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COOPERATIVE FUZZY GAMES EXTENDED FROM ORDINARY COOPERATIVE GAMES WITH RESTRICTIONS ON COALITIONS

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Cooperative games are very useful in considering profit allocation among multiple decision makers who cooperate with each other. In order to deal with cooperative games in practical situations, however, we have to deal with two additional factors. One is some restrictions on coalitions. This first factor has been taken into consideration through feasibility of coalitions. The other is partial cooperation of players. In order to describe this second factor, we consider fuzzy coalitions which permit partial participation in a coalition to a player. In this paper we take both of these factors into account in cooperative games. Namely, we analyze and discuss cooperative fuzzy games extended from ordinary cooperative games with restrictions on coalitions in two approaches. For the purpose of comparison of these two approaches, we define two special classes of extensions called U-extensions which satisfy linearity and W-extensions which satisfy U-extensions and two additional conditions, restriction invariance and monotonicity. Finally, we show sufficient conditions under which these obtained games in two approaches coincide.

Keywords: cooperative games, cooperative fuzzy games, restricted games, coalitions
AMS Subject Classification: 91A12

1. INTRODUCTION

A cooperative game (or coalitional game) is usually described by a set of players and a characteristic function. In a transferable utility game only one number is attached to each coalition. In order to deal with practical situations in cooperative game theory, we have to take into account two important factors: restrictions on coalitions and partial cooperation by players.

The first factor has been taken into consideration through feasibility of coalitions. In a cooperative game, it is generally assumed that an arbitrary coalition is feasible, i. e., each player can form a coalition with arbitrary players. However, situations where some of coalitions are impossible or prohibited may occur, that is, infeasible coalitions may occur. In order to deal with these situations, the concept of feasible coalition systems has been introduced (cf. Algaba et al. [1] and Bilbao [3]).

The second factor has been studied through fuzzy coalitions instead of ordinary coalitions which are subsets of players and leads to cooperative fuzzy games. In a traditional cooperative game, it is generally assumed that each player participates in a coalition fully or not. On the contrary, Aubin [2] introduced a fuzzy coalition which permits partial participation in a coalition to a player. As a game with fuzzy coalitions, a cooperative fuzzy game has been studied [8]. Each fuzzy coalition is identified with a point in the hypercube $[0,1]^n$, while an ordinary coalition is regarded as a vertex of this hypercube, i. e., a point in $\{0,1\}^n$. Another application of fuzzy theory to cooperative games is in studies of cooperative games with fuzzy worth (coalitional values) [6]. In this paper, however, we deal only with cooperative games with fuzzy coalitions as cooperative fuzzy games.

The outline of this paper is as follows. Section 2 introduces cooperative games and feasible coalition systems as restrictions on coalitions. Section 3 introduces cooperative fuzzy games and feasible fuzzy coalition systems as restrictions on fuzzy coalitions. Section 4 deals with cooperative fuzzy games obtained by extending cooperative games. We consider two special classes of extensions called U-extensions which satisfy linearity and W-extensions which satisfy U-extensions and two additional conditions, restriction invariance and monotonicity. In Section 5, we take two approaches to cooperative fuzzy games extended from ordinary cooperative games with restrictions on coalitions. We show sufficient conditions under which these obtained games in two approaches coincide.

2. COOPERATIVE GAMES WITH RESTRICTIONS ON COALITIONS

In this paper we deal with transferable utility games as cooperative games (coalitional games), and their extensions as cooperative fuzzy games. Let $N = \{1, 2, \ldots, n\}$ be a set of players and $S \subseteq N$ be a coalition which is a subset of N. A transferable utility game $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The function v is also called characteristic function. The set of all games with the player set N is denoted as Γ^N.

Superadditivity and convexity are defined as follows:

Definition 1. A game $v \in \Gamma^N$ is said to be

1. *superadditive* if $v(S) + v(T) \le v(S \cup T), \quad \forall\, S, T \subseteq N \;$ s.t. $S \cap T = \emptyset$.
2. *convex* if $v(S) + v(T) \le v(S \cup T) + v(S \cap T), \quad \forall\, S, T \subseteq N$.

The sum of two games $v, w \in \Gamma^N$, and the scalar multiplication of v by $\alpha \in \mathbb{R}$ are defined by

$$\begin{cases} (v + w)(S) = v(S) + w(S), & \forall\, S \subseteq N, \\ (\alpha v)(S) = \alpha v(S), & \forall\, S \subseteq N, \end{cases}$$

respectively. Since every transferable utility game is uniquely determined by collection of its worth $\{v(S) : S \subseteq N, S \ne \emptyset\}$, the vector space Γ^N of all cooperative games on N will be identified with $\mathbb{R}^{2^N - 1}$. In fact, for any $T \subseteq N, T \ne \emptyset$, we define the unanimity game $u_T \in \Gamma^N$ by

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

Then, every game $v \in \Gamma^N$ is represented by a unique linear combination of unanimity games:

$$v = \sum_{T \subseteq N, T \neq \emptyset} d_T(v) u_T,$$

where

$$d_T(v) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$$

is called the dividend of T in the game v. The dividends satisfy the following recursive formula:

$$d_T(v) = \begin{cases} 0, & \text{if } T = \emptyset, \\ v(T) - \sum_{S \subset T} d_S(v), & \text{if } T \neq \emptyset. \end{cases}$$

Now we consider some restrictions on coalitions. It is usually described by a set system on N (see e.g. [3]).

Definition 2. A set $\mathcal{F} \subseteq 2^N$ is said to be a *feasible coalition system* (FCS for short) if it satisfies the following two conditions:

1. $\emptyset \in \mathcal{F}$.
2. $\{i\} \in \mathcal{F}, \quad \forall\, i \in N$.

We introduce some basic concepts about FCS. For a coalition $S \subseteq N$, $\{S_k\}_{k \in K}, \emptyset \neq S_k \subseteq S$ is said to be a partition of S if it satisfies

$$\begin{cases} S_k \cap S_{k'} = \emptyset, & k \neq k',\ k, k' \in K, \\ S = \bigcup_{k \in K} S_k. \end{cases}$$

Especially for an FCS $\mathcal{F}$, a partition of S, $\{S_k\}$ such that $S_k \in \mathcal{F}$ for all k, is said to be an $\mathcal{F}$-partition of S. $\mathcal{P}_{\mathcal{F}}(S)$ denotes the set of all $\mathcal{F}$-partitions of S.

For a coalition $S \subseteq N$ and an FCS $\mathcal{F}$, a subset of S is said to be an $\mathcal{F}$-component of S if it is a maximal subset of S in $\mathcal{F}$. $\mathcal{C}_{\mathcal{F}}(S)$ denotes the set of all $\mathcal{F}$-components of S.

Definition 3. Let $\mathcal{F}$ be an FCS and $v \in \Gamma^N$. Then the restricted game of v by $\mathcal{F}$, $v^{\mathcal{F}} \in \Gamma^N$, is defined as follows:

$$v^{\mathcal{F}}(S) = \max \left\{ \sum_{k \in K} v(S_k) \mid \{S_k\}_{k \in K} \in \mathcal{P}_{\mathcal{F}}(S) \right\}, \quad \forall\, S \subseteq N.$$

Special classes of FCS have been investigated. In this paper, we deal with partition systems in the special class.

Definition 4. An FCS $\mathcal{F}$ is said to be a *partition system* (PS for short) if $\mathcal{C}_{\mathcal{F}}(S)$ is a partition of S for all $S \subseteq N$.

Proposition 1. (see [3]) An FCS $\mathcal{F}$ is a PS if and only if $S \cup T \in \mathcal{F}$ for any $S, T \in \mathcal{F}$ such that $S \cap T \neq \emptyset$.

Proposition 2. ([3]) If $v \in \Gamma^N$ is superadditive and FCS $\mathcal{F}$ is a PS, then

$$v^{\mathcal{F}}(S) = \sum_{T \in \mathcal{C}_{\mathcal{F}}(S)} v(T), \quad \forall S \subseteq N.$$

Given an FCS $\mathcal{F}$ and a game $v \in \Gamma^N$, we define the restricted dividends of $T \in \mathcal{F}$ in the game v in the following recursive manner:

$$d_T^{\mathcal{F}}(v) = \begin{cases} 0, & \text{if } T = \emptyset, \\ v(T) - \displaystyle\sum_{S \subset T, S \in \mathcal{F}} d_S^{\mathcal{F}}(v), & \text{if } T \neq \emptyset, T \in \mathcal{F}. \end{cases}$$

The following theorem is useful because we are able to calculate the dividends in the restricted games without calculating the restricted game.

Theorem 1. If a game $v \in \Gamma^N$ is superadditive and an FCS $\mathcal{F}$ is a PS, then

$$d_T(v^{\mathcal{F}}) = \begin{cases} 0, & \text{if } T \notin \mathcal{F}, \\ d_T^{\mathcal{F}}(v), & \text{if } T \in \mathcal{F}. \end{cases}$$

P r o o f. Note that $d_T(v^{\mathcal{F}}) = d_T^{\mathcal{F}}(v) = 0$ for $T = \emptyset$. We prove the theorem by induction. First let $T = \{i\}$ for any $i \in N$. Then $T \in \mathcal{F}$ and, by Proposition 2,

$$d_T(v^{\mathcal{F}}) = v^{\mathcal{F}}(T) = v(T) = d_T^{\mathcal{F}}(v).$$

Next consider the case $|T| > 1$ and $T \in \mathcal{F}$. Then

$$d_T(v^{\mathcal{F}}) = v^{\mathcal{F}}(T) - \sum_{S \subset T} d_S(v^{\mathcal{F}}) = v(T) - \sum_{S \subset T, S \in \mathcal{F}} d_S^{\mathcal{F}}(v) = d_T^{\mathcal{F}}(v).$$

Finally, consider the case $|T| > 1$ and $T \notin \mathcal{F}$ with $\mathcal{C}_{\mathcal{F}}(T) = \{T_1, \ldots, T_l\}$. Then

$$d_T(v^{\mathcal{F}}) = v^{\mathcal{F}}(T) - \sum_{S \subset T} d_S(v^{\mathcal{F}}) = \sum_{j=1}^{l} v(T_j) - \sum_{S \subset T, S \in \mathcal{F}} d_S^{\mathcal{F}}(v).$$

If $S \subset T$ and $S \in \mathcal{F}$, then there exists $k \in \{1, \ldots, l\}$ such that $S \cap T_k \neq \emptyset$. Since $\mathcal{F}$ is a PS, $S \cup T_k \in \mathcal{F}$ by Proposition 1. In view of maximality of T_k, $S \cup T_k = T_k$, which implies that $S \subseteq T_k$. Hence

$$d_T(v^{\mathcal{F}}) = \sum_{j=1}^{l} \{v(T_j) - \sum_{S \subset T_j, S \in \mathcal{F}} d_S^{\mathcal{F}}(v)\} = 0.$$

This complete the proof of the theorem. $\square$

3. COOPERATIVE FUZZY GAMES WITH RESTRICTION ON COALITIONS

In a cooperative game, the coalition S is identified with the vector e^S, defined by $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ otherwise, and the domain 2^N of the characteristic function v is identified with $\{0,1\}^n$, i.e., $v : \{0,1\}^n \to \mathbb{R}$. Hence extending $\{0,1\}^n$ to $[0,1]^n$ implies extending ordinary coalitions to fuzzy coalitions. Thus, given the player set N, a cooperative fuzzy game ξ on N is a function from $[0,1]^n$ to $\mathbb{R}$ with $\xi(0) = 0$. The set of all cooperative fuzzy games is denoted Δ^N. Let $s \in [0,1]^n$ be a fuzzy coalition which is a fuzzy subset of N. Then, for a fuzzy coalition $s = (s_1, \ldots, s_n)$, each element s_i of s indicates the membership grade of i in s, i.e., the rate of ith player's participation in s.

In this paper we use the following notations. First, the vector $e^{\{i\}}$ is simply denoted by e^i. For s, $t \in [0,1]^n$, vectors $s \vee t$ and $s \wedge t \in [0,1]^n$ are defined by

$$\begin{aligned} (s \vee t)_i &= \max\{s_i, t_i\}, \\ (s \wedge t)_i &= \min\{s_i, t_i\}, \end{aligned}$$

For $s \in [0,1]^n$, let $\operatorname{supp} s = \{i \in N \mid s_i > 0\}$. In this paper, for s, $t \in [0,1]^n$, $s \leq t$ means that $s_i \leq t_i$, $\forall\, i$ and $s < t$ means that $s \leq t$ and $s \neq t$.

Then superadditivity and convexity in a cooperative fuzzy game are defined as extensions of them in a cooperative game.

Definition 5. A cooperative fuzzy game $\xi \in \Delta^N$ is said to be

1. *strongly superadditive* if $\xi(s) + \xi(t) \leq \xi(s+t)$, $\forall\, s, t \in [0,1]^n$ s.t. $s+t \in [0,1]^n$.
2. *weakly superadditive* if $\xi(s) + \xi(t) \leq \xi(s \vee t)$, $\forall\, s, t \in [0,1]^n$ s.t. $s \wedge t = 0$.
3. *convex* if $\xi(s) + \xi(t) \leq \xi(s \vee t) + \xi(s \wedge t)$, $\forall\, s, t \in [0,1]^n$.

It is obvious that if the game ξ is strongly superadditive, then it is weakly superadditive. If ξ is convex, it is weakly superadditive. In Brânzei [4], a fuzzy game is said to be convex if it satisfies the coordinate-wise convexity condition in addition to the inequality in the above definition.

The sum of two games ξ, $\xi' \in \Delta^N$ and the scalar multiplication of ξ by $\alpha \in \mathbb{R}$ is defined by

$$\begin{cases} (\xi + \xi')(s) = \xi(s) + \xi'(s), & \forall\, s \in [0,1]^n, \\ (\alpha\xi)(s) = \alpha\xi(s), & \forall\, s \in [0,1]^n, \end{cases}$$

respectively.

Now we deal with the cooperative fuzzy game with restrictions on fuzzy coalitions.

For a set $F \subseteq [0,1]^n$ and a vector $s \in [0,1]^n$, a vector $t \in [0,1]^n$ is said to be F-vector of s if

$$t \leq s,\ t \in F, \text{ and } t \leq t' \leq s \text{ with } t' \in F \text{ imply that } t' = t$$

and $C^F(s)$ is the set of all F-vectors of s.

First, FFCS is defined as follows:

Definition 6. A set $F \subseteq [0,1]^n$ is said to be a *feasible fuzzy coalition system* (FFCS for short) if it satisfies the following two conditions:

F1. $\alpha e^i \in F, \quad \forall \alpha \in [0,1]$.

F2. For any $s \in [0,1]^n$ and $t \in F$ such that $t \leq s$, there exists $\bar{t} \in C^F(s)$ such that $t \leq \bar{t}$.

For a fuzzy coalition $s \in [0,1]^n$, $\{s^1, \ldots, s^l\} \subseteq [0,1]^n$ such that $\sum_{j=1}^{l} s^j = s$ is said to be a partition of s. Especially for an FFCS F, a partition of s, $\{s^1, \ldots, s^l\}$ such that $s^j \in F$ for all $j = 1, \ldots, l$, is said to be an F-partition of s. $P^F(s)$ denotes the set of all F-partitions of s.

We extend the restricted game of cooperative games to cooperative fuzzy games.

Definition 7. Let $\xi \in \Delta^N$ be a cooperative fuzzy game and F be an FFCS. Then the restricted game of ξ by F, $\xi^F \in \Delta^n$, is defined as follows:

$$\xi^F(s) = \sup\left\{ \sum_{j=1}^{l} \xi(s^j) \;\middle|\; \{s^1, \ldots, s^l\} \in P^F(s) \right\}, \quad \forall s \in [0,1]^n.$$

Clearly, if ξ is strongly superadditive and $s \in F$, we have $\xi^F(s) = \xi(s)$.

Definition 8. FFCS F is said to be a *partition fuzzy system* (PFS for short) if $C^F(s)$ is a partition of s for any $s \in [0,1]^n$.

For $s \in [0,1]^n$ and $T \subseteq N$, $s_{|T}$ is defined as follows:

$$(s_{|T})_i = \begin{cases} s_i, & \text{if } i \in T, \\ 0, & \text{if } i \notin T. \end{cases}$$

Proposition 3. (see [7]) The following conditions are equivalent.

1. An FFCS F is a PFS.
2. For any $s \in [0,1]^n$, there exists a partition $\{I_1, \ldots, I_l\}$ of N such that
$$C^F(s) = \{s_{|I_1}, \ldots, s_{|I_l}\}.$$
3. If $s, t \in F$ and $s \wedge t \neq 0$, then $s \vee t \in F$.

Proposition 4. (see [7]) Let $s \in [0,1]^n$ be a fuzzy coalition and let $\xi \in \Delta^N$ be a cooperative fuzzy game. For an FFCS F, if $C^F(s)$ is a partition of s and ξ is strongly superadditive, then the following holds:

$$\xi^F(s) = \sum_{t \in C^F(s)} \xi(t).$$

4. COOPERATIVE FUZZY GAMES AS EXTENSIONS OF ORDINARY COOPERATIVE GAMES

In this section, we introduce cooperative fuzzy games as extensions of ordinary cooperative games.

Let g be a mapping of Γ^N into Δ^N with the following three properties:

- For each $v \in \Gamma^N$, the value $g(v)$ of g at v is an extension of v.
- $g(v+w) = g(v) + g(w)$, $\forall\, v, w \in \Gamma^N$
- $g(\alpha v) = \alpha g(v)$, $\forall\, \alpha \in \mathbb{R}$, $\forall\, v \in \Gamma^N$

Moreover, let G denote the set of all such functions for Γ^N and Δ^N. Now we can define two different concepts in a natural way; namely, "U-extension with respect to a given $g \in G$" and "U-extension".

Definition 9. Let g be an element of G. A cooperative fuzzy game $\xi \in \Delta^N$ is said to be a (U_g)-*extension of* $v \in \Gamma^N$ if $\xi = g(v)$.

Definition 10. A cooperative fuzzy game $\xi \in \Delta^N$ is said to be a U-*extension of* $v \in \Gamma^N$ if there exists $g \in G$ such that $\xi = g(v)$.

Note that according to these definitions, a cooperative fuzzy game $\xi \in \Delta^N$ is a U-extension of $v \in \Gamma^N$ if and only if it is an (U_g)-extension for some $g \in G$. In the following, when we consider U-extensions, we fix some $g \in G$ and denote the (U_g)-extension $g(v)$ of v simply by ξ_v without ambiguity.

If ξ_v is a U-extension of v, it can be represented as a linear combination of the U-extensions ξ_{u_T} of the unanimity games u_T as $\xi_v = \sum_{T \subseteq N} d_T(v)\xi_{u_T}$. Moreover, we assume two additional conditions, restriction invariance and monotonicity, on extensions to obtain W-extensions.

Definition 11. A cooperative fuzzy game $\xi \in \Delta^N$ is said to be a W-extension of $v \in \Gamma^N$ if it is a U-*extension of* $v \in \Gamma^N$ and if

W1. $\xi_{u_T}(s) = \xi_{u_T}(s_{|T}),\ \forall\, s \in [0,1]^n,$

W2. $\xi_{u_T}(s) \leq \xi_{u_T}(t),\ \forall\, s, t \in [0,1]^n$ s.t. $s \leq t.$

Since the space Γ^N of all cooperative games on N is a linear space and the set of unanimity games forms a basis, a U-extension of any game v is specified by those of unanimity games. We obtain a stronger result for a W-extension.

Proposition 5. If $\xi \in \Delta^N$ is a W-extension of $v \in \Gamma^N$, then

$$\xi_v(s) = \sum_{T \subseteq \text{supp } s} d_T(v)\xi_{u_T}(s) \quad \forall\, s \in [0,1]^n.$$

P r o o f. Let $s \in [0,1]^n$. Then, by Definition 11, we have

$$\xi_{u_T}(e^{\text{supp } s}) = \begin{cases} 1, & T \subseteq \text{supp } s, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $T \not\subseteq \text{supp } s$. Then, we have $0 \leq s \leq e^{\text{supp } s}$. Since it follows from W2 that

$$0 = \xi_{u_T}(0) \leq \xi_{u_T}(s) \leq \xi_{u_T}(e^{\text{supp } s}) = 0,$$

we have $\xi_{u_T}(s) = 0$. Therefore, we have

$$\xi_v(s) = \sum_{T \subseteq N} d_T(v)\xi_{u_T}(s) = \sum_{T \subseteq \text{supp } s} d_T(v)\xi_{u_T}(s).$$

□

Two well-known examples of W-extensions are the multilinear extension and the Lovász extension. The multilinear extension m_v of v, introduced by Owen [9], is given by

$$m_{u_T}(s) = \prod_{i \in T} s_i.$$

On the other hand, the Lovász extension [5] l_v of v is given by

$$l_{u_T}(s) = \min_{i \in T} s_i.$$

Some properties of these extensions are given in Bilbao [3] and Tanino [10].

Proposition 6. If a cooperative game $v \in \Gamma^N$ is superadditive, then its Lovász extension $l_v \in \Delta^N$ is weakly superadditive.

Proposition 7. If a cooperative game $v \in \Gamma^N$ is convex, then its Lovász extension $l_v \in \Delta^N$ is convex.

Proposition 8. A cooperative game $v \in \Gamma^N$ is convex if and only if its Lovász extension $l_v \in \Delta^N$ is strongly superadditive.

5. COOPERATIVE FUZZY GAMES EXTENDED FROM COOPERATIVE GAMES UNDER RESTRICTIONS ON COALITIONS

In this section we take two approaches to cooperative fuzzy games extended from ordinary cooperative games with restrictions on coalitions. We show sufficient conditions under which these obtained games in two approaches coincide.

We consider a cooperative game $v \in \Gamma^N$ with an FCS $\mathcal{F}$. In order to deal with both restrictions on coalitions and partial cooperations of players, we would like to extend the game v to a cooperative fuzzy game and also to restrict the game under the FCS. We may consider two approaches:

1. First we define the restricted game $v^{\mathcal{F}}$ and then extend it to a cooperative fuzzy game.

2. First we extend v to a cooperative fuzzy game and then define its restricted game.

For the second approach, we have to consider a fuzzy set system $F(\mathcal{F})$ corresponding to the FCS $\mathcal{F}$ as follows:

$$F(\mathcal{F}) = \{s \in [0,1]^n \mid \text{supp } s \in \mathcal{F}\}.$$

Lemma 1. For any s, $t \in [0,1]^n$, the following hold:

$$\begin{aligned} \text{supp } (s \vee t) &= (\text{supp } s) \cup (\text{supp } t), \\ \text{supp } (s \wedge t) &= (\text{supp } s) \cap (\text{supp } t). \end{aligned}$$

Proof.

$$\begin{aligned} i \in \text{supp } (s \vee t) &\Leftrightarrow (s \vee t)_i > 0 \Leftrightarrow \max\{s_i, t_i\} > 0 \Leftrightarrow i \in \text{supp } s \quad \text{or} \quad i \in \text{supp } t \\ &\Leftrightarrow i \in (\text{supp } s) \cup (\text{supp } t). \end{aligned}$$

$$\begin{aligned} i \in \text{supp } (s \wedge t) &\Leftrightarrow (s \wedge t)_i > 0 \Leftrightarrow \min\{s_i, t_i\} > 0 \Leftrightarrow i \in \text{supp } s \quad \text{and} \quad i \in \text{supp } t \\ &\Leftrightarrow i \in (\text{supp } s) \cap (\text{supp } t). \end{aligned}$$

□

Proposition 9. If $\mathcal{F}$ is an FCS, then the corresponding $F(\mathcal{F})$ is an FFCS.

Proof. Observe that $\emptyset \in \mathcal{F}$ and $i \in \mathcal{F}$, $\forall\, i \in N$ from the definition of FCS.

F1. For $\alpha \in [0,1]$,

$$\begin{aligned} \text{supp } (\alpha e^i) = \text{supp } 0 = \emptyset \in \mathcal{F} \quad &\text{if} \quad \alpha = 0, \\ \text{supp } (\alpha e^i) = \{i\} \in \mathcal{F} \quad &\text{if} \quad \alpha > 0. \end{aligned}$$

F2. Let $s \in [0,1]^n$ and $t \in F(\mathcal{F})$ such that $t \leq s$. There exists $T \in \mathcal{C}_{\mathcal{F}}(\text{supp } s)$ such that supp $t \subseteq T$. We show that $t \leq s_{|T}$ and $s_{|T} \in C^{F(\mathcal{F})}(s)$. We have $t \leq s_{|T}$ since supp $t \subseteq T$ and $t \leq s$. Hence we show that $s_{|T} \in F(\mathcal{F})$ and $s_{|T}$ is maximal in order to show $s_{|T} \in C^{F(\mathcal{F})}(s)$. Since supp $s_{|T} = T \in \mathcal{F}$, we have $s_{|T} \in F(\mathcal{F})$. Now suppose that $s_{|T} \in F(\mathcal{F})$ is not maximal. Then,

$$\begin{aligned} & \exists\, s' \in F(\mathcal{F}) : s_{|T} < s' \leq s, \text{supp } s_{|T} \neq \text{supp } s' \\ \Leftrightarrow \quad & \exists\, s' \in [0,1]^n : \text{supp } s' \in \mathcal{F}, \text{supp } (s_{|T}) \subset \text{supp } s' \subseteq \text{supp } s. \end{aligned}$$

This contradicts $T \in C_{\mathcal{F}}(\text{supp } s)$ from supp $(s_{|T}) = T$. Therefore, $s_{|T} \in F(\mathcal{F})$ and $s_{|T}$ is maximal. □

Proposition 10. If $\mathcal{F}$ is an PS, then the corresponding $F(\mathcal{F})$ is an PFS.

Proof. By Proposition 9, the corresponding $F(\mathcal{F})$ is an FFCS if $\mathcal{F}$ is an FCS. Let $s, t \in F(\mathcal{F})$ such that $s \wedge t \neq 0$. We have supp s, supp $t \in \mathcal{F}$ and supp $s \cap$supp $t =$ supp $(s \wedge t) \neq \emptyset$ by Lemma 1, and we have supp $s \cup$ supp $t =$ supp $(s \vee t) \in \mathcal{F}$ from assumption and Proposition 1. Therefore, we have $s \vee t \in F(\mathcal{F})$, and therefore $F(\mathcal{F})$ is a PFS by Proposition 3. □

Lemma 2. Let $F(\mathcal{F})$ be the FFCS corresponding to an FCS $\mathcal{F}$. For any $s \in [0,1]^n$, let

$$\mathcal{C}_{\mathcal{F}}(\text{supp } s) = \{I_1, \ldots, I_l\}.$$

Then the set of all F($\mathcal{F}$)-vectors of s is given by

$$C^{F(\mathcal{F})}(s) = \{s_{|I_1}, \ldots, s_{|I_l}\}.$$

Proof. We first show that $s_{|I_k} \in C^{F(\mathcal{F})}(s)$ for $k \in \{1, \ldots, l\}$. We have $s_{|I_k} \in F(\mathcal{F})$ from supp $(s_{|I_k}) = I_k \in \mathcal{F}$. Suppose that $s_{|I_k} \in F(\mathcal{F})$ is not maximal. We have

$$\begin{aligned} & \exists\, s' \in F(\mathcal{F}) : s_{|I_k} < s' \leq s, \text{supp } s_{|I_k} \neq \text{supp } s \\ \Leftrightarrow \quad & \exists\, s' \in [0,1]^n : \text{supp } s' \in \mathcal{F}, \text{supp } (s_{|I_k}) \subset \text{supp } s' \subseteq \text{supp } s. \end{aligned}$$

Since supp $(s_{|I_k}) = I_k$, this contradicts the maximality of I_k. Therefore, we have $s_{|I_k} \in F(\mathcal{F})$ and $s_{|I_k}$ is maximal.

We show that for $t \in C^{F(\mathcal{F})}(s)$ there exists $k \in \{1, \ldots, l\}$ satisfying that $t = s_{|I_k}$. We have supp $t \in \mathcal{F}$ from $t \in F(\mathcal{F})$. Then there exists $k \in \{1, \ldots, l\}$ satisfying that supp $t \subseteq I_k$ since supp $t \subseteq$ supp s by $t \leq s$ and I_k is an $\mathcal{F}$-component of supp s. Moreover, we have

$$t = t_{|I_k} \leq s_{|I_k} \leq s,$$

and by the definition of $F(\mathcal{F})$-vector, we have $t = s_{|I_k}$. Therefore,

$$C^{F(\mathcal{F})}(s) = \{s_{|I_1}, \ldots, s_{|I_l}\}.$$

□

Lemma 3. Let $\mathcal{F}$ be a PS. For any $s \in [0,1]^n$, let

$$\mathcal{C}_{\mathcal{F}}(\text{supp } s) = \{I_1, \ldots, I_l\}.$$

Then the following holds:

$$\{T \in \mathcal{F} \mid \emptyset \neq T \subseteq \text{supp } s\} = \{T \in \mathcal{F} \mid T \neq \emptyset, \exists_1 k \in \{1, \ldots, l\} : T \subseteq I_k\}.$$

Proof. ($\supseteq$) Suppose that there exists a unique $k \in \{1, \ldots, l\}$ satisfying that $T \subseteq I_k$ for $T \in \mathcal{F}$. Since $(N, \mathcal{F})$ is a PS and $\mathcal{C}_{\mathcal{F}}(\text{supp } s) = \{I_1, \ldots, I_l\}$, we have $T \subseteq I_k \subseteq \text{supp } s$. Therefore, we have

$$\{T \in \mathcal{F} \mid \emptyset \neq T \subseteq \text{supp } s\} \supseteq \{T \in \mathcal{F} \mid T \neq \emptyset, \exists_1\ k \in \{1, \ldots, l\} : T \subseteq I_k\}.$$

($\subseteq$) Let $T \in \mathcal{F}$ such that $\emptyset \neq T \subseteq \text{supp } s$. There exists $k \in \{1, \ldots, l\}$ satisfying that $T \subseteq I_k$ since $\mathcal{C}_{\mathcal{F}}(\text{supp } s) = \{I_1, \ldots, I_l\}$. In order to prove that this k is unique, we suppose $k_1, k_2 \in \{1, \ldots, l\}$ such that $T \subseteq I_{k_1}$,$T \subseteq I_{k_2}$ and $k_1 \neq k_2$. Then we have

$$I_{k_1} \cap I_{k_2} \supseteq T \neq \emptyset.$$

But this contradicts that $\{I_1, \ldots, I_l\}$ is a partition of supp s since $(N, \mathcal{F})$ is a PS. Therefore, we have

$$\{T \in \mathcal{F} \mid \emptyset \neq T \subseteq \text{supp } s\} \subseteq \{T \in \mathcal{F} \mid T \neq \emptyset, \exists_1\ k \in \{1, \ldots, l\} : T \subseteq I_k\}. \qquad \square$$

Let $v \in \Gamma^N$ be a superadditive game, $\mathcal{F}$ be a PS on N and ξ be a W-extension of $v \in \Gamma^N$. For $s \in [0,1]^n$, let $\mathcal{C}_{\mathcal{F}}(\text{supp } s) = \{I_1, \ldots, I_l\}$. Since $\xi_{v^{\mathcal{F}}}$ is a W-extension of $v^{\mathcal{F}}$, we have, by Proposition 5,

$$\xi_{v^{\mathcal{F}}}(s) = \sum_{T \subseteq \text{supp } s} d_T(v^{\mathcal{F}}) \xi_{u_T}(s_{|T}).$$

Notice that $d_T(v^{\mathcal{F}}) = 0$ for $T \notin \mathcal{F}$ by Theorem 1 since $\mathcal{F}$ is a PS, and we have

$$\xi_{v^{\mathcal{F}}}(s) = \sum_{T \in \mathcal{F}, T \subseteq \text{supp } s} d_T(v^{\mathcal{F}}) \xi_{u_T}(s_{|T}).$$

Then we have

$$\xi_{v^{\mathcal{F}}}(s) = \sum_{j=1}^{l} \sum_{T \in \mathcal{F}, T \subseteq I_j} d_T(v^{\mathcal{F}}) \xi_{u_T}(s_{|T}). \tag{1}$$

On the other hand, suppose ξ_v is strongly superadditive. Now we have $C^{F(\mathcal{F})}(s) = \{s_{|I_1}, \ldots, s_{|I_l}\}$ by Lemma 2 since $\mathcal{C}_{\mathcal{F}}(\text{supp } s) = \{I_1, \ldots, I_l\}$. $F(\mathcal{F})$ is a PFS since $\mathcal{F}$ is a PS and Proposition 10. Notice that $F(\mathcal{F})$ is an FFCS and a PFS and ξ_v is strongly superadditive. Then, from Proposition 4, we have

$$\begin{aligned}(\xi_v)^{F(\mathcal{F})}(s) &= \sum_{t \in C^{F(\mathcal{F})}(s)} \xi_v(t) = \sum_{t \in C^{F(\mathcal{F})}(s)} \sum_{T \subseteq N} d_T(v) \xi_{u_T}(t) \\ &= \sum_{j=1}^{l} \sum_{T \subseteq N} d_T(v) \xi_{u_T}(s_{|I_j}).\end{aligned}$$

Therefore, by Proposition 5, we have

$$(\xi_v)^{F(\mathcal{F})}(s) = \sum_{j=1}^{l} \sum_{T \subseteq \text{supp } s_{|I_j}} d_T(v)\xi_{u_T}(s_{|I_j}).$$

Notice that supp $s_{|I_j} = I_j$ and $(s_{|I_j})_{|T} = s_{|T}$ for $T \subseteq$ supp $s_{|I_j}$, and we have

$$(\xi_v)^{F(\mathcal{F})}(s) = \sum_{j=1}^{l} \sum_{T \subseteq I_j} d_T(v)\xi_{u_T}(s_{|T}). \tag{2}$$

Therefore, (1) and (2) are generally different. We consider sufficient conditions for the equality of these two values. In order to show sufficient conditions, we introduce subcomplete system as follows.

Definition 12. An FCS $\mathcal{F}$ is said to be subcomplete if it satisfies

$$T \in \mathcal{F}, S \subseteq T \;\Rightarrow\; S \in \mathcal{F}.$$

A subcomplete FCS is simply written as SCS.

Lemma 4. If $v \in \Gamma^N$ is superadditive and $\mathcal{F}$ is a PS and an SCS, then for any $T \in \mathcal{F}$

$$d_T(v^{\mathcal{F}}) = d_T(v)$$

Proof. Since $(N, \mathcal{F})$ is an SCS and $v^{\mathcal{F}}(S) = v(S)$ for $S \in \mathcal{F}$, we have

$$d_T(v^{\mathcal{F}}) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v^{\mathcal{F}}(S) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S) = d_T(v). \qquad \square$$

Theorem 2. Let ξ be a W-extension of a cooperative game $v \in \Gamma^N$ and $F(\mathcal{F})$ be the FFCS corresponding to an FCS $\mathcal{F}$. If v is superadditive, ξ_v is strongly superadditive and $\mathcal{F}$ is an SCS and a PS, then

$$\xi_{v^{\mathcal{F}}} = (\xi_v)^{F(\mathcal{F})}.$$

Proof. Since $\mathcal{F}$ is an SCS, $T \subseteq I_j$ and $I_j \in \mathcal{F}$, we have $T \in \mathcal{F}$. Then, from (1), we have

$$\xi_{v^{\mathcal{F}}}(s) = \sum_{j=1}^{l} \sum_{T \in \mathcal{F}, T \subseteq I_j} d_T(v^{\mathcal{F}})\xi_{u_T}(s_{|T}) = \sum_{j=1}^{l} \sum_{T \subseteq I_j} d_T(v^{\mathcal{F}})\xi_{u_T}(s_{|T}). \tag{3}$$

From (2) and (3), we have

$$(\xi_v)^{F(\mathcal{F})}(s) = \sum_{j=1}^{l} \sum_{T \subseteq I_j} d_T(v)\xi_{u_T}(s_{|T}) = \xi_{v^{\mathcal{F}}}(s). \qquad \square$$

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