. the location of a moving detector. A pioneering work in this field is [7] which developes the notion of a conditional intensity (cf. [6]) and a representation theorem for the conditional characteristics. In the applications besides filtering also prediction and smoothing are considered and results for Itô process obtained. The aim of the present paper is to develop analogous results for the broad class of driving random functions of OU type including the case of infinite activity underlying Lévy process with finite variation. In this new setting a substantial extension of models is obtained.

The structure of the paper is as follows: in Section 2 the necessary background from Lévy and OU type processes is gathered. In Section 3 the filtering problem is defined and some comments to previous solutions given. In Section 4 we turn to the spatio-temporal situation. The doubly stochastic analytic point processes are defined and their properties recalled. Section 5 yields the main result, the filtering equation for a spatio-temporal point process of given type is developed.

2. LÉVY PROCESSES AND PROCESSES OF ORNSTEIN–UHLENBECK TYPE

In this section we review the basic definitions and properties of the Lévy processes and the Ornstein–Uhlenbeck type processes derived from them. The general theory about Lévy processes was taken from [12] and [13], and the theory about Ornstein–Uhlenbeck type processes derived from Lévy processes and some more specialized results come from [1] and [4].

Definition 1. Let $Z = \{Z(t)\}_{t\geq 0}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ taking values in $\mathbb{R}^d$ whose realizations are right continuous with limits from the left almost surely (rcll). Suppose that Z is stochastically continuous with stationary and independent increments and $Z(0) = 0$ a. s. Then Z is called a Lévy process.

In the paper we will denote the limit from the left by $Z(t_-) = \lim_{s\uparrow t} Z(s)$.

Definition 2. Let X be a random variable taking values in $\mathbb{R}^d$. The cumulant transform $C\{\cdot \ddagger X\}$ of X is defined by

$$C\{v \ddagger X\} = \log \mathsf{E}\,[e^{i\langle v,\, X\rangle}], \qquad v \in \mathbb{R}^d,$$

where $\langle\cdot,\,\cdot\rangle$ denotes the scalar product.

The following convenient property holds for the cumulant transform of any Lévy process.

Theorem 1. (Sato [13], Theorems 1.1 and 1.3) Let $Z = \{Z(t)\}_{t\geq 0}$ be a Lévy process taking values in $\mathbb{R}^d$. Then for the cumulant transform of Z holds

$$C\{v \ddagger Z(t)\} = t\, C\{v \ddagger Z(1)\}, \quad \text{for any } t \geq 0,\ v \in \mathbb{R}^d, \tag{1}$$

and $Z(1)$ has the Lévy–Khintchin representation

$$C\{v\ddagger Z(1)\} = i\,\langle a,\, v\rangle - \frac{1}{2}\,\langle v,\, Av\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle v,\, x\rangle} - 1 - i\,\langle v,\, x\rangle\, \mathbf{1}\{|x| \leq 1\}\right) \mu(\mathrm{d}x), \tag{2}$$

where $|x|$ denotes the Euclidean norm of x, A is a symmetric nonnegative-definite $d \times d$ matrix, $a \in \mathbb{R}^d$, and μ is a measure on $\mathbb{R}^d$ satisfying $\mu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\mu(\mathrm{d}x) < \infty.$$

The triplet (a, A, μ) is unique.

Definition 3. We call the triplet (a, A, μ) from the previous theorem the generating triplet, μ is called the Lévy measure of the process $Z(t)$.

The Lévy–Itô decomposition of a Lévy process into a deterministic, a Brownian diffusion and a pure jump part is now formulated (it is a special case of Theorem 1.4 from [13]).

Theorem 2. Let $Z(t)$ be a Lévy process taking values in $\mathbb{R}^d$ with the generating triplet (a, A, μ). For any $G \in \mathcal{B}((0, \infty) \times \mathbb{R}^d)$ let $J_Z(G) = J_Z(G, \omega)$ be the number of jumps at time s with height $Z(s, \omega) - Z(s_-, \omega)$ such that $(s, Z(s, \omega) - Z(s_-, \omega)) \in G$. Then $J_Z(G)$ has Poisson distribution with mean $\tilde{\mu}(G)$. If $G_1, \ldots, G_n$ are disjoint, then $J_Z(G_1), \ldots, J_Z(G_n)$ are independent. We can define, a. s.,

$$\begin{aligned} Z_1(t, \omega) \;&=\; \lim_{\epsilon \to 0} \int_{(0,t]\times\{\epsilon<|x|\leq 1\}} \{x J_Z(\mathrm{d}s, \mathrm{d}x, \omega) - x\tilde{\mu}(\mathrm{d}s, \mathrm{d}x)\} \\ &\quad + \int_{(0,t]\times\{|x|>1\}} x J_Z(\mathrm{d}s, \mathrm{d}x, \omega), \end{aligned} \tag{3}$$

where the convergence on the right-hand side is uniform in t in any finite interval a. s. The process $\{Z_1(t)\}$ is a Lévy process with the generating triplet $(0, 0, \mu)$. Let

$$Z_2(t, \omega) = Z(t, \omega) - Z_1(t, \omega).$$

Then $\{Z_2(t)\}$ is an a. s. continuous Lévy process with the generating triplet $(a, A, 0)$, $Z_2(t) = at + B(t)$, $B(t)$ is the Brownian motion with covariance matrix A. The processes $\{Z_1(t)\}$ and $\{Z_2(t)\}$ are independent.

Corollary 3. (Cont and Tankov [4] Proposition 3.7) Each Lévy process can be represented for any $\epsilon > 0$ as $Z(t) = Z^\epsilon(t) + R^\epsilon(t)$ where $Z^\epsilon(t)$ is a process with finite number of jumps on each bounded time interval and $R^\epsilon(t)$ is a mean-zero square integrable martingale such that its variation $\mathrm{Var}(R^\epsilon(t)) \to 0$ as $\epsilon \to 0$.

The properties of J_Z in Theorem 2 show that it is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $\tilde{\mu}(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s\mu(\mathrm{d}x)$. Thus for a deterministic measurable function f on $[0,t] \times \mathbb{R}^d$ the integral with respect to J_Z

$$\int_0^t \int_{\mathbb{R}^d} f(s,y) J_Z(\mathrm{d}s, \mathrm{d}y) = \sum_{n, t_n \in [0,t]} f(t_n, y_n),$$

is a stochastic process with jumps $y_n \in \mathbb{R}^d$ at times t_n, where $J_Z = \sum_{n \geq 1} \delta_{(t_n, y_n)}$ (δ denotes the Dirac measure).

We will consider the case $A = 0$. Then $\mu(\mathbb{R}^d) < \infty$ leads to a pure jump process with finitely many jumps on each finite time interval while when $\mu(\mathbb{R}^d) = \infty$ (infinite activity case) the jump times form a countable dense set in $\mathbb{R}_+$. Nevertheless, even when $\mu(\mathbb{R}^d) = \infty$ the trajectories of Z may have finite variation. This happens if and only if

$$\int_{|x| \leq 1} |x| \mu(\mathrm{d}x) < \infty. \tag{4}$$

In the sequel we will always assume that $A = 0$ and (4) holds for the process $Z(t)$. When $a = \int_{|x| \leq 1} x \mu(\mathrm{d}x)$ it holds

$$C\{v \ddagger Z(t)\} = t \int_{\mathbb{R}^d} (e^{i\langle v, x \rangle} - 1) \mu(\mathrm{d}x), \tag{5}$$

and

$$Z(t) = \int_{\mathbb{R}^d} \int_0^t x J_Z(\mathrm{d}s, \mathrm{d}x), \tag{6}$$

is a purely jump process with finite variation. If moreover $\mu(\mathbb{R}^d) < \infty$ then $Z(t)$ is a compound Poisson process, i. e. it has only finite number of jumps in any bounded interval.

Now we can introduce the Ornstein–Uhlenbeck type (OU type) processes. Suppose first that $d = 1$.

Definition 4. Let $Z(t)$ be a one-dimensional Lévy process, $\gamma > 0$ and consider the stochastic differential equation for $X(t)$, $t \geq 0$

$$\mathrm{d}X(t) = -\gamma X(t) \, \mathrm{d}t + \mathrm{d}Z(\gamma t). \tag{7}$$

The stationary solution of (7) is called a process of Ornstein–Uhlenbeck type (OU type process). $Z(t)$ is called the background driving Lévy process (BDLP) for $X(t)$.

To be able to specify the conditions under which (7) has a desired solution we need one more definition.

Definition 5. A random variable Y with characteristic function ψ has a self-decomposable distribution if for all $c \in (0,1)$ there exists a characteristic function ψ_c such that

$$\psi(v) = \psi(cv)\psi_c(v) \qquad \text{for all } v \in \mathbb{R}.$$

Theorem 4. (Barndorff-Nielsen and Shephard [1], Theorem 1) Let ψ be the characteristic function of a random variable X. If X is self-decomposable then there is a stationary stochastic process $X(t)$ and a Lévy process $Z(t)$ such that $X(t) \overset{D}{=} X$ (equality in distribution) and

$$X(t) = e^{-\gamma t} X(0) + \int_0^t e^{-\gamma(t-s)} \,\mathrm{d}Z(\gamma s), \tag{8}$$

for all $\gamma > 0$, thus $X(t)$ satisfies (7).

The process (8) is a unique stochastically continuous Markov process and it has a modification with right-continuous realizations with left limits. We will always work with this rcll modification of $X(t)$.

We can also start with the BDLP $Z(t)$ – there exists a sufficient condition on $Z(t)$ for the existence of a stationary solution of the equation (7). For the general case see e. g. [9] Theorem 3.6.6. Here we discuss the case of purely jump $Z(t)$ with finite variation.

Lemma 5. (Barndorff-Nielsen and Shephard [1], Lemma 1) Let $Z(t)$ be a Lévy process specified by (5) and assume that for its Lévy measure holds

$$\int_1^\infty \log(x)\mu(\,\mathrm{d}x) < \infty. \tag{9}$$

Then there exists a unique solution of the equation (7) and $X(t)$ can be written as (8). For the cumulant transform of $X(t)$ holds

$$C\{\zeta \ddagger X(t)\} = \int_0^\infty C\{e^{-s}\zeta \ddagger Z(1)\} \,\mathrm{d}s. \tag{10}$$

If we moreover suppose that μ has a differentiable density w and we define a function u by

$$u(x) = \int_1^\infty w(vx) \,\mathrm{d}v, \tag{11}$$

then u is the Lévy density (density of the Lévy measure) of the marginal distribution of the process $X(t)$ and w can be computed from u by

$$w(x) = -u(x) - xu'(x), \tag{12}$$

where $u'(x)$ denotes the derivative of u.

For $d > 1$ let $Z(t)$ be the d-dimensional Lévy process with characteristic function given by (5) and suppose for simplicity that μ has the density w with respect to d-dimensional Lebesgue measure and denote by $w_i(x_i)$ the ith marginal of w, i. e.

$$w_i(x_i) = \int_{\mathbb{R}^{(d-1)}} w(x)\,\mathrm{d}x_1 \dots \mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1} \dots \mathrm{d}x_d.$$

If each w_i satisfies condition (9) then we may (on account of Lemma 5) define the stationary processes $X_i(t)$ by

$$X_i(t) = e^{-\gamma t} X_i(0) + \int_0^t e^{-\gamma(t-s)}\,\mathrm{d}Z_i(\gamma s).$$

The vector process $X(t) = (X_1(t), \dots, X_d(t))$ is then the solution of the vector stochastic differential equation

$$\mathrm{d}X(t) = -\gamma X(t)\,\mathrm{d}t + \mathrm{d}Z(\gamma t), \tag{13}$$

where $\gamma > 0$, and it is a vector OU type process.

3. FILTERING PROBLEM

In this section we will briefly review some solutions of the filtering problem for the temporal Cox point processes. Let Φ be a Cox point process on $\mathbb{R}_+$ with driving random measure Λ. That means conditionally $\Phi \mid (\Lambda = \bar{\Lambda})$ is a Poisson point process with intensity measure $\bar{\Lambda}$. Assume that Λ is absolutely continuous with respect to the Lebesgue measure, with density λ. The random process λ is called the driving intensity (function) of Φ. We will make the following assumptions on λ

(L1) $\{X(t)\}_{t \geq 0}$ is an $\mathbb{R}^d$ valued stochastic process and the driving intensity of Φ has the form

$$\lambda(t) = \lambda(t, X(t)),$$

where $\lambda : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}_+$ is a positive function, $X(t)$ will be also called the driving process.

(L2) It holds

$$\mathsf{E}\,(\lambda(t, X(t)) < \infty, \qquad \text{for all } t \in \mathbb{R}_+.$$

Now we can formulate the filtering problem for Φ. Denote the number of points of Φ in $[0, t]$ as

$$N(t) = \Phi([0, t]), \qquad t \in \mathbb{R}_+.$$

In the filtering problem our aim is to find the MSE optimal estimator $\widehat{\lambda}$ of $\lambda(t, X(t))$ given $\{N(s), 0 \leq s < t\}$, i. e. the random function $\widehat{\lambda}$ that minimizes

$$\mathsf{E}\,[|\lambda(t, X(t)) - \widehat{\lambda}|^2 \mid N(s), 0 \leq s < t].$$

The solution is the conditional expectation of the driving intensity

$$\widehat{\lambda}(t) = \mathsf{E}\,[\lambda(t, X(t)) \mid N(s),\, 0 \leq s < t]. \tag{14}$$

Thus our problem reduces to the evaluation of (14).

To shorten the notation we will denote by hat all the conditional characteristics given the sample path of Φ, i. e.

$$\widehat{\mathsf{E}}\,[\,\cdot\,] = \mathsf{E}\,[\cdot \mid N(s),\ 0 \leq s < t].$$

For example for the conditional characteristic function of $X(t)$ we write

$$\widehat{\psi}_t(v) = \widehat{\mathsf{E}}\,[e^{i\langle v,\, X(t)\rangle}].$$

Definition 6. Suppose $X(t)$ is a Markov process and there exists a nonnegative function $g_t(v, X(t))$ with finite mean such that for all $\triangle t > 0$ it holds

$$\frac{1}{\triangle t}\left|\mathsf{E}\left[e^{i\langle v,\, \triangle X(t)\rangle} - 1 \mid X(t)\right]\right| \leq g_t(v, X(t)), \tag{15}$$

where $\triangle X(t) = X(t + \triangle t) - X(t)$.

The characteristic form for the differential generator of $X(t)$ is defined as

$$\Psi_t(v \mid X(t)) = \lim_{\triangle t \downarrow 0} \frac{1}{\triangle t}\mathsf{E}\left[e^{i\langle v,\, \triangle X(t)\rangle} - 1 \mid X(t)\right], \tag{16}$$

if the right hand side exists.

In [15] the following theorem is proved.

Theorem 6. (Snyder and Miller [15], Theorem 7.4.2) Let Φ be a Cox process on $\mathbb{R}_+$ with driving intensity λ. Suppose (L1) and (L2) are satisfied for the $\mathbb{R}^d$-valued driving process $\{X(t)\}_{t\geq 0}$ which is Markov stochastically continuous and suppose that $\lambda(t, X(t))$ is left continuous. Then the following differential equation holds for the conditional characteristic function of $X(t)$

$$\begin{aligned} \mathrm{d}\widehat{\psi}_t(v) &= \widehat{\mathsf{E}}\left[e^{i\langle v,\, X(t)\rangle}\Psi_t(v \mid X(t))\right]\mathrm{d}t \\ &\quad + \widehat{\mathsf{E}}\left[e^{i\langle v,\, X(t)\rangle}(\lambda(t, X(t)) - \widehat{\lambda}(t))\right]\frac{1}{\widehat{\lambda}(t)}(\mathrm{d}N(t) - \widehat{\lambda}(t)\,\mathrm{d}t), \\ \widehat{\psi}_0(v) &= \mathsf{E}\ e^{i\langle v,\, X(0)\rangle}. \end{aligned} \tag{17}$$

Equations of this type for temporal Cox point processes with driving Markov jump or diffusion processes of special types were obtained by [3, 10, 11, 14]. Inverse Fourier transform of (17) leads to a differential equation for the conditional distribution of $X(t)$ given $\{N(s),\ 0 \leq s < t\}$. This equation can be solved numerically recursively in time, cf. [15]. The desired estimate of $\widehat{\lambda}(t)$ is obtained by evaluating the expectation of $\lambda(t, X(t))$ with respect to this distribution. Our aim is not the numerical solution but the extension of the class of models for the theoretical study of the filtering problem.

Consider a Cox point process Φ driven by an OU type vector process $X(t)$. The characteristic form of the differential operator for such $X(t)$ was derived in [2].

Lemma 7. (Beneš and Prokešová [2], Lemma 1) Let $X(t)$ be an OU type d-dimensional process given as a solution of (13) with $\gamma > 0$ and with the background driving Lévy process $Z(t)$ satisfying equation (5). If $X(t)$ has finite mean then the characteristic form for the differential generator for $X(t)$ is

$$\Psi_t(v \mid X(t)) = -i\gamma \langle v, X(t)\rangle + \gamma \int_{\mathbb{R}^d} (e^{i\langle v, x\rangle} - 1)\mu(\mathrm{d}x). \tag{18}$$

The main result of [2] is the following theorem which yields the differential equation for the probability density of $\widehat{X}(t)$.

Theorem 8. (Beneš and Prokešová [2], Theorem 2) Under the assumptions of Theorem 6 and Lemma 7 let the Lévy measure μ have density w with respect to the Lebesgue measure. If $\mu(\mathbb{R}^d) < \infty$ the conditional probability density of $X(t)$ given $\{N(s), 0 \le s < t\}$, satisfies

$$\begin{aligned} \mathrm{d}\widehat{p}_t(x) &= \gamma\left(\widehat{p}_t(x)(1-\mu(\mathbb{R}^d)) + (\widehat{p}_t * w)(x) + \left\langle x, \frac{\mathrm{d}\widehat{p}_t(x)}{\mathrm{d}x}\right\rangle\right)\mathrm{d}t \\ &\quad + \widehat{p}_t(x)\left(\lambda(t,x) - \widehat{\lambda}(t)\right)\frac{1}{\widehat{\lambda}(t)}\left(\mathrm{d}N(t) - \widehat{\lambda}(t)\,\mathrm{d}t\right), \end{aligned} \tag{19}$$

where $\widehat{p}_t * w$ denotes the convolution of $\widehat{p}_t$ and w.
Generally (also for $\mu(\mathbb{R}^d) = \infty$) it holds

$$\begin{aligned} \mathrm{d}\widehat{p}_t(x) &= \gamma\left(F^{-1}(\widehat{\psi}_t \cdot \log(\psi_{Z(1)})) + \widehat{p}_t(x) + \left\langle x, \frac{\mathrm{d}\widehat{p}_t(x)}{dx}\right\rangle\right)\mathrm{d}t \\ &\quad + \widehat{p}_t(x)\left(\lambda(t,x) - \widehat{\lambda}(t)\right)\frac{1}{\widehat{\lambda}(t)}\left(\mathrm{d}N(t) - \widehat{\lambda}(t)\,\mathrm{d}t\right), \end{aligned} \tag{20}$$

where F^{-1} denotes the inverse Fourier transform.

In the following examples we demonstrate both situations from the Theorem 8 for $d = 1$.

Example 1. a) The gamma OU process $X(t)$ has marginal probability density

$$p(x) = \frac{\alpha^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\alpha x}, \quad x \ge 0, \qquad p(x) = 0, \quad x < 0, \tag{21}$$

with parameters $\alpha > 0$, $\nu > 0$. The Lévy density of the gamma distribution is

$$u(x) = \nu \frac{1}{x}\, e^{-\alpha x}, \qquad x \ge 0,$$

Now using the equation (12) from Lemma 5 we can evaluate the density w of the Lévy measure μ of $Z(1)$

$$w(x) = \alpha \nu e^{-\alpha x}, \quad x \ge 0, \qquad w(x) = 0, \quad x < 0,$$

μ is finite. From Lemma 7 we get

$$\Psi_t(v \mid X(t)) = \nu\gamma \left(\frac{\alpha}{\alpha - iv} - 1 \right) - iv\gamma X(t) \tag{22}$$

and from Theorem 8 we get for the gamma OU type process

$$\begin{aligned} \mathrm{d}\widehat{p}_t(x) &= -\nu\gamma\, (\widehat{p}_t(x) - (\widehat{p}_t * \mathcal{E}_\alpha)(x)))\,\mathrm{d}t \\ &\quad + \gamma \left(\widehat{p}_t(x) + x \frac{\mathrm{d}\widehat{p}_t(x)}{\mathrm{d}x} \right) \mathrm{d}t \\ &\quad + \widehat{p}_t(x) \left(\lambda(t,x) - \widehat{\lambda}(t) \right) \frac{1}{\widehat{\lambda}(t)} \left(\mathrm{d}N(t) - \widehat{\lambda}(t)\,\mathrm{d}t \right), \end{aligned} \tag{23}$$

where $\mathcal{E}_\alpha$ is the density of the exponential distribution with parameter α.

b) The inverse Gaussian (IG) OU type process $X(t)$ has the marginal probability density

$$p\,(x) = \frac{\delta}{\sqrt{2\pi}}\, e^{-\delta\gamma}\, x^{-\frac{3}{2}}\, e^{-\frac{1}{2}(\,\frac{\delta^2}{x}\,+\gamma^2 x)}, \qquad \delta > 0, \quad \gamma \ge 0, \quad x \ge 0$$

and the Lévy density

$$u(x) = \frac{1}{\sqrt{2\pi}}\, \delta\, x^{-\frac{3}{2}}\, e^{-\frac{\gamma^2 x}{2}}, \qquad x \ge 0.$$

Using Lemma 5 we obtain the Lévy density of the Lévy measure μ of $Z(1)$

$$w(x) = \frac{1}{\sqrt{2\pi}}\, \frac{\delta}{2}\, (\,\frac{1}{x} + \gamma^2)\, \frac{1}{\sqrt{x}}\, e^{-\frac{\gamma^2 x}{2}}, \qquad x \ge 0.$$

Here we can see that $\int_{\mathbb{R}} w(x)\,\mathrm{d}x = \infty$ (infinite activity case with finite variation – (4) holds) and thus equation (20) from Theorem 8 applies.

4. SPATIO–TEMPORAL DOUBLY STOCHASTIC ANALYTIC POINT PROCESSES

Spatio-temporal point processes can be defined in several ways (cf. [5]). We follow the approach of Fishman and Snyder [7] based on the notion of an analytic point process. Further doubly stochastic spatio-temporal point processes are defined and the problem of filtering is studied.

A spatio-temporal point process is a point process Φ defined on $\mathcal{X} = [0,\infty) \times \mathbb{R}^k$. Let $\mathcal{N}$ be the set of all locally finite simple counting measures on $\mathcal{X}$ (simple means that $\phi(\{x\}) \leq 1$ for all $x \in \mathcal{X}, \phi \in \mathcal{N}$). Denote $\mathfrak{N}$ the smallest σ-algebra on $\mathcal{N}$ which makes mappings $\phi \mapsto \phi(B)$ measurable for each Borel set $B \subset \mathcal{X}$. We will put $(\Omega, \mathcal{F}) = (\mathcal{N}, \mathfrak{N})$, with Φ being the identity map from $(\Omega, \mathcal{F}, \mathsf{P})$ to $\mathcal{N}$, P is the probability distribution of Φ and let

$$\mathsf{E}\,\Phi([0,t) \times \mathbb{R}^k) < \infty \qquad \text{for any} \quad t > 0. \tag{24}$$

The symbol $N(t),\ t > 0$ will now denote the random variable equal to $\Phi([0,t) \times \mathbb{R}^k)$, $N(0) = 0$.

A simple counting measure is characterized by its support. Thus we can identify the realizations $\Phi(\omega), \omega \in \Omega$ as

$$\Phi(\omega) = \{(t_1, r_1), (t_2, r_2), \dots\},$$

where $0 \leq t_1 \leq t_2 \leq \dots$ are the times of events and $r_i \in \mathbb{R}^k$ their locations. For $j = 1, 2, \dots$ let $U_j = \{(t_1, r_1), (t_2, r_2), \dots, (t_j, r_j)\}$ and $\mathcal{F}_j \subset \mathcal{F}$ be the $\sigma-$algebra generated by U_j. Further let $\mathcal{F}_t = \sigma\{\Phi \mid_{[0,t)\times\mathbb{R}^k}\} \subset \mathcal{F}$ be the $\sigma-$algebra generated by the past of the process up to time t, $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Definition 7. A spatio-temporal point process Φ is analytic if the following conditions hold:

a) $\mathsf{P}(N(t) < \infty) = 1$ for all $t \geq 0$ finite.

b) The measure $P_j(Q) = \mathsf{P}(U_j \in Q),\ Q \in \mathcal{B}(\mathbb{R}^{(k+1)j})$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}^{(k+1)j}$, $j = 1, 2, \dots$

c) The conditional distribution

$$F_{j+1}(t \mid \mathcal{F}_j) = \mathsf{P}(t_{j+1} < t \mid t_1, r_1, \dots, t_j, r_j),$$

$j = 0, 1, 2, \dots$ satisfies $F_{j+1}(t \mid \mathcal{F}_j) < 1$ for all finite t a.s.

According to the condition b) of Definition 7 there exists $f_j[(t_1, r_1), \dots, (t_j, r_j)]$ the density of the first j points of Φ. The conditional density

$$f_{j+1}(t, r \mid \mathcal{F}_j) = \frac{f_{j+1}[(t_1, r_1), \dots, (t_j, r_j), (t, r)]}{f_j[(t_1, r_1), \dots, (t_j, r_j)]}, \quad j \geq 1, \qquad f_1(t, r \mid \mathcal{F}_0) = f_1(t, r),$$

enables to define

$$g_j(t, r; \omega) = \begin{cases} 0, & t_0 \leq t < t_j \\ f_{j+1}(t, r \mid \mathcal{F}_j)[1 - \int_{t_j}^{t} \int_{\mathbb{R}^k} f_{j+1}(s, q \mid \mathcal{F}_j)\, \mathrm{d}q\, \mathrm{d}s]^{-1}. \end{cases}$$

Definition 8. The conditional intensity of an analytic spatio-temporal point process is defined by

$$\lambda^*(t,r) = g_{N(t)}(t,r).$$

Under the condition

$$\mathsf{E}\left[\int_0^t \int_{\mathbb{R}^k} \lambda^*(u,v)\,\mathrm{d}v\,\mathrm{d}u\right]^2 < \infty \tag{25}$$

for any $B \in \mathcal{B}(\mathbb{R}^k)$ and $0 \le t < u < \infty$ it holds a. s.

$$\mathsf{E}\left[\Phi([t,u)\times B) \mid \mathcal{F}_t\right] = \mathsf{E}\left[\int_t^u \int_B \lambda^*(u,v)\,\mathrm{d}v\,\mathrm{d}u \mid \mathcal{F}_t\right]. \tag{26}$$

Also, the likelihood of a realization of the process observed up to time t is expressed by means of the conditional intensity as

$$L_t(\omega) = \prod_{i=1}^{N(t)} \lambda^*(t_i,r_i)\exp\left[-\int_0^t \int_{\mathbb{R}^k} \lambda^*(u,v)\,\mathrm{d}v\;\mathrm{d}u\right], \tag{27}$$

assuming the product to be equal to 1 if $N(t) = 0$.

Lemma 9. Suppose that for the intensity Λ of a spatio-temporal Poisson point process Π holds (24) and that Λ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}^k$. Then Π is analytic and its conditional intensity is equal to the intensity function $\lambda^* = \lambda$.

P r o o f. Condition a) from the Definition 7 follows from (24), b) and c) from the absolute continuity of Λ with respect to the Lebesgue measure on $\mathcal{X}$. $\lambda^* = \lambda$ follows from (26) and the definition of the Poisson point process. □

It is just the Poisson process for which the conditional intensity is non-random. Now we proceed to the doubly stochastic processes.

Let $(\Omega^*, \mathcal{S}, P^*)$ be another probability space and

$$P : \Omega^* \times \mathcal{F} \mapsto [0,1]$$

a probability kernel such that for each $\omega^* \in \Omega^*$ $P(\omega^*,.)$ is the distribution of a spatio-temporal point process, denoted Φ^*.

Definition 9. A doubly stochastic (spatio-temporal) point process is a process with distribution $\bar{P}(A) = P'(A \times \Omega^*)$, $A \in \mathcal{F}$, where

$$P'(A \times S) = \int_S P(\omega^*, A) P^*(\mathrm{d}\omega^*),,\ S \in \mathcal{S}.$$

If for each $\omega^* \in \Omega^*$ $P(\omega^*,\cdot)$ is the distribution of a Poisson point process on $\mathcal{X}$, the doubly stochastic process with distribution $\bar{P}$ is called a Cox process.

Theorem 10. (Fishman and Snyder [7], Theorem 3) Let Φ be a doubly stochastic spatio-temporal point process such that Φ^* is analytic for each $\omega^* \in \Omega^*$ and the corresponding conditional intensities $\lambda^*(t, r; \omega, \omega^*)$ are jointly measurable in the arguments t, r, ω and ω^* and

$$\int_{\Omega\times\Omega^*} \left[\int_0^t \int_{\mathbb{R}^d} \lambda^*(s, q; \omega, \omega^*)\,\mathrm{d}q\,\mathrm{d}s\right] P'(\mathrm{d}\omega \times \mathrm{d}\omega^*) < \infty. \tag{28}$$

Then (i) Φ is analytic,

(ii) the conditional intensity $\widehat{\lambda^*}(t, r; \omega)$ of the point process Φ is

$$\widehat{\lambda^*}(t, r; \omega) = \mathsf{E}'[\lambda^*(t, r; \omega, \omega^*)|\mathcal{F}_t \times \mathcal{S}_0],$$

where $\mathcal{S}_0 = \{\emptyset, \Omega^*\}$ is the trivial σ-algebra and E' the expectation w.r.t. P'.

Combining the theorem with Lemma 9 a direct consequence for the Cox processes is

Corollary 11. Let Φ be a Cox process on $\mathcal{X}$ satisfying (24), such that its driving measure $\Lambda(\omega^*)$ is absolutely continuous with respect to the Lebesgue measure on $\mathcal{X}$ a. s. Then Φ is an analytic doubly stochastic point process.

5. FILTERING FOR SPATIO–TEMPORAL DOUBLY STOCHASTIC ANALYTIC POINT PROCESSES DRIVEN BY OU TYPE PROCESSES

In the above setting used for doubly stochastic spatio-temporal point processes, the filtering problem means doing inference about the value of the unobserved ω^* if we observe the realization $\{(t_1, r_1), \ldots, (t_{N(t)}, r_{N(t)})\}$ in $[0, t) \times \mathbb{R}^k$. We can express the conditional probability

$$P^*(S \mid \mathcal{F}_t)(\omega) = P'(\Omega \times S \mid \mathcal{F}_t \times \mathcal{S}_0)(\omega, \omega^*), \qquad S \in \mathcal{S},$$

using the likelihood. The likelihood of a general doubly stochastic process is equal to

$$\bar{L}_t(\omega) = \Pi_{i=1}^{N(t)} \widehat{\lambda^*}(t_i, r_i) \exp\left[-\int_0^t \int_{\mathbb{R}^k} \widehat{\lambda^*}(u, v)\;\mathrm{d}v\;\mathrm{d}u\right].$$

Theorem 12. (Fishman and Snyder [7], Theorem 5) For a doubly stochastic process Φ satisfying the conditions of Theorem 10 it holds for all $S \in \mathcal{S}$ and $t \in [0, \infty)$

$$P^*(S \mid \mathcal{F}_t)(\omega) = \frac{1}{\bar{L}_t(\omega)} \int_S L_t(\omega, \omega^*) P^*(\mathrm{d}\omega^*),$$

$\bar{P}$ a. s. on $\mathcal{F}_t$. Here $L_t(\omega, \omega^*)$ are likelihoods for Φ^* given ω^*, cf. (27).

This so-called representation theorem has important consequences.

Corollary 13. Let Φ be a doubly stochastic analytic point process satisfying conditions of Theorem 10 and suppose that the conditional intensities $\lambda^*(t,r,\omega,\omega^*)$ are left continuous in t and continuous in r. For an $(\mathcal{F}_t \times \mathcal{S})$-measurable spatio-temporal vector random field $Y(t,r;\omega,\omega^*)$ defined on $[0,\infty) \times \mathbb{R}^k \times \Omega \times \Omega^*$ denote $\widehat{Y}(t,r) = \mathsf{E}'[Y(t,r;\omega,\omega^*)|\mathcal{F}_t \times \mathcal{S}_0]$. It holds

$$\widehat{Y}(t,r) = \mathsf{E}^*(Y(t,r)\exp(\xi(t))),$$

where E^* is the mean value with respect to P^* and

$$\xi(t) = -\int_0^t \int_{\mathbb{R}^k} (\lambda^*(s,r) - \widehat{\lambda^*}(s,r))\,\mathrm{d}r\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^k} \log \frac{\lambda^*(s,r)}{\widehat{\lambda^*}(s,r)}\,\Phi(\mathrm{d}s,\,\mathrm{d}r). \tag{29}$$

Now we have at our disposal all we need to attack the problem of filtering for a doubly stochastic point process driven by OU type temporal stochastic process $\{X(t)\}$. We are interested in the filtered estimate $\widehat{X}(t) = \mathsf{E}'[X(t,\omega,\omega^*)|\mathcal{F}_t \times \mathcal{S}_0]$. To be able to derive the differential equation for $\widehat{X}$ we need the Itó differential formula for a special type of vector process. Stochastic integral with respect to the Poisson random measure is understood in the sense of [4], Subsection 8.1.4.

Lemma 14. Let Φ be a doubly stochastic point process from Theorem 10. Let for the vector process $\zeta(t) = (\zeta^1(t),\dots,\zeta^m(t))$ hold

$$\mathrm{d}\zeta(t;\omega,\omega^*) = \alpha(t;\omega,\omega^*)\,\mathrm{d}t + C\,\mathrm{d}Z(\gamma t;\omega^*) + \int_{\mathbb{R}^k} \delta(t,r;\omega,\omega^*)\Phi((\mathrm{d}t,\,\mathrm{d}r);\omega,\omega^*), \tag{30}$$

where α, δ are random m-dimensional vectors, C is an $m\times d$ matrix of real numbers, and $\{Z(t)\}$ is a d-dimensional Lévy process with the characteristic function given by (5). We assume that α, δ have sample paths which are left continuous in t and continuous in r and are $(\mathcal{F}_t \times \mathcal{S}_t)$-measurable, where $\mathcal{S}_t = \sigma\{Z(s); s \le t\} \subset \mathcal{S}$. Let η be a $C^1(\mathbb{R}^m)$ scalar function. Then η satisfies the following stochastic differential equation a. s.

$$\begin{aligned} \mathrm{d}\eta(\zeta(t)) &= \left\langle \alpha(t),\, \frac{\partial \eta}{\partial \zeta}(t) \right\rangle \mathrm{d}t + \int_{\mathbb{R}^k} (\eta(\zeta(t_-) + \delta(t,r)) - \eta(\zeta(t_-)))\,\Phi(\mathrm{d}t,\,\mathrm{d}r) \\ &\quad + \int_{\mathbb{R}^d} (\eta(\zeta(t_-) + Cy) - (\eta(\zeta(t_-))))\,J_Z(\mathrm{d}\gamma t,\,\mathrm{d}y), \end{aligned} \tag{31}$$

where J_Z is a Poisson process satisfying (6).

P r o o f. We need to show that

$$\begin{aligned} \eta(\zeta(t)) - \eta(\zeta(0)) &= \int_0^t \left\langle \alpha(t),\, \frac{\partial \eta}{\partial \zeta}(t) \right\rangle \mathrm{d}t \\ &\quad + \int_0^t \int_{\mathbb{R}^k} (\eta(\zeta(t_-) + \delta(t,r)) - \eta(\zeta(t_-)))\,\Phi(\mathrm{d}t,\,\mathrm{d}r) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} (\eta(\zeta(t_-) + Cy) - (\eta(\zeta(t_-))))\,J_Z(\mathrm{d}\gamma t,\,\mathrm{d}y), \end{aligned} \tag{32}$$

holds a. s.

Let us first suppose that the Lévy process Z has finite Lévy measure μ, i. e. $\{Z(t)\}$ has only finite number of jumps on every interval $[0,t)$ a. s. Then we can proceed like in the classical proof of the Itô formula in [8]. We consider a sequence of partitions $\mathcal{T}_j = \{\tau_{1,j},\dots,\tau_{j+1,j}\}$ of the interval $[0,t)$ defined for $j\in\mathcal{N}$ by

$$\tau_{i,j} = \frac{i-1}{j}\,t, \qquad i=1,2,\dots,j+1.$$

For any such partition we can write

$$\eta(\zeta(t))-\eta(\zeta(0)) = \sum_{i=1}^{j}\eta(\zeta(\tau_{i+1,j}))-\eta(\zeta(\tau_{i,j})) = \sum_{i=1}^{j}\eta(\zeta(\tau_{i,j})+\triangle\zeta(\tau_{i,j}))-\eta(\zeta(\tau_{i,j})),$$

where

$$\triangle\zeta(\tau_{i,j}) = \zeta(\tau_{i+1,j})-\zeta(\tau_{i,j}) = \triangle A_{i,j}+\triangle B_{i,j}+\triangle D_{i,j},$$

and

$$\begin{aligned}
\triangle A_{i,j} &= \int_{\tau_{i,j}}^{\tau_{i+1,j}}\alpha(s;\omega,\omega^*)\,\mathrm{d}s,\\
\triangle B_{i,j} &= C\int_{\tau_{i,j}}^{\tau_{i+1,j}}\mathrm{d}Z(\gamma s;\omega,\omega^*),\\
\triangle D_{i,j} &= \int_{\tau_{i,j}}^{\tau_{i+1,j}}\int_{\mathbb{R}^k}\delta(t,r;\omega,\omega^*)\Phi((\mathrm{d}t,\,\mathrm{d}r);\omega,\omega^*).
\end{aligned}$$

Let us write $\eta(\zeta(t))-\eta(\zeta(0)) = \sum_{A,j}+\sum_{B,j}+\sum_{D,j}$ where

$$\begin{aligned}
\sum\nolimits_{A,j} &= \sum_{i=1}^{j}\eta(\zeta(\tau_{i,j})+\triangle\zeta(\tau_{i,j}))-\eta(\zeta(\tau_{i,j})+\triangle B_{i,j}+\triangle D_{i,j}),\\
\sum\nolimits_{B,j} &= \sum_{i=1}^{j}\eta(\zeta(\tau_{i,j})+\triangle B_{i,j}+\triangle D_{i,j})-\eta(\zeta(\tau_{i,j})+\triangle D_{i,j}),\\
\sum\nolimits_{D,j} &= \sum_{i=1}^{j}\eta(\zeta(\tau_{i,j})+\triangle D_{i,j})-\eta(\zeta(\tau_{i,j})).
\end{aligned}$$

From the mean-value theorem we have

$$\lim_{j\to\infty}\sum\nolimits_{A,j} = \int_0^t\left\langle\alpha(s),\,\frac{\partial\eta}{\partial\zeta}(s)\right\rangle\mathrm{d}s. \tag{33}$$

Because Φ is an analytic doubly stochastic point process it has only finite number of points in $[0,t)\times\mathbb{R}^k$ thus $\int_0^t\int_{\mathbb{R}^k}\Phi(\mathrm{d}s,\,\mathrm{d}r)$ has only finite number of jumps. The same holds for

$$C\int_0^t\mathrm{d}Z(\gamma s) = C\int_0^t\int_{\mathbb{R}^d}yJ_Z(\mathrm{d}\gamma s,\,\mathrm{d}y))$$

when the Lévy measure μ of Z is finite on $\mathbb{R}^d$. Moreover the jumps of Φ and J_Z arise at different times a. s. Thus for big enough $m(\omega, \omega^*)$ there is at most one jump of Φ and J_Z in each $[\tau_{i,j}, \tau_{i+1,j})$ for any $j \geq m(\omega, \omega^*)$ and

$$\begin{aligned}\sum\nolimits_{D,j} &= \sum_{l=1}^{\Phi([0,t)\times\mathbb{R}^k;\omega,\omega^*)} \eta(\zeta(\tau_{l,j}) + \delta(t_l, r_l)) - \eta(\zeta(\tau_{l,j}))\\ \sum\nolimits_{B,j} &= \sum_{q=1}^{\int_0^t \mathbf{1}[\int_{\mathbb{R}^d} J_Z(\gamma s,\, \mathrm{d}y)\neq 0]\,\mathrm{d}s} \eta(\zeta(\tau_{q,j}) + Cy_q) - \eta(\zeta(\tau_{q,j})).\end{aligned}$$

Here (t_l, r_l), $(\gamma t_q, y_q)$ are the occurrence points of Φ, J_Z, respectively and the intervals $[\tau_{l,j}, \tau_{l+1,j})$ and $[\tau_{q,j}, \tau_{q+1,j})$ are the elements of the partition containing the points t_l, t_q respectively. From the continuity of η and the existence of left limits for ζ it follows

$$\begin{aligned}\lim_{j\to\infty} \sum\nolimits_{D,j} &= \sum_{l=1}^{\Phi([0,t)\times\mathbb{R}^k;\omega,\omega^*)} \eta(\zeta(t_{l-}) + \delta(t_l, r_l)) - \eta(\zeta(t_{l-}))\\ &= \int_0^t \int_{\mathbb{R}^k} (\eta(\zeta(s_-) + \delta(s,r)) - \eta(\zeta(s_-)))\, \Phi(\,\mathrm{d}s,\, \mathrm{d}r) \qquad (34)\end{aligned}$$

$$\begin{aligned}\lim_{j\to\infty} \sum\nolimits_{B,j} &= \sum_{q=1}^{\int_0^t \mathbf{1}[\int_{\mathbb{R}^d} J_Z(\gamma s,\, \mathrm{d}y)\neq 0]\,\mathrm{d}s} \eta(\zeta(t_{q_-}) + Cy_q) - \eta(\zeta(t_{q_-}))\\ &= \int_0^t \int_{\mathbb{R}^d} (\eta(\zeta(s_-) + Cy) - \eta(\zeta(s_-)))\, J_Z(\,\mathrm{d}\gamma s,\, \mathrm{d}y), \qquad (35)\end{aligned}$$

almost surely. By combining (33), (35) and (34) we get (32).

For general Lévy processes $\{Z(t)\}$ we know from Corollary 3 that every Lévy process can be decomposed as $Z(t) = Z^\epsilon(t) + R^\epsilon(t)$, where $Z^\epsilon(t)$ is a Lévy process with finite number of jumps in any bounded interval and $R^\epsilon(t)$ is a mean-zero square integrable martingale with $\mathrm{Var}(R^\epsilon(t)) \to 0$ as $\epsilon \to 0$. Denote by ζ^ϵ the process defined by (30) but with the Lévy process $\{Z^\epsilon(t)\}$ instead of $\{Z(t)\}$.

Suppose that η and its first derivatives are bounded by a constant K. Then

$$|\eta(\zeta(t_-) + Cy) - \eta(\zeta(t_-))| \leq KCy$$

thus the right hand side of (32) is finite since $Z(t) = \int_0^t \int_{\mathbb{R}^d} yJ_Z(\,\mathrm{d}(y,s))$ is. Moreover

$$|\eta(\zeta(t)) - \eta(\zeta^\epsilon(t))|^2 \leq K^2(R^\epsilon(t))^2.$$

Thus

$$\lim_{\epsilon\to 0} \eta(\zeta^\epsilon(t)) = \eta(\zeta(t)),$$

in $L^2(\mathsf{P})$. But the equation (32) holds for ζ^ϵ and taking the limits on both sides we get the equality (32) also for $Z(t)$.

A general Lévy process fulfilling the assumptions of Lemma 14 is of finite variation thus if we define the sets $A_M = \{(\omega, \omega^*) : \zeta(s; \omega, \omega^*) \leq M \text{ for all } s \leq t\}$, then $A_M \to (\Omega \times \Omega^*)$ as $M \to \infty$. But η is bounded with bounded first derivatives on A_M and (32) holds on A_M. Thus taking limit $M \to \infty$ the validity of (32) follows for any ζ satisfying the assumptions of the theorem. □

Now we are ready to derive the differential equation for the conditional mean $\hat{X}$ of $X(t)$.

Theorem 15. Let Φ be a doubly stochastic point process from Theorem 10 driven by an OU type d-dimensional process $X(t)$ given by (13) with $\gamma > 0$ and with the Lévy process $Z(t)$ satisfying equation (5). Suppose that the conditional intensities $\lambda^*(t,r)$ are left continuous in t and continuous in r. Then the conditional mean $\widehat{X}(t)$ satisfies

$$\begin{aligned} \mathrm{d}\widehat{X}(t) &= -\gamma \widehat{X}(t)\,\mathrm{d}t - \int_{\mathbb{R}^k} (\widehat{\mathsf{E}}[X(t_-)\lambda^*(t,r)] - \widehat{X}(t_-)\widehat{\lambda^*}(t,r))\,\mathrm{d}r\,\mathrm{d}t \\ &\quad + \int_{\mathbb{R}^k} (\widehat{\mathsf{E}}[X(t_-)\lambda^*(t,r)] - \widehat{X}(t_-)\widehat{\lambda^*}(t,r)) \frac{1}{\widehat{\lambda^*}(t,r)}\,\Phi(\mathrm{d}t,\mathrm{d}r) + \mathrm{d}\widehat{Z}(\gamma t), \end{aligned} \tag{36}$$

Proof. Let $\zeta(t) = [X(t), \xi(t)]$ be a vector process with $(d+1)$ components. Combining (13) and (29) we get the stochastic differential equation for ζ

$$\mathrm{d}\zeta(t) = \begin{pmatrix} -\gamma X(t) \\ -\int_{\mathbb{R}^k} (\lambda^* - \widehat{\lambda^*})\,\mathrm{d}r \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ \int_{\mathbb{R}^k} \log \frac{\lambda^*}{\widehat{\lambda^*}}\,\Phi(\mathrm{d}t,\mathrm{d}r) \end{pmatrix} + \begin{pmatrix} \mathrm{d}Z(\gamma t) \\ 0 \end{pmatrix}.$$

For $i = 1, \dots, d$ let

$$\eta_i(\zeta(t)) = X_i(t)\exp(\xi(t)).$$

Then η_i are continuously differentiable functions and we have

$$\frac{\partial \eta_i}{\partial \zeta_j} = \begin{cases} e^{\xi(t)} & i = j \\ 0 & j \neq i, d+1 \\ \eta_i & j = d+1. \end{cases}$$

Thus we can use Lemma 14 for ζ and each η_i with

$$\begin{aligned} \alpha(t) &= [-\gamma X(t), -\int_{\mathbb{R}^k} (\lambda^* - \widehat{\lambda^*})\,\mathrm{d}r], \\ \delta(t,r) &= \left[0, \dots, 0, \log \frac{\lambda^*}{\widehat{\lambda^*}}\right], \\ C &= \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}, \end{aligned}$$

where $\mathbf{1}$ is $d \times d$ identity matrix and $\mathbf{0}$ is $1 \times d$ matrix of zeros. Using the equalities

$$\eta_i(\zeta(t_-) + Cy) - (\eta_i(\zeta(t_-))) = (X(t) + y)e^{\xi(t_-)} - X(t)e^{\xi(t_-)} = ye^{\xi(t_-)},$$

and

$$\begin{aligned}\eta_i(\zeta(t_-)+\delta(t,r))-\eta_i(\zeta(t_-)) &= X_i(t_-)e^{\xi(t_-)+\log\frac{\lambda^*}{\widehat{\lambda^*}}}-X_i(t_-)e^{\xi(t_-)}\\ &= \eta_i(\zeta(t_-))\left(\frac{\lambda^*}{\widehat{\lambda^*}}-1\right),\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{d}\eta_i(\zeta(t)) &= -\gamma X_i(t)e^{\xi(t_-)}\,\mathrm{d}t-\eta_i(\zeta(t_-))\int_{\mathbb{R}^k}(\lambda^*-\widehat{\lambda^*})\,\mathrm{d}r\,\mathrm{d}t \\ &\quad +\eta_i(\zeta(t_-))\int_{\mathbb{R}^k}(\lambda^*-\widehat{\lambda^*})\frac{1}{\widehat{\lambda^*}}\,\Phi(\mathrm{d}t,\mathrm{d}r)+\int_{\mathbb{R}^d}ye^{\xi(t)}J_Z(\mathrm{d}\gamma t,\mathrm{d}y).\end{aligned} \tag{37}$$

Now from Lemma 13 we have $\widehat{X}_i(t)=\mathsf{E}^*(\eta_i(\zeta(t)))$ and from Fubini's theorem

$$\mathrm{d}\widehat{X}_i(t)=\mathsf{E}^*(\mathrm{d}\eta_i(\zeta(t))).$$

Therefore taking expectations on both sides of (37) and writing

$$\mathsf{E}^*\left[\int_{\mathbb{R}^d}ye^{\xi(t)}J_Z(\mathrm{d}\gamma t,\mathrm{d}y)\right]=\mathsf{E}^*\left[e^{\xi(t)}\int_{\mathbb{R}^d}yJ_Z(\mathrm{d}\gamma t,\mathrm{d}y)\right]=\mathrm{d}\widehat{Z}(\gamma t),$$

we obtain the equation (36). □

Corollary 16. Let Φ be a spatio-temporal Cox process from Corollary 11 driven by an OU type d-dimensional process $X(t)$ from the Theorem 15. Suppose that the conditional intensities $\lambda^*(t,r)$ are left continuous in t and continuous in r. Then the conditional mean $\widehat{X}(t)$ satisfies equation (36).

ACKNOWLEDGEMENT

The research was supported by the Ministry of Education, Youth and Sports of the Czech Republic under project MSM 0021620839 and by the Grant Agency of the Academy of Sciences of the Czech Republic under Grant IAA101120604.

(Received December 12, 2005.)

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