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QL–IMPLICATIONS VERSUS D–IMPLICATIONS

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This paper deals with two kinds of fuzzy implications: QL and Dishkant implications. That is, those defined through the expressions $I(x, y) = S(N(x), T(x, y))$ and $I(x, y) = S(T(N(x), N(y)), y)$ respectively, where T is a t-norm, S is a t-conorm and N is a strong negation. Special attention is due to the relation between both kinds of implications. In the continuous case, the study of these implications is focused in some of their properties (mainly the contrapositive symmetry and the exchange principle). Finally, the case of non continuous t-norms or non continuous t-conorms is studied, deriving new implications of both kinds and showing that they remain strongly connected.

Keywords: t-norm, t-conorm, implication operator, QL-implication, D-implication

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1. INTRODUCTION

It is well known that t-norms and t-conorms are useful not only in fuzzy logic but also in other fields. In particular, they are an important tool in the frame of aggregation operators. Moreover, t-norms and t-conorms as well as fuzzy implication functions defined from them, have been also used in several topics related to the aggregation problem. For instance, implication functions are useful in the construction of RET operators ([15]), in mathematical morphology ([11]) and others.

Implications are generally performed by suitable functions $I : [0, 1]^2 \to [0, 1]$ derived from t-norms, t-conorms and strong negations. The four most usual ways to define these implications are:

i) $I(x, y) = \sup\{z \in [0, 1] \mid T(x, z) \leq y\}$ for a given left-continuous t-norm T, called *R-implications*,

ii) $I(x, y) = S(N(x), y)$ for a given t-conorm S and a strong negation N, called *S-implications*,

iii) $I(x, y) = S(N(x), T(x, y))$ for a given t-norm T, a t-conorm S and a strong negation N, called *QL-implications*, and

iv) $I(x, y) = S(T(N(x), N(y)), y)$ for a given t-norm T, a t-conorm S and a strong negation N, called *Dishkant implications* or *D-implications* in short.

R-implications and S-implications have been extensively developed (see [2, 4, 5, 7, 12]). The other two kinds of implications have been studied in [9] and [10] in the framework of a finite chain. In the framework of [0,1], an initial study of QL-implications can be found in [13] and some properties of both, QL and D-implications, are developed in [14]. However, it is also stated in the conclusions of [13] that "much work still remains to be done on the subject..." and the main purpose of this paper consists in partially making this "remaining" work.

Specifically, in this paper we want to deal with these two kinds of implications and mainly from the point of view of their interrelationship. It is proved in general that if a QL-operator is a QL-implication, the same happens for the corresponding D-operator and vice versa. When the involved t-conorm is continuous, a particular case is mainly studied. In this case it is proved that both implications coincide if and only if they satisfy contrapositive symmetry, and this occurs if and only if the φ^{-1}-transform of the t-norm T and its dual satisfy the Frank equation. It is also characterized when they satisfy the exchange principle. The noncontinuous case is also introduced deriving new implications of both kinds and showing that they remain strongly connected.

2. PRELIMINARIES

We suppose the reader to be familiar with basic results concerning t-norms and t-conorms that can be found in [8]. In any case, we will use the following notations. Given any increasing bijection $\varphi : [0,1] \to [0,1]$, N_φ stands for the strong negation given by $N_\varphi(x) = \varphi^{-1}(1 - \varphi(x))$ for all $x \in [0,1]$, whereas for any binary operator $F : [0,1]^2 \to [0,1]$, F_φ will denote the operator given by

$$F_\varphi(x, y) = \varphi^{-1}(F(\varphi(x), \varphi(y)))$$

for all $x, y \in [0,1]$, usually called the φ-transform of F.

Definition 1. (See for instance Definition 4 in [3]) A binary operator $I : [0,1] \times [0,1] \to [0,1]$ is said to be an implication operator, or an implication, if it satisfies:

I1) I is nonincreasing in the first variable and nondecreasing in the second one.

I2) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ cannot be derived from the definition.

Definition 2. (See for instance Definition 12 in [3]) We will say that an implication $I : [0,1] \times [0,1] \to [0,1]$ is a border implication if it satisfies $I(1,x) = x$ for all $x \in [0,1]$.

S-implications were characterized in [3] in the general framework of bounded partially ordered sets. Translated to [0,1] such characterization is given in the following proposition that derives from Proposition 28 in [3].

Proposition 1. An implication $I : [0,1]^2 \to [0,1]$ is an S-implication if and only if it satisfies

- $I(1,x) = x$ for all $x \in [0,1]$ (i. e., is a border implication),
- $I(N(y),N(x)) = I(x,y)$ for all $x,y \in [0,1]$ (contrapositive symmetry), and
- $I(x,I(y,z)) = I(y,I(x,z))$ for all $x,y,z \in [0,1]$ (exchange principle).

Definition 3. (See for instance page 108 in [8]) A t-norm T is called a Frank t-norm if it is in the family, denoted $(T_\lambda)_{\lambda\in[0,+\infty]}$, given by

$$T_\lambda(x,y) = \begin{cases} \min(x,y) & \text{if } \lambda = 0 \\ \Pi(x,y) & \text{if } \lambda = 1 \\ W(x,y) & \text{if } \lambda = \infty \\ \log_\lambda\left(1 + \frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}\right) & \text{otherwise} \end{cases}$$

where Π, W denote the product and Łukasiewicz t-norms, respectively. Frank t-conorms are the dual of Frank t-norms and they are denoted by S_λ.

Next proposition was firstly proved in [6], see also Theorem 5.14 in [8] for the following version. Recall that, given a t-norm T, its dual t-conorm is given by

$$S(x,y) = 1 - T(1-x,1-y) \quad \text{for all} \quad x,y \in [0,1].$$

Proposition 2. A t-norm T and a t-conorm S satisfy the Frank equation, i. e.

$$T(x,y) + S(x,y) = x + y \quad \text{for all } x,y \in [0,1]$$

if and only if T is an ordinal sum of Frank t-norms $T = (\langle a_\alpha, b_\alpha, T_{\lambda_\alpha}\rangle)_{\alpha\in A}$ and S is the ordinal sum of the corresponding dual Frank t-conorms, $S = (\langle a_\alpha, b_\alpha, S_{\lambda_\alpha}\rangle)_{\alpha\in A}$.

Finally, recall that a t-norm T satisfies the Lipschitz condition if

$$T(x,y) - T(x',y) \le x - x' \quad \text{whenever } x' \le x.$$

3. QL AND D–IMPLICATIONS

Let us begin this section by introducing the operators that we want to study.

Definition 4. A binary operator $I : [0,1]^2 \to [0,1]$ is called a

- QL-operator when there are a t-norm T, a t-conorm S and a strong negation N such that I is given by
$$I(x,y) = S(N(x),T(x,y)) \quad \text{for all } x,y \in [0,1]. \tag{1}$$
- D-operator when there are a t-norm T, a t-conorm S and a strong negation N such that I is given by
$$I(x,y) = S(T(N(x),N(y)),y) \quad \text{for all } x,y \in [0,1]. \tag{2}$$

Definition 5. A binary operator $I : [0,1]^2 \to [0,1]$ is called a QL-implication (D-implication) when it is both a QL-operator (D-operator) and an implication.

Remark 1. Note that any QL-operator always satisfies condition (I2) and monotonicity in the second variable and thus, only monotonicity in the first one can fail in order to be an implication. On the other hand, the monotonicity in the second variable is the only condition that can fail in order to be any D-operator an implication. Note also that any QL or D-operator I always satisfies $I(1,y) = y$ for all $y \in [0,1]$ and consequently, any QL or D-implication will be in fact a border implication.

We will denote QL and D-operators by I_Q and I_D respectively. The following proposition gives a necessary condition for a QL-operator, as well as for a D-operator, to be an implication. The proof in each case is trivial and can be found in [13] and [14] respectively.

Proposition 3. Let T be a t-norm, S a t-conorm, N a strong negation and I the operator given by (1) (or by (2)). If I is a QL-implication (or a D-implication) then

$$S(x, N(x)) = 1 \qquad \text{for all} \quad x \in [0,1]. \tag{3}$$

The condition given in the previous proposition is necessary but not sufficient. Infinitely many counterexamples will follow from Proposition 9.

Let us now give the following general result that we will apply to our special case of QL and D-operators.

Proposition 4. Let N be a strong negation. A function $I : [0,1]^2 \to [0,1]$ is an implication if and only if their contraposition J, given by $J(x,y) = I(N(y), N(x))$ for all $x, y \in [0,1]$, is an implication.

Proof. Suppose first that I is an implication, then clearly

$$J(0,0) = I(1,1) = 1, \quad J(1,1) = I(0,0) = 1 \quad \text{and} \quad J(1,0) = I(1,0) = 0.$$

Moreover, given $x < x'$, we have $N(x') < N(x)$ and then, since I is nondecreasing in the second variable,

$$J(x,y) = I(N(y), N(x)) \geq I(N(y), N(x')) = J(x', y).$$

This proves that J is nonincreasing in the first variable and the nondecreasingness in the second one follows similarly. The proof of the converse can be viewed in the same way. $\square$

The following propositions give the first signals about the strong connection between QL and D-implications.

Proposition 5. Let T be a t-norm, S a t-conorm and N a strong negation. Then, the corresponding QL-operator, I_Q, is a QL-implication if and only if the corresponding D-operator, I_D, is a D-implication.

Proof. Just note that I_D is the contraposition of I_Q and apply the previous proposition. □

Proposition 6. Let T be the t-norm minimum, N a strong negation and S a t-conorm satisfying equation (3). Then the corresponding QL and D-operators are always implications, they coincide and are given by

$$I(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ S(N(x), y) & \text{otherwise.} \end{cases} \tag{4}$$

Proof. Note that when $x \leq y$ we have

$$I_Q(x,y) = S(N(x), \min(x,y)) = S(N(x), x) = 1$$

and

$$I_D(x,y) = S(\min(N(x), N(y)), y) = S(N(y), y) = 1.$$

Similarly, when $y < x$,

$$I_Q(x,y) = S(N(x), \min(x,y)) = S(N(x), y) = S(\min(N(x), N(y)), y) = I_D(x,y).$$

This proves that I_Q and I_D coincide and are given by (4). Finally, the monotonicity of this expression in each variable is obvious and so it gives an implication. □

Two important properties of implications, that are commonly required depending on the context, are contrapositive symmetry and exchange principle. So let us give some results about them that we will use later in the case when S is continuous.

For a binary operator, contrapositive symmetry with respect to a strong negation N_1, is defined by:

$$I(x,y) = I(N_1(y), N_1(x)) \quad \text{for all } x, y \in [0,1],$$

and it is an important property for implications. It is proved in [14], that if a QL-implication satisfies this property with respect to a strong negation N_1 then $N_1 = N$. The same happens for D-implications (see [14]) and thus, only contrapositive symmetry with respect to the same negation N can be done. Note that we trivially have the following result.

Proposition 7. Let T be a t-norm, S a t-conorm and N a strong negation. Let I_Q and I_D be the corresponding QL and D-operators. The following statements are equivalent:

i) I_Q satisfies contrapositive symmetry with respect to N.

ii) I_D satisfies contrapositive symmetry with respect to N.

iii) $I_Q = I_D$.

P r o o f. It is trivial from the fact that I_Q and I_D are the contraposition one of each other. □

With respect to the exchange principle, that is

$$I(x, I(y, z)) = I(y, I(x, z)) \quad \text{for all } x, y, z \in [0, 1],$$

we have the following results.

Remark 2. Given any QL or D-operator I it is trivially satisfied that $I(x, 0) = N(x)$. Thus, if any QL or D-operator satisfies the exchange principle then it also satisfies contrapositive symmetry with respect to N, since

$$I(N(y), N(x)) = I(N(y), I(x, 0)) = I(x, I(N(y), 0)) = I(x, y).$$

Proposition 8. Let T be a t-norm, S a t-conorm and N a strong negation such that the corresponding QL-operator I_Q (equivalently the D-operator I_D) is an implication. The following statements are equivalent:

i) I_Q satisfies the exchange principle.

ii) I_D satisfies the exchange principle.

iii) I_Q is an S-implication.

iv) I_D is an S-implication.

v) There exists a t-conorm S_1 such that

$$S(N(x), T(x, y)) = S_1(N(x), y) \quad \text{for all} \quad x, y \in [0, 1]. \tag{5}$$

P r o o f. Let us prove only the equivalence among i), iii) and v) since the equivalence among ii), iv) and v) follows similarly.

- i) $\Rightarrow$ iii) If I_Q satisfies the exchange principle, it also satisfies contrapositive symmetry by Remark 2 and then, by Proposition 1, I_Q is an S-implication.
- iii) $\Rightarrow$ v) If I_Q is an S-implication, there exist a t-conorm S_1 and a strong negation N_1 such that

  $$I_Q(x, y) = S(N(x), T(x, y)) = S_1(N_1(x), y) \quad \text{for all} \quad x, y \in [0, 1].$$

  But taking $y = 0$ in the above equation we obtain $N(x) = N_1(x)$ for all $x \in [0, 1]$ and consequently equation (5) follows.
- v) $\Rightarrow$ i) If equation (5) is satisfied, I_Q is in fact an S-implication and so it satisfies the exchange principle. □

We divide the rest of our study in two subsections, one devoted to the continuous case and the other devoted to the non-continuous one.

3.1. Continuous case

For both types of implications we want to deal in this subsection with the case when S is continuous. Note that in this case equation (3) implies that (see for instance [8]):

- S must be a nilpotent t-conorm, that is a W^*_φ for some increasing bijection $\varphi : [0,1] \to [0,1]$, where W^* denotes the Łukasiewicz t-conorm.
- $N \geq N_\varphi$.

We want to deal specially with the case $N = N_\varphi$. Assuming this condition we have for any QL-operator:

$$\begin{aligned} I(x,y) &= \varphi^{-1}\big(\min(1-\varphi(x)+\varphi(T(x,y)),1)\big) \\ &= \varphi^{-1}(1-\varphi(x)+\varphi(T(x,y))) \end{aligned}$$

and similarly for any D-operator:

$$\begin{aligned} I(x,y) &= \varphi^{-1}\big(\min(\varphi(T(N_\varphi(x),N_\varphi(y)))+\varphi(y),1)\big) \\ &= \varphi^{-1}(\varphi(T(N_\varphi(x),N_\varphi(y)))+\varphi(y)). \end{aligned}$$

In both cases, I is totally determined by the function φ and the t-norm T. For this reason, we will denote from now on the corresponding QL-operator by $I^{\varphi,T}$ and the corresponding D-operator by $I_{\varphi,T}$.

The following proposition characterizes all t-norms for which $I^{\varphi,T}$ and $I_{\varphi,T}$ are implications. Note that, from the result, it is deduced in particular that the involved t-norm T must be also continuous in order to obtain QL or D-implications.

Proposition 9. Let T be any t-norm and $\varphi : [0,1] \to [0,1]$ an increasing bijection. The following statements are equivalent:

i) $I^{\varphi,T}$ is a QL-implication.

ii) $I_{\varphi,T}$ is a D-implication.

iii) $T_{\varphi^{-1}}$ satisfies the Lipschitz condition.

Proof. We begin with the equivalence between i) and iii). Given $a_1 \leq a_2$ and $b \in [0,1]$, take $x_i = \varphi^{-1}(a_i)$ for $i = 1,2$ and $y = \varphi^{-1}(b)$. Thus, $x_1 \leq x_2$ and we have $I^{\varphi,T}(x_2,y) \leq I^{\varphi,T}(x_1,y)$ if and only if

$$\varphi^{-1}(1-\varphi(x_2)+\varphi(T(x_2,y))) \leq \varphi^{-1}(1-\varphi(x_1)+\varphi(T(x_1,y))),$$

that is,

$$\varphi(T(x_2,y)) - \varphi(T(x_1,y)) \leq \varphi(x_2) - \varphi(x_1)$$

or equivalently

$$\varphi\left(T(\varphi^{-1}(a_2),\varphi^{-1}(b))\right) - \varphi\left(T(\varphi^{-1}(a_1),\varphi^{-1}(b))\right) \leq a_2 - a_1$$

and this happens if and only if $T_{\varphi^{-1}}(a_2,b) - T_{\varphi^{-1}}(a_1,b) \leq a_2 - a_1$, that is, if and only if $T_{\varphi^{-1}}$ satisfies the Lipschitz condition.

Now, the equivalence between ii) and iii) follows from the fact that i) and ii) are equivalent by Proposition 5. □

Remark 3. Note that the proof of iii) $\Longrightarrow$ ii) in the previous proposition can be found in [14]. On the other hand, recall that a t-norm satisfies the Lipschitz condition if and only if it is a copula and such t-norms are characterized for instance in [8].

The following result definitively shows the strong connection between QL and D-implications. In particular it shows many QL-implications that are also D-implications and vice versa.

Proposition 10. Let $\varphi : [0,1] \to [0,1]$ be an increasing bijection and let T and T' be t-norms. Then, $I^{\varphi,T} = I_{\varphi,T'}$ if and only if $T_{\varphi^{-1}}$ and the dual t-conorm of $T'_{\varphi^{-1}}$ satisfy the Frank equation.

P r o o f. $I^{\varphi,T} = I_{\varphi,T'}$ if and only if we have

$$\varphi^{-1}(1 - \varphi(x) + \varphi(T(x,y))) = \varphi^{-1}(\varphi(T'(N_\varphi(x), N_\varphi(y))) + \varphi(y))$$

for all $x, y \in [0,1]$ and then the following chain of equivalence holds

$$\begin{aligned} I^{\varphi,T} = I_{\varphi,T'} \quad &\Longleftrightarrow \quad 1 - \varphi(x) + \varphi(T(x,y)) = \varphi(T'(N_\varphi(x), N_\varphi(y))) + \varphi(y) \\ &\Longleftrightarrow \quad \varphi(T(x,y)) + 1 - \varphi(T'(N_\varphi(x), N_\varphi(y))) = \varphi(x) + \varphi(y) \\ &\Longleftrightarrow \quad T_{\varphi^{-1}}(\varphi(x), \varphi(y)) + 1 - T'_{\varphi^{-1}}(1 - \varphi(x), 1 - \varphi(y)) = \varphi(x) + \varphi(y). \end{aligned}$$

Finally, taking $a = \varphi(x)$ and $b = \varphi(y)$, this is equivalent to

$$T_{\varphi^{-1}}(a,b) + 1 - T'_{\varphi^{-1}}(1-a, 1-b) = a + b \quad \text{for all} \quad a, b \in [0,1].$$

That is, $T_{\varphi^{-1}}$ and the dual t-conorm of $T'_{\varphi^{-1}}$ satisfy the Frank equation, which ends the proof. $\square$

Remark 4. Note that all the QL and D-operators $I^{\varphi,T}$ and $I_{\varphi,T'}$ satisfying $I^{\varphi,T} = I_{\varphi,T'}$ are in fact implications because the solutions of Frank's equation always satisfy the Lipschitz condition.

Now we want to deal with contrapositive symmetry and exchange principle in this continuous case. The first property was already studied for both kinds of implications in [5] and [14]. We show in the following Proposition that, Theorem 4 in [5] and Theorem 5 in [14] can be easily derived from the previous Proposition. The corollary also gives new t-norms T (apart from minimum, see Proposition 6) for which both kinds of implications coincide.

Proposition 11. Let $\varphi : [0,1] \to [0,1]$ be an increasing bijection and T a t-norm. The following statements are equivalent:

i) $I^{\varphi,T}$ satisfies contrapositive symmetry with respect to N_φ.

ii) $I_{\varphi,T}$ satisfies contrapositive symmetry with respect to N_φ.

iii) $T_{\varphi^{-1}}$ and its dual t-conorm satisfy the Frank equation.

Proof. It is clearly deduced from Propositions 7 and 10. □

Note that, from Remark 4, all the solutions in the proposition above satisfy $I^{\varphi,T} = I_{\varphi,T}$ and they are implications.

The exchange principle in this continuous case can be also completely characterized. From Proposition 8, we know that this property is equivalent to the existence of a t-conorm S_1 satisfying equation (5). Such equation was solved in [1] in the continuous case and from the results proved there it follows the following corollary.

Corollary 1. Let $\varphi : [0,1] \to [0,1]$ be an increasing bijection and T a t-norm such that $T_{\varphi^{-1}}$ satisfies the Lipschitz condition. The following statements are equivalent:

i) $I^{\varphi,T}$ satisfies the exchange principle.

ii) $I_{\varphi,T}$ satisfies the exchange principle.

iii) $T_{\varphi^{-1}}$ is a Frank t-norm.

In all these cases, $I^{\varphi,T} = I_{\varphi,T}$.

Proof. If $I^{\varphi,T}$ satisfies the exchange principle we know from Proposition 8 that there exists a t-conorm S_1 such that T, S_1 satisfy equation (5) with $S = W_\varphi^*$ and $N = N_\varphi$. Since T is continuous S_1 must be also continuous, and several lemmas in [1] prove that, with these conditions, necessarily $T_{\varphi^{-1}}$ is a Frank t-norm.

Conversely, if $T_{\varphi^{-1}}$ is a Frank t-norm, it suffices to prove that there exists a continuous t-conorm S_1 such that T and S_1 satisfy equation (5) with $S = W_\varphi^*$ and $N = N_\varphi$. From the main theorem in [1], such a t-conorm S_1 is given, depending on $T_{\varphi^{-1}}$, by:

- If $T_{\varphi^{-1}} = \min$, just take $S_1 = S$.
- If $T_{\varphi^{-1}} = \Pi$, then $T = \Pi_\varphi$ and it is enough to take S_1 the φ-transform of the dual t-conorm of Π.
- If $T_{\varphi^{-1}} = W$, then $T = W_\varphi$ and it is enough to take $S_1 = \max$.
- If $T_{\varphi^{-1}} = T_\lambda$ with $\lambda \neq 0, 1, \infty$, then $T = (T_\lambda)_\varphi$ and it is enough to take S_1 the φ-transform of the dual t-conorm of $T_{1/\lambda}$. □

Among other interesting properties note that the generalized modus ponens and modus tollens for these types of implications were recently studied in [14]. Finally we want to deal with two more properties that are easily characterized.

Proposition 12. Let T be a t-norm and φ be an increasing bijection. The following statements are equivalent:

i) $I^{\varphi,T}(x,y) = 1$ if and only if $x \leq y$.

ii) $I^{\varphi,T}(x,x) = 1$ for all $x \in [0,1]$.

iii) $T = \min$.

The same holds for D-operators $I_{\varphi,T}$.

Proof. i) $\Rightarrow$ ii) is obvious. If $I^{\varphi,T}(x,x) = 1$ for all $x \in [0,1]$, we have $1 - \varphi(x) + \varphi(T(x,x)) = 1$ which implies $T(x,x) = x$ for all $x \in [0,1]$. Thus, $T = \min$ and this proves ii) $\Rightarrow$ iii). Finally, if $T = \min$, we have

$$I^{\varphi,T}(x,y) = 1 \quad \Longleftrightarrow \quad 1 - \varphi(x) - \varphi(\min(x,y)) = 1$$

and this occurs if and only if $\min(x,y) = x$, i. e., $x \leq y$, proving iii) $\Rightarrow$ i).

The result for D-operators follows similarly. $\square$

Proposition 13. Let $\varphi : [0,1] \to [0,1]$ be an increasing bijection and T a t-norm. The following statements are equivalent:

i) $I^{\varphi,T}(x, N_\varphi(x)) = N_\varphi(x)$ for all $x \in [0,1]$.

ii) $I_{\varphi,T}(x, N_\varphi(x)) = N_\varphi(x)$ for all $x \in [0,1]$.

iii) $T(x, N_\varphi(x)) = 0$ for all $x \in [0,1]$.

Proof. Again we only prove the equivalence between i) and iii) since the equivalence between ii) and iii) is similar. Note that,

$$I^{\varphi,T}(x, N_\varphi(x)) = N_\varphi(x) \quad \Longleftrightarrow \quad 1 - \varphi(x) + \varphi(T(x, N_\varphi(x))) = 1 - \varphi(x)$$

if and only if $T(x, N_\varphi(x)) = 0$. $\square$

3.2. Non-continuous case

Recall that a necessary condition for QL or D-operators to be implications is given by (3). For any strong negation N, if S is not necessarily continuous, there are many other possibilities for S than nilpotent t-conorms. For instance, taking S the nilpotent maximum with respect to N, that is,

$$S(x,y) = \begin{cases} \max(x,y) & \text{if } y < N(x) \\ 1 & \text{otherwise} \end{cases}$$

and $T = \min$, it is easy to see that the corresponding QL and D-implications coincide and they are given by

$$I(x,y) = \begin{cases} \max(N(x),y) & \text{if } y < x \\ 1 & \text{otherwise.} \end{cases} \tag{6}$$

Note that this implication was extensively studied in [12].

We are now interested in studying the t-norms such that the QL and D-operators derived from them and the nilpotent maximum, give implications.

Proposition 14. Let N be a strong negation with fixed point s and $\max_N$ the corresponding nilpotent maximum. Let T be a t-norm such that the corresponding QL-operator $I_Q(x, y) = \max_N(N(x), T(x, y))$ is an implication. Then

i) $T(x, y) = \min(x, y)$ whenever $\max(x, y) > s$, and

ii) If $a \in (0, s]$ is T-idempotent (that is $T(a, a) = a$), then $T(x, y) = \min(x, y)$ for all x, y such that $\min(x, y) \leq a \leq \max(x, y) \leq s$.

Proof. i) Since T is commutative, it suffices to show that $T(x, y) = x$ when $x \leq y$ and $y > s$. We do this in two cases:

- When $s < x \leq y$. Suppose on the contrary that there are $s < x_0 \leq y_0$ such that $T(x_0, y_0) < x_0$, we will also have $N(x_0) < s < y_0$. By one hand, since I_Q is a border implication we have $I_Q(1, y_0) = y_0$. By the other hand, we have

$$I_Q(x_0, y_0) = \max_N(N(x_0), T(x_0, y_0)) = \max(N(x_0), T(x_0, y_0)) < y_0$$

contradicting the non-increasingness on the first variable.

- When $x \leq s < y$. Suppose again that there are $0 < x_0 \leq s < y_0$ such that $T(x_0, y_0) < x_0$ and take x_1 such that $s < x_1 < y_0$. By the previous step, $T(x_1, y_0) = x_1$ and then,

$$I_Q(x_1, y_0) = \max_N(N(x_1), T(x_1, y_0)) = \max_N(N(x_1), x_1) = 1$$

whereas

$$I_Q(x_0, y_0) = \max_N(N(x_0), T(x_0, y_0)) = \max(N(x_0), T(x_0, y_0)) < 1$$

obtaining again a contradiction.

ii) Let $a \in (0, s]$ be a T-idempotent element. Again by commutativity we only need to prove that $T(x, y) = x$ for all x, y such that $x \leq a \leq y \leq s$. First of all note that for these values

$$I_Q(x, y) \geq I_Q(a, y) \geq I_Q(a, a) = \max_N(N(a), T(a, a)) = \max_N(N(a), a) = 1.$$

Suppose now that there are $0 < x_0 \leq a \leq y_0 \leq s$ such that $T(x_0, y_0) < x_0$, then

$$I_Q(x_0, y_0) = \max_N(N(x_0), T(x_0, y_0)) = \max(N(x_0), T(x_0, y_0)) < 1$$

which is a contradiction. $\square$

Now, we can characterize all continuous t-norms with this property.

Corollary 2. Let N be a strong negation with fixed point s and $\max_N$ the corresponding nilpotent maximum. Let T be a continuous t-norm and I_Q the corresponding QL-operator. Then I_Q is a QL-implication if and only if T satisfies $T(x, x) = x$ for all $x \geq s$. In these cases I_Q is given by

$$I_Q(x, y) = \begin{cases} 1 & \text{if } x, s \leq y \text{ or } (x \leq y < s \text{ and } T(x, y) = x) \\ y & \text{if } N(x) \leq y < x \\ N(x) & \text{otherwise.} \end{cases} \tag{7}$$

Proof. If I_Q is a QL-implication, we have $T(x,x) = x$ for all $x > s$ from Proposition 14 and, by continuity, it will verify $T(x,x) = x$ for all $x \geq s$.

Conversely, if $T(x,x) = x$ for all $x \geq s$ we have $T(x,y) = \min(x,y)$ whenever $\max(x,y) \geq s$, because T is continuous. Since equation (7) clearly gives an implication, to finish the proof it suffices to show that with these conditions on T, I_Q is given by such equation. We do this by distinguishing two cases.

i) When $x \leq y$:

- If $s \leq y$ or $y < s$ but $T(x,y) = x$. Then $I_Q(x,y) = \max_N(N(x), x) = 1$.
- When $y < s$ and $T(x,y) < x$. Then, since $T(x,y) < x < s < N(x)$, $I_Q(x,y) = \max(N(x), T(x,y)) = N(x)$.

ii) When $y < x$:

- If $N(x) \leq y$. Then $I_Q(x,y) = \max_N(N(x), y) = \max(N(x), y) = y$.
- When $N(x) > y$. Then, since $T(x,y) \leq y < x, N(x)$, we have $I_Q(x,y) = \max(N(x), T(x,y)) = N(x)$. □

Similarly for D-implications we have the following result.

Proposition 15. Let N be a strong negation with fixed point s and $\max_N$ the corresponding nilpotent maximum. Let T be a t-norm and $I_D(x,y) = \max_N(T(N(x), N(y)), y)$ the corresponding D-operator. Then

i) If I_D is an implication, then $T(x,y) = \min(x,y)$ whenever $\max(x,y) > s$.

ii) If I_D is an implication and $a \in (0,s]$ is T-idempotent, then $T(x,y) = \min(x,y)$ for all x, y such that $\min(x,y) \leq a \leq \max(x,y) \leq s$.

iii) If T is continuous then I_D is a D-implication if and only if T satisfies $T(x,x) = x$ for all $x \geq s$. In these cases I_D is given by

$$I_D(x,y) = \begin{cases} 1 & \text{if } x \leq s, y \text{ or } (s < x \leq y \text{ and } T(N(x), N(y)) = N(y)) \\ N(x) & \text{if } y < x \leq N(y) \\ y & \text{otherwise.} \end{cases} \tag{8}$$

Proof. If I_D is an implication we know, by Proposition 5, that the corresponding QL-operator I_Q is also an implication and then i) and ii) follow from Proposition 14. To prove iii), just use Proposition 5 and Corollary 2. Finally, equation (8) can be obtained again from Corollary 2 and the equality $I_D(x,y) = I_Q(N(y), N(x))$. □

Remark 5. Note that when S is the nilpotent maximum and T is a continuous t-norm, we obtain that $I_Q = I_D$ only when $T = \min$ from equations (7) and (8). On the other hand, contrapositive symmetry with respect to N for I_Q derived from nilpotent maximum $\max_N$ was studied in [5]. It is proved there that $T = \min$ is again the only t-norm satisfying it. Note that from Proposition 7 the same happens for D-implications I_D.

Example 1. Let T be a continuous t-norm such that the operator I_Q (I_D) derived from $\max_N$ is an implication. From the results above, $T_{/[0,s]^2}$ can be any continuous t-norm on $[0,s]^2$. Suppose that such a restriction is an ordinal sum. Take for instance, $N(x) = 1 - x$ and

$$T = \big(\langle 0, a, T_1\rangle, \langle a, b, T_2\rangle, \langle b, 1/2, T_3\rangle\big)$$

where $0 < a < b < 1/2$ and T_i are Archimedean t-norms for $i = 1, 2, 3$. Then I_Q and I_D are shown in Figure 1.

Given any strong negation N, there exists a family of right-continuous t-conorms satisfying (3) whose corresponding family of N-dual t-norms appeared for the first time in [5] and was characterized in [7]. Namely, for any $a \in [0,1]$ such that $a \le N(a)$ the operator $S_{N,a}$ given by

$$S_{N,a}(x,y) = \begin{cases} 1 & \text{if } \; y \ge N(x) \\ a + (N(a)-a)S_1\left(\frac{x-a}{N(a)-a}, \frac{y-a}{N(a)-a}\right) & \text{if } x, y \in [a, N(a)] \\ & \text{and } y < N(x) \\ \max(x,y) & \text{otherwise} \end{cases} \tag{9}$$

where S_1 is the nilpotent t-conorm with associated negation N_1 given by

$$N_1(x) = \frac{N((N(a)-a)x + a) - a}{N(a) - a}$$

for all $x \in [0,1]$. Note that $S_{N,a}$ is continuous only for the case $a = 0$ since then $N_1 = N$ and $S_1 = S_{N,a}$. On the other hand, the case $a = s$ where s is the fixed point of N gives the nilpotent maximum with respect to N. Thus, which continuous t-norms satisfy that the corresponding QL-operator and D-operator are implications? In which cases do they coincide?

These are questions for a future work, but note that the behavior in these cases is different from the one for the nilpotent maximum. For instance, the following example shows that in the Łukasiewicz case there are continuous t-norms, apart from the minimum, for which both kinds of implications coincide.

Example 2. Take the particular strong negation $N(x) = 1 - x$ for all $x \in [0,1]$, denoted by $1 - j$, and the Łukasiewicz t-conorm $S_1 = W^*$. With a straightforward computation we obtain the following family of t-conorms:

$$S_{1-j,a}(x,y) = \begin{cases} 1 & \text{if } x + y \ge 1 \\ x + y - a & \text{if } x + y < 1 \text{ and } x, y \in [a, 1-a] \\ \max(x,y) & \text{otherwise.} \end{cases}$$

From them the following implications can be easily derived:

i) When $T = \min$,

$$I_Q(x,y) = I_D(x,y) = \begin{cases} 1 & \text{if } x \le y \\ 1 + y - x - a & \text{if } a \le y < x \le 1 - a \\ \max(1-x, y) & \text{otherwise.} \end{cases}$$

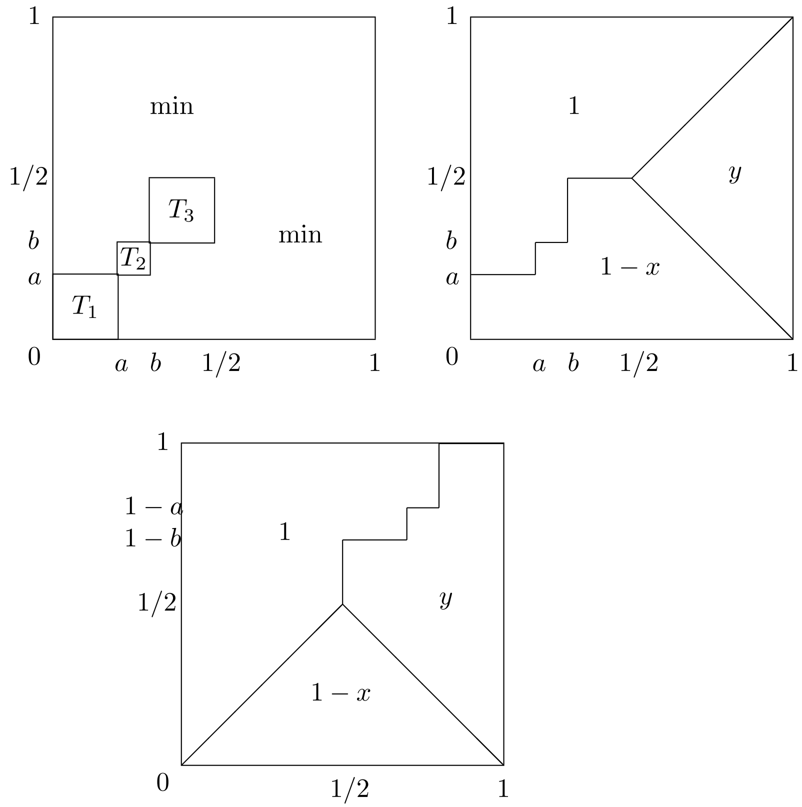

Fig. 1. The t-norm (up left) of Example 1 and its derived implications I_Q (up right) and I_D (bottom).

ii) When T is the ordinal sum $T = (a, 1-a, W)$,

$$I_Q(x,y) = I_D(x,y) = \begin{cases} 1 & \text{if } x \leq y \text{ and } x \notin (a, 1-a) \\ & \text{or } x \leq y \text{ and } y \notin (a, 1-a) \\ \max(1-x, y) & \text{otherwise.} \end{cases}$$

The t-conorms given in the previous example can be viewed in Figure 2. The derived implications given in cases i) and ii) can be viewed in Figure 3.

CONCLUSIONS

QL-operators, given by $I_Q(x,y) = S(N(x), T(x,y))$ (and D-operators that are their contraposition), were introduced in fuzzy logic by analogy with quantum mechanic

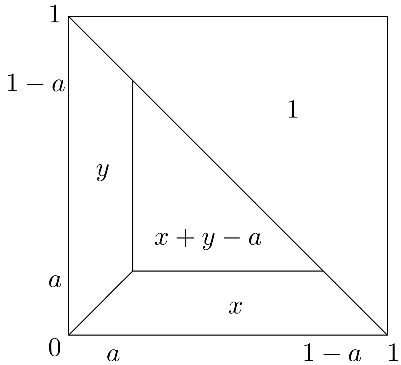

Fig. 2. The family of t-conorms given in Example 2.

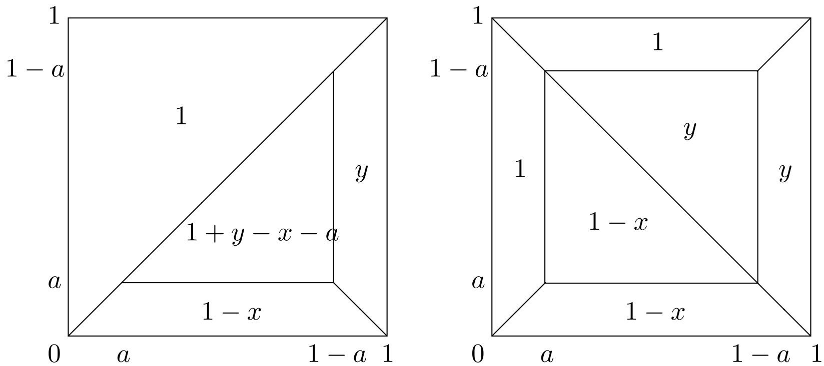

Fig. 3. The implications given in cases i) (left) and ii) (right) of Example 2.

logic. However, in order to be used in fuzzy inference processes, it is important to require these operators to be also implications. In this paper, it is investigated in which cases this fact holds and then several properties are studied. In particular, contrapositive symmetry, exchange principle and other usual properties are characterized in a special case in which the involved t-conorm S is continuous. For non-continuous t-conorms, the case of the nilpotent maximum is studied, and all continuous t-norms for which the corresponding QL and D-operators are implications, are characterized. However, there are many other non-continuous t-conorms to be investigated (for instance those given by equation (9) or more particularly those given in Example 2) that can lead to a future work.

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