Wolfgang Sander; Jens Siedekum

Multiplication, distributivity and fuzzy-integral. I

Kybernetika, Vol. 41 (2005), No. 3, [397]--422

Persistent URL: http://dml.cz/dmlcz/135664

KYBERNETIKA — VOLUME 41 (2005), NUMBER 3, PAGES 397–422

MULTIPLICATION, DISTRIBUTIVITY AND FUZZY-INTEGRAL I

WOLFGANG SANDER AND JENS SIEDEKUM

The main purpose is the introduction of an integral which covers most of the recent integrals which use fuzzy measures instead of measures. Before we give our framework for a fuzzy integral we motivate and present in a first part structure- and representation theorems for generalized additions and generalized multiplications which are connected by a strong and a weak distributivity law, respectively.

Keywords: fuzzy measures, distributivity law, restricted domain, pseudo-addition, pseudo-multiplication, Choquet integral, Sugeno integral

AMS Subject Classification: 28A25, 20M30

1. INTRODUCTION

In the last years many books and survey articles on so-called fuzzy-integrals appeared, where the name fuzzy-integral comes from the fact that these integrals are defined with respect to a fuzzy measure (which is an increasing set function which disappears on the empty set) instead of an additive set function.

We mention here [2], all articles in [8, 9, 16], and [21].

The two most well-known fuzzy integrals are the Choquet integral and the Sugeno integral. Having at least two different integrals a natural question is always to look for a more general notion of an integral which covers the known ones.

In the above mentioned literature there are many proposals for such 'general integrals'. We are here interested in the different methods for arriving at 'general integrals', in structural aspects and of course, in the general integrals themselves.

The two main approaches in [15] and [2], which lead to the most general fuzzy-integrals which are known at present, use the same basic ideas. We first want to point out these ideas since an understanding of these ideas is essentially for getting a deeper insight into more technical problems which occur naturally in our considerations. The main idea is to consider generalized operations as functional equations not on the whole domain but on a restricted domain. Thus we arrive at a larger variety of solutions. This idea of using 'restricted domains' was already used in [15] but it was not mentioned that this is a special successful technique used rather often in the theory of functional equations.

2. ADDITION, MULTIPLICATION, AND DIFFERENCE

Let $(X, \mathcal{A})$ be a measurable space, let $f : X \to [0,1]$ be $\mathcal{A}$-measurable, and let $\mu : \mathcal{A} \to [0,1]$ be a fuzzy measure ($\mu(A) \leq \mu(B)$ if $A \subset B$ and $\mu(\emptyset) = 0$).

If f is a simple function then there are several equivalent representations, namely

$$f = \sum_{i=1}^{n} 1_{E_i} \cdot \alpha_i, \tag{1}$$

$$f = \bigvee_{1 \leq i \leq n} (1_{F_i} \wedge \alpha_i), \tag{2}$$

$$f = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) \cdot 1_{F_i}. \tag{3}$$

Here the $E_i \in \mathcal{A}$, $1 \leq i \leq n$, are pairwise disjoint, $E_1 \cup \ldots \cup E_n = X$, $\alpha_0 = 0 \leq \alpha_1 < \alpha_2 < \ldots \alpha_n \leq 1$ and $F_i = \bigcup_{j=i}^{n} E_j$, so that $X = F_1 \supset F_2 \supset \ldots \supset F_n = E_n$ and $F_i = \{x \in X : f(x) \geq \alpha_i\}$.

From (1) - (3) we get

$$\text{(L)} \qquad \int f \mathrm{d}\mu = \sum_{i=1}^{n} \mu(E_i) \cdot \alpha_i, \tag{4}$$

$$\text{(S)} \qquad \int f \mathrm{d}\mu = \bigvee_{i=1}^{n} \mu(F_i) \wedge \alpha_i, \tag{5}$$

$$\text{(C)} \qquad \int f \mathrm{d}\mu = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) \cdot \mu(F_i), \tag{6}$$

that is, the Lebesgue integral, the Sugeno integral and the Choquet integral of a simple function f.

Now, (4) and (5) with $n = 2$ imply for $f = 1_{E_1 \cup E_2}$ and $\alpha_1 = \alpha_2 = 1$

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) \quad \text{and} \quad \mu(F_1 \cup F_2) = \mu(F_1) \vee \mu(F_2), \tag{7}$$

that is, μ is additive and maxitive, respectively.

Using (7) we get from (4) and (5) with $n = 2$ and $f = 1_{E_1 \cup E_2} \cdot x$

$$\begin{aligned} (\mu(E_1) + \mu(E_2)) \cdot x &= \mu(E_1) \cdot x + \mu(E_2) \cdot x, \\ (\mu(F_1) \vee \mu(F_2)) \cdot x &= \mu(F_1) \cdot x \vee \mu(F_2) \cdot x \end{aligned} \tag{8}$$

whereas (6) with $f = a \cdot 1_{E_1} + b \cdot 1_{\emptyset}$, $a \geq b \geq 0$ leads to

$$(a - b) \cdot \mu(F_1) + (b - 0) \cdot \mu(F_1) = (a - 0) \cdot \mu(F_1). \tag{9}$$

This means that (8) and (9) lead to distributivity laws of the two types

$$(a + b) \cdot x = a \cdot x + b \cdot x, \tag{10}$$

$$((a - b) \cdot x) + ((b - c) \cdot x) = (a - c) \cdot x, \quad a \geq b \geq c, \tag{11}$$

respectively.

If we replace in (1) and (2) sum + and max $\vee$ for example by a t-conorm $\triangle$ and multiplication $\cdot$ by a t-norm $\diamond$, then we have a so-called generalized addition and a generalized multiplication, which we will call from now on pseudo-addition and pseudo-multiplication.

In (3) and (6) we need – and this is an essential fact – an additional operation, a generalized difference or pseudo-difference $-_\triangle$ (with respect to the pseudo-addition $\triangle$). From (10) and (11) we see that $\triangle, \diamond$ and $-_\triangle$ satisfy

$$(a \triangle b) \diamond x = (a \diamond x) \triangle (b \diamond x) \quad \text{and} \tag{12}$$

$$((a -_\triangle b) \diamond x) \triangle ((b -_\triangle c) \diamond x) = (a -_\triangle c) \diamond x, \quad a \geq b \geq c, \tag{13}$$

respectively, that is, they satisfy distributivity laws for pseudo-additions and pseudo-differences.

Thus we have seen that

a) the Lebesgue integral of a simple function depends on the usual addition + and multiplication $\cdot$,

b) the Sugeno integral of a simple function depends on the operations $\vee$ and $\wedge$,

c) the Choquet integral of a simple function depends on the usual addition +, difference − and multiplication $\cdot$.

In a more general framework we need a pseudo-addition $\triangle$, a pseudo-multiplication $\diamond$, but in addition we need a pseudo difference $-_\triangle$, too.

It is clear that $\triangle, \diamond$ and $\triangle, \diamond, -_\triangle$ must satisfy (12) and (13), respectively. So the problem is to find appropriate big classes of operations $\triangle, \diamond, -_\triangle$ satisfying (12) and (13), and perhaps some additional requirements which are compatible with nice properties of a 'generalized integral'.

In Section 3 and 4 we give an overview of the results presented in [15] and [2], respectively.

Starting from Section 5, our results will be presented.

Let us remark, that the pseudo-addition $\triangle$ is – up to some modifications – essentially a t-conorm in all approaches. The problem of getting a so-called $\triangle$-fitting pseudo-multiplication satisfying (at first) (12) will be attacked differently and is explained in the following sections.

3. t–CONORM INTEGRAL

In [20] a fuzzy integral for continuous t-conorms as pseudo-additions is presented, whereas in [15] a fuzzy integral for Archimedean t-conorms as pseudo-additions is presented. Nevertheless we want to report on the last mentioned paper since it presents a progress in comparison to the results of the first mentioned paper (Let us add a remark: The two authors mentioned in [15] that they restrict their investigations to Archimedean t-conorms because of their decomposition theorem in [20] and because of the ordinal sum representation for a t-conorm. Exactly this small remark was essentially the reason for starting our investigations: It turns out that it is not so obvious to restrict the investigations only to Archimedean t-conorms).

In [15] so-called t-conorm systems for integration are introduced.

A t-conorm system is a quadruplet $(\triangle, \perp, \amalg, \diamond)$ consisting of three pseudo-additions $\triangle, \perp, \amalg : [0,1]^2 \to [0,1]$, which are continuous Archimedean t-conorms, and a pseudo-multiplication $\diamond : [0,1]^2 \to [0,1]$ satisfying

$$\diamond \quad \text{is increasing in both places,} \tag{14}$$

$$\diamond \quad \text{is continuous on } (0,1]^2, \tag{15}$$

$$a \diamond x = 0 \Longleftrightarrow a = 0 \quad \text{or} \quad x = 0, \tag{16}$$

$$a \triangle b < 1 \Longrightarrow (a \triangle b) \diamond x = (a \diamond x) \amalg (b \diamond x), \tag{17}$$

$$x \perp y < 1 \Longrightarrow a \diamond (x \perp y) = (a \diamond x) \amalg (a \diamond y). \tag{18}$$

Thus we see that the pseudo-multiplication satisfies the 3 conditions (14)–(16), whereas (17) and (18) express the compatibility between pseudo-addition and pseudo-multiplication.

It is interesting that a generalized left-distributivity law and a generalized right-distributivity law is required on a restricted domain. The idea behind this is that functional equations (like (17) or (18)) have rather often a bigger variety of solutions if they are required on a restricted domain and not on the whole domain (see also Theorem 5.21 in [11]).

If some trivial and/or useless solutions are ignored the authors present two essential solutions:

— Maximum type solutions:
All three t-conorms $\triangle, \perp, \amalg$ are the maximum operator $\vee$.

In this case there are a lot of pseudo-multiplications, for example, a strict t-norm (note that in this case (17) is automatically satisfied (because of (14)).

— Archimedean type solutions:
If the t-conorms $\triangle, \perp, \amalg$ have additive (increasing, continuous) generators k, g and h from $[0,1]$ into $[0,\infty]$ respectively, then

$$a \diamond x = h^{(-1)}(k(a) \cdot \bar{g}(x)) \quad a, x \in [0,1], \tag{19}$$

where $\bar{g}$ is an appropriate generator of $\perp$ and $h^{(-1)}(x) = h^{-1}(x \wedge h(1))$ is the quasi-inverse of h (see for example [12]).

Note that g and $\bar{g}$ differ only by a multiplicative constant.

The fuzzy-measures μ considered in [15] are called normal with respect to $\perp$ if μ is $\perp$-decomposable (that is, $\mu(X) = 1$, $\mu(A \cup B) = \mu(A) \perp \mu(B)$ if A and B are disjoint sets of X), and either ($\perp = \vee$) or ($\perp$ has a generator h and $h \circ \mu$ is a measure).

Finally the pseudo-difference $-_\triangle$ of an Archimedean t-conorm $\triangle$ is defined by

$$a -_\triangle b := \inf\{c \in [0,1] \mid b \triangle c \geq a\}. \tag{20}$$

Then $-_\triangle$ satisfies

$$(a -_\triangle b)\triangle b = a \vee b \qquad \text{and} \tag{21}$$
$$(a -_\triangle b)\triangle(b -_\triangle c) = a -_\triangle c \tag{22}$$

(if $a \geq b \geq c$ and $a \wedge b < 1$). Obviously (22) implies (13).

Now the t-conorm integral of the quadruplet $(\triangle, \perp, \amalg, \diamond)$ is defined in analogy to (6) (we use the notations of Section 2):

Let a simple function f be given by (1) and let the fuzzy measure μ be normal with respect to $\perp$.

Then the triple $(\triangle, \amalg, \diamond)$ will be used to define the t-conorm integral

$$\text{(F)} \qquad \int f \mathrm{d}\mu := \amalg_{i=1}^{n}((\alpha_i -_\triangle \alpha_{i-1}) \diamond \mu(F_i)). \tag{23}$$

If f is measurable, and (f_n) is an increasing sequence of simple functions which converges pointwise to f, then

$$\text{(F)} \qquad \int f \mathrm{d}\mu := \lim_{n \to \infty} (F) \int f_n \mathrm{d}\mu. \tag{24}$$

Here we need that the fuzzy measure is continuous from below, since otherwise it is not defined.

Now, in the case of maximum-type solutions the integral has the form

$$\text{(F)} \qquad \int f \mathrm{d}\mu = \sup_{\alpha \in [0,1]} [\alpha \diamond \mu(f > \alpha)], \tag{25}$$

which is a generalization of the Sugeno integral (which we get by choosing $\diamond = \wedge$). In the case of Archimedean-type solutions there exists an additive generator $\bar{g}$ such that

$$\text{(F)} \qquad \int f \mathrm{d}\mu = h^{(-1)} \left[\int k(f)\, \mathrm{d}(\bar{g} \circ \mu) \right], \tag{26}$$

where the integral on the right hand side in (26) is the Choquet integral, which reduces to the Lebesgue integral if the fuzzy measure μ is a measure.

Thus the t-conorm integral is a generalization of the Choquet integral, too (take $h = k = \bar{g} = id$ and for $\diamond$ the usual product).

Let $\perp$ be a continuous, Archimedean t-conorm with additive generator g. Moreover let μ be $\perp$-decomposable. Then there are three cases:

(S): $\perp$ is strict and $g \circ \mu$ is thus a measure,

(NSA): $\perp$ is not strict and $g \circ \mu$ is a measure,

(NSP): $\perp$ is not strict and $g \circ \mu$ is not a measure.

Defining

$$a \diamond x := g^{-1}(a \cdot g(x)), \quad a, x \in [0,1], \tag{27}$$

then $(\hat{+}, \perp, \perp, \diamond)$ (where $\hat{+}$ is the Lukasiewicz t-conorm) is a t-conorm system (with $k = id$) and we get (26), which is the Weber integral (see [22]). We remark that in the case (NSP) Weber introduced a modification of his integral which doesn't coincide with the t-conorm integral.

For further examples and properties of the t-conorm integral we refer to [9].

4. MONOTONE SET FUNCTIONS–BASED INTEGRALS

In [2] the authors take as pseudo-addition $\triangle$ a function $\triangle : [0,B]^2 \to [0,B]$, $0 < B \le \infty$ satisfying

$$\begin{array}{lll} (x\triangle y)\triangle z = x\triangle(y\triangle z) & \text{associativity,} & (28) \\ x\triangle 0 = 0\triangle x = x & \text{neutral element,} & (29) \\ \triangle \ \text{is increasing in each place} & \text{monotonicity,} & (30) \\ \triangle \ \text{is continuous in each place} & \text{continuity.} & (31) \end{array}$$

Equations (28) – (31) imply that $([0,B], \triangle)$ is an ordinal sum of Archimedean t-conorms $([a_n, b_n], S_n), n \in K_\triangle$, where $K_\triangle$ is at most countable.

Thus each S_n has an additive generator $g_n : [a_n, b_n] \to [0, \infty]$, which is continuous, strictly increasing and which satisfies $g_n(a_n) = 0$ and

$$S_n(a,b) = g_n^{(-1)}(g(a) + g(b)), \quad a,\, b \in [a_n, b_n]. \tag{32}$$

Especially, $\triangle$ is a t-conorm (and thus commutative, so that commutativity must not be required). Equivalently, we can say that $([0,B], \triangle)$ is an I-semigroup in the sense of Mostert and Shields (see [14]), and again we get that $\triangle$ is a t-conorm (see Theorem 2.43 in [11]).

As pseudo-multiplication a function $\diamond : [0,M]^2 \to [0,M], \quad 0 < M \le \infty$ is chosen so that the following holds:

$$\begin{array}{lll} \diamond \ \text{is increasing in both places,} & & (33) \\ \diamond \ \text{is left continuous,} & & (34) \\ a \diamond 0 = 0 \diamond a = 0, & \text{zero element} & (35) \\ (a\triangle b) \diamond y = (a \diamond y)\triangle(b \diamond y), & \text{left distributivity law (see (12)).} & (36) \end{array}$$

In comparison with Section 3 we see, that (15) and (16) are weakened to (34) and (35), whereas the left distributivity law is required on the whole domain. Since we have only one pseudo-addition $\triangle$, we say that $\diamond$ is a $\triangle$-fitting pseudo-multiplication (instead of: $\triangle$ and $\diamond$ satisfy the compatibility condition (36)).

Note that we have two ranges here, $[0,B]$ as range of the functions, which we want to integrate, and $[0,M]$ as range of the fuzzy measure μ. This is a little bit more general than in Section 3, but when defining an integral the most important case is $B = M$.

The idea of determining $\diamond$ is to interpret (35) – for fixed y – as a Cauchy functional equation:

$$f_y(a \triangle b) = f_y(a) \triangle f_y(b), \quad \text{where} \quad f_y(x) := x \diamond y. \tag{37}$$

Theorem 1. The function $\diamond$, defined by (37) is a $\triangle$-fitting pseudo-multiplication iff $\{f_y : y \in [0, M]\}$ is a non-decreasing system of left-continuous solutions of the Cauchy equation (37) satisfying

$$f_y(0) = 0, \; f_0(x) = 0, \; f_y(x) = \sup_{z<y} f_z(x) \tag{38}$$

for all $x \in [0, B]$ and for all $y \in (0, M]$).

Since the structure of the solutions f_y of (37) is known (see [3]) the $\triangle$-fitting pseudo-multiplication $\diamond$ can explicitly be described in specific cases, that is, if $\triangle$ as ordinal sum of Archimedean t-conorms is explicitly given.

Let us take the most simple example to point out the idea how to get the pseudo-multiplication $\diamond$:

If in the ordinal sum $n = 1$, $[a_1, b_1] = [0, B]$ and the generator $k : [0, B] \to [0, \infty]$ is an increasing bijection, that is

$$a \triangle b = k^{-1}(k(a) + k(b)),$$

then

$$f_y(x) = k^{-1}(k(x) \cdot h)\, , \; h \in [0, \infty].$$

Letting y vary again, we arrive at $k : [0, M] \to [0, \infty]$ and

$$x \diamond y = k^{-1}(k(x) \cdot h(y))\, , \; x \in [0, B]\, , \; y \in [0, M]. \tag{39}$$

for some function $h : [0, M] \to [0, \infty]$. For further, more complicated and interesting examples (which can be constructed in the same manner like in the last example) we refer to [2].

In [2] the following is needed to define an integral:

A measurable space $(X, \mathcal{A})$ together with a fuzzy measure $\mu : \mathcal{A} \to [0, M], M = \mu(X)$.

Moreover, let $\triangle$ and $\diamond$ be the above pseudo-addition and the $\triangle$-fitting pseudo-multiplication, respectively. Thus to each pair $(\triangle, \diamond)$ a fuzzy integral will be associated as follows.

If f is a simple function with the representation (1), then – like in Section 3 – the integral is defined by (using the notations of Section 1)

$$\int^{\triangle} f \diamond \mathrm{d}\mu = \triangle_{i=1}^{n}((\alpha_i -_{\triangle} \alpha_{i-1}) \diamond \mu(F_i)). \tag{40}$$

Then it is shown that this integral in (40) satisfies the following 4 properties (we use the notation $f \sim g$ to express that f and g are comonotone (here comonotone stands for common monotone (see [6]), moreover a is a real constant, $A \in \mathcal{A}$, f and g are simple):

$$\int^{\triangle} (a \cdot 1_A) \diamond \mathrm{d}\mu = a \diamond \mu(A). \tag{41}$$

$$f \le g \Longrightarrow \int^{\triangle} f \diamond \mathrm{d}\mu \le \int^{\triangle} g \diamond \mathrm{d}\mu. \tag{42}$$

$$f \sim g \Longrightarrow \int^{\triangle} (f \triangle g) \diamond \mathrm{d}\mu = \int^{\triangle} f \diamond \mathrm{d}\mu \triangle \int^{\triangle} g \diamond \mathrm{d}\mu. \tag{43}$$

$$f = (f \wedge a) \triangle (f -_{\triangle} a) \tag{44}$$

$$\Longrightarrow \int^{\triangle} f \diamond \mathrm{d}\mu = \int^{\triangle} (f \wedge a) \diamond \mathrm{d}\mu \triangle \int^{\triangle} (f -_{\triangle} a) \diamond \mathrm{d}\mu.$$

Note that (44) is the so-called 'horizontal $\triangle$-additivity' which is a weaker form of comonotone $\triangle$-additivity (42) because the two functions $f \wedge a$ and $f -_{\triangle} a$ are comonotone.

Then, supposing that μ is continuous from below, the integral can be defined on $\mathcal{F} = \{f : X \to [0, B] : f \text{ is } \mathcal{A}\text{-measurable}\}$:

$$f \in \mathcal{F} \Longrightarrow \int^{\triangle} f \diamond \mathrm{d}\mu := \sup \left\{ \int^{\triangle} s \diamond \mathrm{d}\mu : s \le f,\ s \text{ simple} \right\}.$$

For the example after Theorem 1 we arrive at the following integral

$$\int^{\triangle} s \diamond \mathrm{d}\mu := k^{-1} \left[\int k(f) \mathrm{d}(h \circ \mu) \right], \tag{45}$$

where the integral on the right hand side is the Choquet integral.

There are a lot of nice and interesting examples in [2]:

They show for example that this integral is a generalization of the Sugeno integral, the Choquet integral, and the Weber integral, too.

In [2] we also find the following nice characterization theorem:

Theorem 2. Let $I : \mathcal{F} \to [0, B]$ be a functional. Moreover let $\mu : \mathcal{A} \to [0, B]$ be a fuzzy measure which is continuous from below.

Then $I(f) = \int^{\triangle} f \diamond \mathrm{d}\mu$ for all $f \in \mathcal{F}$ iff

$$I(a \cdot 1_A) = a \diamond I(1_A),\ a \in [0, B],\ A \in \mathcal{A},$$
$$f = (f \wedge a) \triangle (f -_{\triangle} a) \Longrightarrow I(f \wedge a) \triangle I(f -_{\triangle} a),\ a \in [0, B],$$
I is continuous from below.

Note that the Sugeno integral and the Choquet integral are characterized by the three properties of Theorem 2 (take $\triangle = \vee$, $\diamond = \wedge$ and $\triangle = \hat{+}, \diamond = \cdot$).

For further properties and extension problems we refer to [2].

If we compare the results of Section 3 with the results in Section 4 we see:

(I) If the pseudo-addition $\triangle$ is $\triangle = \vee$ then we can take as pseudo-multiplication $\diamond$ every function $\diamond$ satisfying the minimum requirements for a pseudo-multiplication. This result holds in Section 3 and Section 4.

In [2] it is thus proposed to take as appropriate pseudo-multiplication a left continuous t-norm or a left continuous uni-norm (see for example [11]).

We remark that for Archimedean t-norms T the left-continuity of T is equivalent to the continuity of T (see [11], p. 30).

(II) If the pseudo-addition is a continuous, Archimedean t-conorm (that is, the ordinal sum representation for continuous t-conorms is degenerated to one interval) then the pseudo-multiplication has the form (19) in Section 3 whereas the pseudo-multiplication in Section 4 is given by (39).

The result in Section 3 is more general because of working with two pseudo-additions and by using restricted domains for the distributivity laws.

(III) In Section 4 pseudo-additions are always continuous, but they are no longer restricted to be Archimedean t-conorms like in Section 3. The result is: If the ordinal sum representation of the pseudo-addition $\triangle$ is given then (if possible) the fitting $\triangle$-pseudo-multiplication can be calculated by using Theorem 1.

We should mention that Sugeno and Murofushi have also considered arbitrary continuous t-conorms as pseudo-additions (see [20]) whereas their pseudo-multiplications satisfy (14) – (16), but in addition they have a left unit (which is a strong condition, as we will still see).

Since in Section 4 it can happen that there is no $\triangle$-fitting pseudo-multiplication or the only $\triangle$-fitting pseudo-multiplication is the usual multiplication the question is how to improve this situation. In our approach we can give an answer:

We require distributivity laws on appropriate restricted domains.

So the process of integration consists of three steps:

(a) Introduce a generalized distributivity law on an appropriate restricted domain to get a big variety of possible pseudo-additions and pseudo-multiplications.

(b) Determine possible pseudo-additions and possible pseudo-multiplications and show that pseudo-differences are compatible with the introduced distributivity law.

(c) Using (a) and (b) define a general integral which satisfies 'nice' properties.

These three steps correspond to the three parts of our paper:

In Multiplication, Distributivity and Fuzzy–Integral I – III we investigate (a) – (c), respectively.

We remark that when doing this approach, which is mainly contained in the Ph.D. Thesis of Siedekum (see [19]), we were not aware of the paper [2].

5. DISTRIBUTIVITY

From now on we want to report on our main results.

Let us start with pseudo-additions in the sense of Section 4, but now defined on arbitrary intervals (which will be done to formulate structure theorems for pseudo-additions, pseudo-multiplications and pseudo-differences on arbitrary intervals, independent upon applications to integration theory).

Definition 1. Let $-\infty \leq A < B \leq \infty$. A mapping $\triangle : [A,B]^2 \to [A,B]$ is called a pseudo-addition on $[A,B]$ iff

$(x\triangle y)\triangle z = x\triangle(y\triangle z)$	associativity,
$x\triangle A = A\triangle x = x$	neutral element,
$\triangle$ is increasing in each place	monotonicity,
$\triangle$ is continuous in each place	continuity.

It is again important that $([A,B],\triangle)$ is an ordinal sum of Archimedean t-conorms $([a_n,b_n],S_n)$, $n \in K_\triangle$ (cf. Chapter 4).

In the following (throughout the rest of the paper) we consider 3 pseudo-additions $\triangle, \perp, \amalg : [A,B]^2 \to [A,B]$ with generator sets

$$\{k_m : [a_m^\triangle, b_m^\triangle] \to [0,\infty] : m \in K_\triangle\}, \tag{46}$$
$$\{g_k : [a_k^\perp, b_k^\perp] \to [0,\infty] : k \in K_\perp\}, \tag{47}$$
$$\{h_l : [a_l^\amalg, b_l^\amalg] \to [0,\infty] : l \in K_\amalg\}, \tag{48}$$

respectively (see (32) for the properties of k_m, g_k and h_l).

If $\triangle$ is especially an Archimedean pseudo-addition with generator k then $\triangle_{i=1}^n u_i := u_1\triangle\ldots\triangle u_n$, $u_i \in [A,B]$, $1 \leq i \leq n$, has the representation

$$\triangle_{i=1}^n u_i = k^{(-1)}\left(\sum_{i=1}^n k(u_i)\right). \tag{49}$$

Moreover we remember the following two statements:

$$\bigwedge_{u\in(A,B]} \lim_{n\to\infty} \triangle_{i=1}^n u = B, \tag{50}$$

$$\triangle \text{ strict} \Leftrightarrow k(B) = \infty \Leftrightarrow \bigwedge_{u\in(A,B)} \bigwedge_{n\in\mathbb{N}} \triangle_{i=1}^n u < B \Leftrightarrow \bigvee_{u\in(A,B)} \bigwedge_{n\in\mathbb{N}} \triangle_{i=1}^n u < B. \tag{51}$$

Only the last equivalence seems to be new. (Let $\bigvee_{u\in(A,B)} \bigwedge_{n\in\mathbb{N}} \triangle_{i=1}^n < B$ be valid, but we assume that $k(B) < \infty$. Now for arbitrary $u \in (A,B)$ we get from $k(u) > 0$ that there is $n \in \mathbb{N}$ such that $nk(u) \geq k(B)$. Thus we get the contradiction $\triangle_{i=1}^n = k^{(-1)}(nk(u)) \geq k^{(-1)}k(B) = B$).

In all theorems and definitions we assume that $\triangle$ and/or $\perp$ and/or $\amalg$ are pseudo-additions satisfying (50) – (52).

We now give the definition of a pseudo-multiplication.

Definition 2. A mapping $\diamond : [A,B]^2 \to [A,B]$, $-\infty \leq A < B \leq \infty$ is called a pseudo-multiplication iff

$$\diamond \quad \text{is increasing in both places,} \tag{52}$$

$$(\cdot)\diamond x \quad \text{and} \quad a \diamond (\cdot) \quad \text{are continuous on}\,(A,B], \quad x,a \in (A,B) \tag{53}$$

$$(\cdot)\diamond B \quad \text{or} \quad B \diamond (\cdot) \quad \text{is continuous in } B. \tag{54}$$

This means that a pseudo-multiplication is only an increasing function of two variables which satisfies a weak regularity assumption.

Let us point out the following consequences of Definition 2, which are used rather often in proofs (and to which we don't refer) and which are easy to prove (by representing the boundary point B through a monotone increasing sequence of points of (A,B)):

$$(\cdot)\diamond x \text{ and } x \diamond (\cdot) \text{ are left-continuous on } (A,B] \text{ for all } x \in (A,B],$$

$$(\ (\cdot)\diamond B \text{ is continuous in B}) \text{ and } (B \diamond (\cdot) \text{ is continuous in B}).$$

The problem of characterizing pseudo-multiplications $\diamond$ is (from our point of view) to add to the minimal requirements (56) – (58) a list of desiderata for pseudo-multiplications $\diamond$, so that we get a characterization with a minimal number of 'essential' and 'natural' properties. As certain properties will occur repeatedly in our results, we present the following list of additional properties for pseudo-multiplications.

(CLB)	$(\cdot)\diamond B$	is continuous on (A,B]	(55)
(CRB)	$B\diamond(\cdot)$	is continuous on (A,B]	(56)
(CLZ)	$(\cdot)\diamond x$	is continuous in A, continuity in zero, first place	(57)
(CRZ)	$a\diamond(\cdot)$	is continuous in A, continuity in zero, second place	(58)
(C)	$A\diamond a = a\diamond A = A$	zero element	(59)
(RU)	$a\diamond e = a$	for some $e \in (A,B]$, right unit	(60)
(LU)	$\tilde{e}\diamond a = a$	for some $\tilde{e} \in (A,B]$, left unit.	(61)

Let us briefly comment these properties (C, L, R, Z, U stand for continuity, left, right, zero and unit, respectively).

We first remark that we require continuity (instead of left-continuity in comparison with Section 3) because a main tool in our considerations is the intermediate value theorem which is not valid for one-sided continuity.

The conditions (CLB) and (CRB) seem to be rather naturally.

The conditions (CLZ) and (CRZ) will be assumed often, but for many statements these conditions are not necessary (but simplify the proofs).

The condition (Z) is necessary to describe pseudo-multiplications on $\{A\}\times[A,B]\cup[A,B]\times\{A\}$.

If (Z) is valid we get for each unit immediately that $e \in (A,B]$ and $\tilde{e} \in (A,B]$.

The asymmetry of our assumptions will be continued also in the following definition.

Definition 3. Let $\diamond$ be a pseudo-multiplication.

(a) $\diamond$ satisfies the left distributivity law with respect to $(\triangle, \amalg)$ iff

$$\text{(DL)} \quad \bigwedge_{a,b,x\in[A,B]} (a\triangle b)\diamond y = (a\diamond y) \amalg (b\diamond y), \text{ left distributivity law.} \tag{62}$$

(b) $\diamond$ satisfies the right distributivity law with respect to $(\perp, \amalg)$ iff

$$\text{(DR)} \quad \bigwedge_{a,b,x\in[A,B]} a\diamond(x\perp y) = (a\diamond x) \amalg (a\diamond y), \text{ right distributivity law.} \tag{63}$$

(c) A pseudo-multiplication $\diamond$ satisfies the left-right distributivity law with respect to $(\triangle, \perp, \amalg)$ iff $\diamond$ satisfies the left distributivity law (DL) and the right distributivity law (DR).

Let us mention that we always get two results because of the asymmetry of our assumptions: A 'left distributivity version' and a 'right distributivity version'. Often we will only mention one result, the other one follows obviously in the same manner.

We present now the following structure theorem for pseudo-additions which is surprising since the ordinal sum structure simplifies dramatically. Moreover there are no nonstrict Archimedean t-conorms in the ordinal sum.

Theorem 3. Let $\triangle, \perp$ and $\amalg$ be pseudo-additions and let $\diamond$ be a pseudo-multiplication satisfying (DL), (CRB) and (RU).

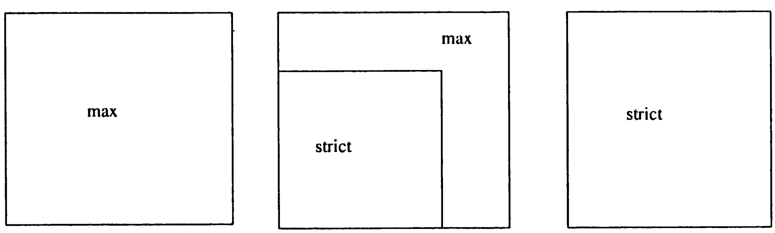
(a) If $\diamond$ satisfies (DR) and if e is idempotent with respect to $\perp$ then $\triangle = \amalg = \vee$.

(b) If the right unit (RU) is also a left unit (LU), then there are two possibilities:

— $\triangle = \amalg = \vee$ and e is idempotent with respect to $\triangle$, or

— $\triangle = \amalg$, $|K_\triangle| = 1$, $a_1 = A$, $e \in (A, b_1)$, $\triangle\rceil_{[A,b_1]^2}$ is strict, $\bigwedge_{a,x\in[A,b_1]} a\diamond x \in [A, b_1]$ (multiplication is compatible with the structure of $\triangle$).

If in addition $\diamond$ satisfies (Z) and (CLZ) then $\triangle = \amalg$ is a strict t-conorm on $[A, B]^2$.



We remark that (also in the following results) in the case $\triangle = \amalg$ we denote the common generator set by $\{h_l : [a_l, b_l] \to [0, \infty] | l \in K\}$ where $K = K_\triangle = K_\amalg$.

The result in Theorem 3 shows that distributivity laws and unit elements are strong conditions:

A right unit and the left distributivity law (DL) imply the equality of the pseudo-additions under consideration.

If the right unit and left unit coincide, then the corresponding ordinal sum consists of at most one Archimedean t-conorm, which must be strict (see the above 3 pictures).

To prove Theorem 3 we first prove the following Lemma which is also of independent interest.

Here, and in the following, we say that 'a $\triangle$-idempotent' instead of 'a is idempotent with respect to $\triangle$.'

Lemma 1. Let $\triangle, \perp$ and $\amalg$ be pseudo-additions and let $\diamond$ be a pseudo-multiplication satisfying (DL), (CRB) and (RU). Then we have:

(a) $\bigwedge\limits_{a,x\in[A,B]} (a\ \triangle-\text{idempotent} \Rightarrow a\diamond x\ \amalg-\text{idempotent});$
$a \in (A,B]\ \triangle - \text{idempotent},\ x_0 \in (A,B],\ \bigvee_{l\in K_\amalg} a\diamond x_0 = a_l^\amalg \Rightarrow \bigwedge_{x\in[x_0,B]} a\diamond x_0 = a_l^\amalg;\ a \in (A,B]\ \triangle - \text{idempotent},\ x_0 \in (A,B], \bigvee_{l\in K_\amalg} a\diamond x_0 = b_l^\amalg \Rightarrow \bigwedge_{x\in(A,x_0]} a\diamond x_0 = b_l^\amalg.$

(b) $\triangle = \amalg$ (so that $K_\triangle = K_\amalg = K,\ K = \mathbb{N}$ or $K = \{1,2,\ldots,M\}$, and we put $b_0 = A$ and (if K is finite) $a_{M+1} = B$);
$a\ \triangle - \text{idempotent} \Leftrightarrow a\ \amalg-\text{idempotent} \Leftrightarrow a$ idempotent.

(c) $\bigwedge\limits_{x\in(A,B]} \bigwedge\limits_{l\in K} [x \in (A,e] \vee (a_l > A) \vee (\mathrm{Z}) \Rightarrow a_l \diamond x = a_l].$

(d) $\bigwedge\limits_{x\in(A,B]} \bigwedge\limits_{l\in K} [x \in (A,e] \vee (a_l > A) \vee ((\mathrm{Z}) \wedge (\mathrm{CLZ})) \Rightarrow b_l \diamond x = b_l].$

(e) $\bigwedge\limits_{x\in(A,B]} \bigwedge\limits_{l\in K} \bigwedge\limits_{a\in[a_l,b_l]} [x \in (A,e] \vee (a_l > A) \vee ((\mathrm{Z}) \wedge (\mathrm{CLZ}) \Rightarrow a\diamond x \in [a_l, b_l]]$

(f) $\bigwedge\limits_{x\in(A,B]} \bigwedge\limits_{l\in K} \bigwedge\limits_{a\in[a_l,b_l]} a\diamond x \in [a_l, a_{l+1}].$

(g) $\bigwedge\limits_{x\in(A,B]} \bigwedge\limits_{l\in K} \bigwedge\limits_{a\in[b_l,a_{l+1}]} a\diamond x \in [b_l, a_{l+1}].$

(h) $\bigwedge\limits_{l\in K} \left[\left(\bigvee\limits_{a\in(a_l,b_l)} \bigvee\limits_{x\in(a,B]} (a\triangle a)\diamond x \le a\right) \Rightarrow \triangle|_{[a,b]^2} \text{ strict}\right].$

(i) $\bigwedge\limits_{l\in K} \left[\left(\bigvee\limits_{a\in(a_l,b_l)} (a\diamond A = A) \wedge a\diamond(\cdot) \text{ continuous in } A\right) \Rightarrow \triangle|_{[a,b]^2} \text{ strict}\right].$

(j) (Z) $\wedge$ (CRZ) $\Rightarrow \bigwedge_{l \in K} \triangle|_{[a_l,b_l]^2}$ strict.

P r o o f o f L e m m a 1. We remark that for the proof of the 3 statements of (a) the assumption (RU) is not needed.

(a) Using (DL) we get $(a \diamond x) \amalg (a \diamond x) = (a \triangle a) \diamond x = a \diamond x$.

To prove the second statement of (a) we assume: $\bigvee_{x \in (x_0,B]} a \diamond x \neq a_l^{\amalg}$. If we show

$$\bigvee_{x' \in (x_0,B]} a \diamond x' \in (a_l^{\amalg}, b_l^{\amalg}), \tag{64}$$

then we are done, since (68) contradicts that $a \diamond x'$ is $\amalg$-idempotent (by (a), since a is $\triangle$-idempotent).

To show (68) we first consider the case that $a \diamond x < b_l^{\amalg}$. Then we can choose $x' = x$ because of $a_l^{\amalg} = a \diamond x_0 < a \diamond x < b_l^{\amalg}$.

If $a \diamond x \geq b_l^{\amalg}$ then $a \diamond x \geq b_l^{\amalg} > \frac{a_l^{\amalg} + b_l^{\amalg}}{2} > a_l^{\amalg} = a \diamond x_0$. By the intermediate value theorem there is $x' \in (x_0, x)$ such that $a \diamond x' = \frac{a_l^{\amalg} + b_l^{\amalg}}{2} \in (a_l^{\amalg}, b_l^{\amalg})$.

The third statement of (a) can be proven in the same manner like (b).

(b) If $a, b \in (A, B]$ then $a \triangle b = (a \triangle b) \diamond e = (a \diamond e) \amalg (b \diamond e) = a \amalg b$.

If $(a = A) \vee (b = A)$ then twice application of the theorem on ordinal sums yields $a \triangle B = a \vee b = a \amalg b$.

To prove (c) and (d), which are the basis for the statements (f) and (g), we first prove 4 partial results (I) – (IV) (here we make use of (54)):

(I) $\bigwedge_{x \in [e,B]} \bigwedge_{l \in K} [(a_l > A) \vee (Z) \Rightarrow a_l \diamond x = a_l]$.

Proof of (I): In the case $(a_l > A)$ we apply the second statement of (a) with $a := a_l, x_0 := e$. The case $(a_l = A) \wedge (Z)$ is obvious.

(II) $\bigwedge_{x \in (A,e]} \bigwedge_{l \in K} b_l \diamond x = b_l$.

To prove (II) use the third statement of (a) with $a := b_l, x_0 := e$.

(III) $\bigwedge_{x \in [e,B]} \bigwedge_{l \in K} [(a_l > A) \vee ((Z) \wedge (CLZ)) \Rightarrow b_l \diamond x = b_l]$.

First case: Let $x < B$. Using (I) we get $b_l \diamond x \geq b_l \diamond e = b_l > a_l = a_l \diamond x$.

Applying (CLZ) for $a_l = A$ and the intermediate value theorem we obtain an element $a \in (a_l, b_l]$ such that $a \diamond x = b_l$. This implies $b_l \diamond x = [\lim_{n\to\infty}(\triangle_{i=1}^n a)] \diamond x = \lim_{n\to\infty}[(\triangle_{i=1}^n a) \diamond x] = \lim_{n\to\infty}[\amalg_{i=1}^n (a \diamond x)] = \lim_{n\to\infty}[\amalg_{i=1}^n b_l] = b_l$.

In the last two steps of the calculation we made use of (DL) and the theorem on ordinal sums.

Second case: Let $x = B$. We simply apply the first case to arrive at $b_l \diamond B = b_l \diamond \sup_{x<B} x = \sup_{x<B}(b_l \diamond x) = b_l$.

(IV) $\bigwedge_{x\in(A,e]} \bigwedge_{l\in K} a_l \diamond x = a_l.$

Since $A \diamond x \le A \diamond e \le A$ we may suppose w.l.o.g. $a_l > A$.
Assume that (IV) is not true, then we have $\bigvee_{x\in(A,e)} \bigvee_{l\in K} a_l \diamond x \neq a_l$.
Using (II) we get $a_l \diamond x < a_l \diamond e = a_l < b_l = b_l \diamond x$, and again there exists $a \in (a_l, b_l)$ with $a \diamond x = a_l$.
We use again (DL) and the theorem on ordinal sums to obtain
$b_l \diamond x = [\lim_{n\to\infty}(\triangle_{i=1}^{n} a)] \diamond x = \lim_{n\to\infty}[(\triangle_{i=1}^{n} a) \diamond x] = \lim_{n\to\infty}[\amalg_{i=1}^{n}(a \diamond x)] = \lim_{n\to\infty}[\amalg_{i=1}^{n} a_l] = a_l$, which is a contradiction to (II).

Now (I) and (IV) imply (c) whereas (II) and (III) imply (d).

To prove (e) to (h) we prove further statements (V) – (VIII).

(V) $\bigwedge_{x\in(A,b]} \bigwedge_{l\in K} \bigwedge_{a\in[a_l,b_l]} a \diamond x \ge a_l.$

(V) is valid since w.l.o.g. we may assume $a_l > A$ (otherwise (V) is trivial) so that $a \diamond x \ge a_l \diamond x = a_l$ (here we have used (c)).

(VI) $\bigwedge_{x\in(A,b]} \bigwedge_{l\in K} \bigwedge_{a\in[a_l,b_l]} [x \in (A,e] \vee (a_l > A) \vee ((\mathrm{Z}) \wedge (\mathrm{CLZ})) \Rightarrow a \diamond x \le b_l].$

Proof for (VI): Using (d) we obtain $a \diamond x \le b_l \diamond x = b_l$.

(VII) $\bigwedge_{x\in(A,b]} \bigwedge_{l\in K} \bigwedge_{a\in[a_l,a_{l+1}]} a \diamond x \le a_{l+1}.$

By (c) we have $a \diamond x \le a_{l+1} \diamond x \le a_{l+1}$ (also if K is finite and if $a_{l+1} = B$).

(VIII) If $x \in [e, B]$ then $a \diamond x \ge b_l \diamond e = b_l$.
If $x \in (A, e]$ then $a \diamond x \ge b_l \diamond x = b_l$ (see (II)).

Using (V) and (VI) we get (e), using (V) and (VII) we arrive at (f), and (VII) and (VIII) imply (g).

To prove (h) we assume that $\triangle|_{[a_l,b_l]^2}$ is not strict. Then (55) yields

$$\bigwedge_{u\in(a_l,b_l)} \bigvee_{n\in N} \triangle_{i=1}^{n} u = b_l.$$

We define $m := min\{n \in \mathbb{N} \mid \triangle_{i=1}^{n} a = b_l\}$ and $\lceil r \rceil := \inf\{n \in \mathbb{Z} | n \ge r\}$, $r \in \mathbb{R}$.
Since $a < b_l$ we have $m \ge 2$ and $\lceil \frac{m}{2} \rceil < m$. Using (d) we get the contradiction

$$b_l \le b_l \diamond x = [\triangle_{i=1}^{m} a] \diamond x \le [\triangle_{i=1}^{2\lceil \frac{m}{2} \rceil} a] \diamond x = \amalg_{i=1}^{\lceil \frac{m}{2} \rceil}[(a \triangle a) \diamond x] \le \amalg_{i=1}^{\lceil \frac{m}{2} \rceil} a < b_l.$$

(i), (j) We show that the suppositions of (h) are satisfied: Choose $a' = h_l^{-1}(\frac{h_l(a)}{2}) \in (a_l, b_l)$. Because of $a \diamond A = A$ and $a \diamond (\cdot)$ is continuous in A there is $x \in (A, B]$ such that $a' \ge a \diamond x = (a' \triangle a') \diamond x$. Finally (j) follows immediately from (i).

Thus Lemma 1 is proven. □

Now we prove Theorem 3.

Proof of Theorem 3. By Lemma 1 (b) we have $\triangle = \amalg$.

Proof of (a). For arbitrary $a \in (A, B]$ we get $a \amalg a = (a \diamond e) \amalg (a \diamond e) = a \diamond (e \perp e) = a \diamond e = a$. Since also $A \amalg A = A$, each element $a \in [A, B]$ is $\amalg$-idempotent, and thus $\amalg = \vee$.

Proof of (b). Note that (DR) now is not supposed, but (LU) and (RU) imply $\tilde{e} = \tilde{e} \diamond e = e$. We distinguish two cases:

Case A: e is $\triangle$-idempotent. Then for all $a \in (A, B]$ we obtain $a \amalg a = (a \diamond e) \amalg (a \diamond e) = (e \triangle e) \diamond a = e \diamond a = a$. Like in (a) we get $\amalg = \vee$.

Case B: e is not $\triangle$-idempotent. Then (by the theorem on ordinal sums) there exists $L \in K$ such that $e \in (a_l, b_l)$. Thus Lemma 1 (f) implies for all $x \in (A, B] : x = e \diamond x \in [a_L, a_{L+1}]$. But this means $a_L = A$ and $a_{L+1} = B$, so that $K = \{L\}$.

If in addition $\diamond$ satisfies (Z) and (CLZ) then Lemma 1 (e) yields for all $x \in (A, B] : x = e \diamond x \in [a_L, b_L]$, so that $b_L = B$.

Thus we have shown: $|K| = 1, a_1 = A, e \in (A, b_1)$, moreover $b_1 = B$ if (Z) and (CLZ).

To prove the final statement, let $(a_n)_{n\in\mathbb{N}} \subset (A, B]$ with $\lim_{n\to\infty} a_n = A$. Then $A \leq e \diamond A \leq \lim_{n\to\infty}(e \diamond a_n) = \lim_{n\to\infty} a_n = A$. Thus $(e \diamond A = A)$ and $e \diamond (\cdot)$ is continuous in A so that Lemma 1 (i) shows that $\triangle|_{A,b_1]^2}$ is strict.

Finally $\bigwedge_{a,x\in[A,b_1]} a \diamond x \leq b_1 \diamond b_1 = b_1$ so that $a \diamond x \in [A, b_1]$ for all $a, x \in [A, b_1]$.

Thus Theorem 3 is proven. □

Let us remark that the weak but complicated looking assumption in Lemma 1 cannot be omitted. For an example we refer to [19].

Example 1. If we take for $\diamond$ the classical multiplication and for $\triangle = \amalg$ the classical addition (and restrict both operations to a finite interval) then the left distributivity law is not satisfied: Let $A = 0$ and $1 \leq B < \infty$, and define:

$$a \triangle b = a \amalg b = (a + b) \wedge B, \quad a \diamond x = (a \cdot x) \wedge B.$$

Then (57) is not satisfied. Take $a = b = \frac{3}{4}B$ and $x = \frac{1}{2}$ to get:

$$(a \triangle b) \diamond x = B \diamond \frac{1}{2} = \frac{1}{2}B < \frac{3}{4}B = \frac{3}{8}B \amalg \frac{3}{8}B = (a \diamond x) \amalg (b \diamond x).$$

This example shows how much the distributivity laws restrict the choice of possible pseudo-additions. Thus it seems reasonable to restrict the domain of the distributivity laws.

To do this we define for each pseudo-addition $\triangle, \perp$ and $\amalg$ the sets (see (50) – (52)):

$$D_{\triangle} = \{b_m^{\triangle} : m \in K_{\triangle}\}, \quad (65)$$
$$D_{\perp} = \{b_k^{\perp} : k \in K_{\perp}\}, \quad (66)$$
$$D_{\amalg} = \{b_l^{\amalg} : l \in K_{\amalg}\}, \quad (67)$$

respectively.

These notions will be used in the rest of the paper without any further reference.

Definition 4. (a) A pseudo multiplication $\diamond$ satisfies the weak left distributivity law with respect to $(\triangle, \amalg)$ iff

$$(\mathrm{DL}^*)\, a \triangle b \notin D_{\triangle} \Longrightarrow (a \triangle b) \diamond x = (a \diamond x) \amalg (b \diamond x) \text{ for all } a, b, x \in (A, B]. \quad (68)$$

(b) A pseudo multiplication $\diamond$ satisfies the weak right distributivity law with respect to $(\perp, \amalg)$ iff

$$(\mathrm{DR}^*)\, x \perp y \notin D_{\perp} \Longrightarrow a \diamond (x \perp y) = (a \diamond x) \amalg (a \diamond y) \text{ for all } x, y, a \in (A, B]. (69)$$

(c) A pseudo-multiplication $\diamond$ satisfies the weak left-right distributivity law with respect to $(\triangle, \perp, \amalg)$ iff $\diamond$ satisfies the weak left distributivity law (DL*) and the weak right distributivity law (DR*).

Note that (72) and (73) mean that we omit in the domain of validity of the distributivity laws all 'right boundary points of Archimedean intervals' in the ordinal sum of $\triangle$ and $\perp$, respectively. This condition seems to be much weaker than the condition

$$a \triangle b < B \Longrightarrow (a \triangle b) \diamond x = (a \diamond x) \amalg (b \diamond x) \quad \text{for all} \quad a, b, x \in (A, B]. \quad (70)$$

which could be considered as natural generalization of (17). Indeed, let us show that (74) implies (DL*).

The case $\bigvee_{m \in K_{\triangle}} b_m^{\triangle} = B$ is trivial. So let $\bigwedge_{m \in K_{\triangle}} b_m^{\triangle} < B$. Then there is a sequence $(a_n) \subset (A, B)$ satisfying $a_n \uparrow B$ and a_n is $\triangle$-idempotent for all $n \in \mathbb{N}$. Now let $a, b, x \in (A, B]$. Thus we get $(a \wedge a_n) \triangle (b \wedge a_n) \leq a_n \triangle a_n = a_n < B$ and

$$\begin{aligned}
(a \triangle b) \diamond x &= \left(\left[\sup_{n \in \mathbb{N}} (a \wedge a_n) \right] \triangle \left[\sup_{n \in \mathbb{N}} (b \wedge a_n) \right] \right) \diamond x \\
&= \left(\sup_{n \in \mathbb{N}} [(a \wedge a_n) \triangle (b \wedge a_n)] \right) \diamond x = \sup_{n \in \mathbb{N}} ([(a \wedge a_n) \triangle (b \wedge a_n)] \diamond x) \\
&= \sup_{n \in \mathbb{N}} ([(a \wedge a_n) \diamond x] \amalg [(b \wedge a_n) \diamond x]) \\
&= \left(\sup_{n \in \mathbb{N}} [(a \wedge a_n) \diamond x] \right) \amalg \left(\sup_{n \in \mathbb{N}} [(b \wedge a_n) \diamond x] \right) \\
&= \left(\left[\sup_{n \in \mathbb{N}} (a \wedge a_n) \right] \diamond x \right) \amalg \left(\left[\sup_{n \in \mathbb{N}} (b \wedge a_n) \right] \diamond x \right) = (a \diamond x) \amalg (b \diamond x).
\end{aligned}$$

Moreover, in (72) and in (73) we can omit also the left boundary points of Archimedean intervals in the ordinal sum of $\triangle$ (and $\perp$), respectively. Thus (DL*) is equivalently to the following condition (DL**):

$$\text{(DL}^{**}\text{)} \quad a\triangle b \notin D_\triangle \cup \{a_m^\triangle | m \in K_\triangle\} \Longrightarrow (a\triangle b) \diamond x = (a \diamond x) \amalg (b \diamond x) \qquad (71)$$
$$\text{for all} \quad a, b \in (A, B]\,,\, x \in (A, B).$$

To prove this we have only to show that (DL**) $\Rightarrow$ (DL*).

We consider first the case $x \in (A, B)$. Moreover, let $a, b \in (A, B], a\triangle b = a_m^\triangle \notin D_\triangle$. Then obviously $a_m^\triangle > A$ and $a, b \le a_m^\triangle$ $(a = a\triangle A \le a\triangle b = a_m^\triangle$, analogously $b \le a_m^\triangle)$. But now we obtain $(a = a_m^\triangle) \vee (b = a_m^\triangle)$ (otherwise $a, b < a_m^\triangle$ and because of $a_m^\triangle \neq b_{m-1}^\triangle$ there is an element $c \in (a \vee b, a_m^\triangle)$ which is $\triangle$-idempotent. This gives the contradiction $a\triangle b \le c\triangle c = c < a_m^\triangle$).

Because of $(a = a_m^\triangle) \vee (b = a_m^\triangle)$ let w.l.o.g. $b = a_m^\triangle$ and choose a sequence $(c_n) \subset (a_m^\triangle, b_m^\triangle)$ with $c_n \downarrow b$. Using $a \le a_m^\triangle$, the theorem on ordinal sums and (DL**) we arrive at $a\triangle c_n = (a \vee c_n) = c_n \in (a_m^\triangle, b_m^\triangle)$, and finally at
$(a\triangle b) \diamond x = \lim_{n\to\infty}[(a\triangle c_n) \diamond x] = \lim_{n\to\infty}[(a \diamond x) \amalg (c_n \diamond x)] = (a \diamond x) \amalg (b \diamond x)$.

Now we consider the case $x = B$ in which we obtain (using the case $x \in (A, B)$) $(a\triangle b)\diamond B = (a\triangle b)\diamond\lim_{x\to B-} x = \lim_{x\to B-}[(a\triangle b)\diamond x] = \lim_{x\to B-}[((a\diamond x)\amalg(b\diamond x)] = [\lim_{x\to B-}(a \diamond x)] \amalg [\lim_{x\to B-}(b \diamond x)] = (a \diamond B) \amalg (b \diamond B]$.

In the next remark we point out a connection between (DL) and (DL*).

If (CRB), (RU), (Z) and (CRZ) are satisfied then the following equivalence holds:

$$\text{(DL)} \Longleftrightarrow \text{(DL}^*\text{)} \wedge \bigwedge_{m\in K_\triangle} \triangle|_{[a_m^\triangle, b_m^\triangle]^2} \quad \text{is strict.} \qquad (72)$$

The one implication is Lemma 1 (j).

To prove the other implication, we distinguish 5 cases (see (66)):

Case 1: $y = A : (a\triangle b) \diamond A = A = (a \diamond A) \amalg (b \diamond A)$.

Case 2: $a = A : (A\triangle b) \diamond y = b \diamond y = (A \diamond y) \amalg (b \diamond y)$.

Case 3: $b = A$: Analogous to Case 2.

Case 4. $a, b, y \in (A, B] \wedge a\triangle b \notin D_\triangle$: (DL*).

Case 5: $a, b, y \in (A, B] \wedge a\triangle b \in D_\triangle$: Thus let $a\triangle b = b_m^\triangle, m \in K_\triangle$ where w.l.o.g. $a \le b$ (since $\triangle, \amalg$ are commutative).

Then $a, b \le b_m^\triangle$ (for example, $a = a\triangle A \le a\triangle b = b_m^\triangle$). Using the hypothesis of strictness we get

$$k_m^{-1}(\infty) = b_m^\triangle = a\triangle b \le (a_m^\triangle \vee b)\triangle(a_m^\triangle \vee b) = k_m^{-1}(2k_m(a_m^\triangle \vee b)$$

which implies $2k_m(a_m^\triangle \vee b) = \infty$, that is $(a_m^\triangle \vee b) = b_m^\triangle$, or $b = b_m^\triangle$.

Now we choose a sequence $(b_n) \subset (a_m^\triangle, b_m^\triangle)$ with $b_n \uparrow b$ and show, that $(a_n \wedge b_n)\triangle b_n \notin D_\triangle$ for all $n \in \mathbb{N}$: $a_m^\triangle < b_n = A\triangle b_n \le (a \wedge b_n)\triangle b_n \le b_n\triangle b_n = k_m^{-1}(2k_m(b_n)) < k_m^{-1}(\infty] = b_m^\triangle$.

To prove (DL) we finally consider two subcases. If $y \in (A, B)$ then (DL*) implies

$$\begin{aligned}(a\triangle b)\diamond y &= \left(\lim_{n\to\infty}[(a\wedge b_n)\triangle b_n]\right)\diamond y = \lim_{n\to\infty}([(a\wedge b_n)\dot{\triangle} b_n]\diamond x)\\ &= \lim_{n\to\infty}([(a\wedge b_n)\diamond y]\amalg[b_n\diamond y]) = \left(\lim_{n\to\infty}[(a\wedge b_n)\diamond y]\right)\amalg\left(\lim_{n\to\infty}[b_n\diamond y]\right)\\ &= (a\diamond y)\amalg(b\diamond y).\end{aligned}$$

If $y = B$ then the case $y \in (A, B)$ leads to

$$\begin{aligned}(a\triangle b)\diamond B &= \lim_{y\to B-}[(a\triangle b)\diamond y] = \lim_{y\to B-}[(a\diamond y)\amalg(b\diamond y)]\\ &= \left[\lim_{y\to B-}(a\diamond y)\right]\amalg\left[\lim_{y\to B-}(b\diamond y)\right] = (a\diamond B)\amalg(b\diamond B]).\end{aligned}$$

Thus (76) is proven.

We remark that in the above case 5 the following result is included:

$$\text{(DL}^*\text{)}\quad \text{implies}:$$

$$\bigwedge_{m\in K_\triangle}\ \bigwedge_{a,b,x\in(A,B]}[\triangle|_{[a_m^\triangle,b_m^\triangle]^2}\text{strict}\wedge a\triangle b = b_m^\triangle \Rightarrow (a\triangle b)\diamond x = (a\diamond x)\amalg(b\diamond x)]. \quad (73)$$

Example 2. The following example shows that the classical addition and multiplication (restricted to finite intervals) is now not excluded from our considerations: If we take for example $A = 0$ and $1 \le B < \infty$, and $a\triangle b = a \perp b = a \amalg b = (a+b)\wedge B$, $a\diamond x = (a\cdot x)\wedge B$, then $\diamond$ is commutative and left right distributive with respect to $(\triangle, \perp, \amalg)$. For example, $\diamond$ satisfies (DL*) since $a\triangle b < B$ implies $(a\triangle b)\diamond x = (a+b)\diamond x = [(a+b)\cdot x]\wedge B = [ax+bx]\wedge B = [(ax)\wedge B + (bx)\wedge B]\wedge B = [(a\diamond x)+(b\diamond x)]\wedge B = (a\diamond x)\amalg(b\diamond x)$.

Before we present an analogous result to Theorem 3 in the 'weak distributive case (DL*)', we need a technical Lemma: In case of the 'strong distributive case (DL)' the implication '$b \le \triangle_{i=1}^n a \Rightarrow b\diamond x \le \amalg_{i=1}^n(a\diamond x)$' is valid. This implication, which is very important and which will be used very often, will be generalized by the following Lemma.

Lemma 2. Let $\triangle$ be a pseudo-addition and let $\diamond$ be a pseudo-multiplication satisfying (DL**). If $m \in K_\triangle$ then we have for all $a \in (a_m^\triangle, b_m^\triangle)$:

(a) $$\bigwedge_{b\in(a,b_m^\triangle)}\ \bigvee_{c\in(a_m^\triangle,b_m^\triangle)}\ \bigvee_{n,N\in\mathbb{N}}(\triangle_{i=1}^n c = a)\wedge(b_m^\triangle > \triangle_{i=1}^N c \ge b).$$

(b) $$\bigwedge_{b\in(a,b_m^\triangle)}\ \bigvee_{s\in\mathbb{N}}\ \bigwedge_{x\in(A,B)} b\diamond x \le \amalg_{i=1}^s(a\diamond x).$$

(c) $$\triangle|_{[a_m^\triangle,b_m^\triangle]^2}\ \text{not strict} \Rightarrow \bigvee_{s\in\mathbb{N}}\ \bigwedge_{b\in(a,b_m^\triangle)}\ \bigwedge_{x\in(A,B)} b\diamond x \le \amalg_{i=1}^s(a\diamond x).$$

(d) $\bigwedge_{a\in(a_m^\triangle,b_m^\triangle]}\bigwedge_{x\in(A,B]} b_m^\triangle \diamond x \leq \amalg_{i=1}^{\infty}(a\diamond x).$

(e) $\triangle|_{[a_m^\triangle,b_m^\triangle]^2}$ not strict $\Rightarrow \bigwedge_{a\in(a_m^\triangle,b_m^\triangle]}\bigvee_{s\in\mathbb{N}}\bigwedge_{x\in(A,B]} b_m^\triangle \diamond x \leq \amalg_{i=1}^{s}(a\diamond x).$

Proof.

(a) Let $b \in (a, b_m^\triangle)$ be arbitrary, but fixed. Using $b\triangle a_m^\triangle = (b \vee a_m^\triangle) = b < b_m^\triangle$ the continuity of $\triangle$ yields $d \in (a, b_m^\triangle)$ such that $b\triangle d < b_m^\triangle$.

Because of $k_m^{(-1)}(0) = a_m^\triangle$ and the continuity of $k_m^{(-1)}$ there is $n \in \mathbb{N}$ such that $k_m^{(-1)}\left(\frac{k_m(a)}{n}\right) < d$.

Now choose $c := k_m^{(-1)}(\frac{k_m(a)}{n})$ and $N := \lceil\frac{nk_m(b)}{k_m(a)}\rceil \in \mathbb{N}$ (remember that $\lceil r\rceil = \inf\{n \in \mathbb{Z}|n \geq r\}$) and get $\triangle_{i=1}^{n}c = k_m^{(-1)}\left(n\frac{k_m(a)}{n}\right) = a$ and

$$\begin{aligned} b_m^\triangle &> b\triangle d \geq b\triangle k_m^{(-1)}\left(\frac{k_m(a)}{n}\right) = k_m^{(-1)}(k_m(b) + \frac{k_m(a)}{n}) \\ &= k_m^{(-1)}\left(\left[\frac{nk_m(b)}{k_m(a)}+1\right]\cdot\frac{k_m(a)}{n}\right) \geq m^{(-1)}\left(\lceil\frac{nk_m(b)}{k_m(a)}\rceil\cdot\frac{k_m(a)}{n}\right) \\ &= k_m^{(-1)}(N\cdot k_m(c)) = \triangle_{i=1}^{N}c \geq k_m^{(-1)}\left(\frac{nk_m(b)}{k_m(a)}\cdot\frac{k_m(a)}{n}\right) = b. \end{aligned}$$

(b) Let $b \in (a, b_m^\triangle)$. Making use of (a) we get $\bigwedge_{j\in\{1,2,\ldots,N\}} \triangle_{i=1}^{j}c \in (a_m^\triangle, b_m^\triangle)$.
Since $N > n$ there is $q \in \mathbb{N}$ and $t \in \{1,2,\ldots,n\}$ such that $N = q\cdot n + t$.
Choose $s := q + 1 \in \mathbb{N}$ to get for arbitrary $x \in (A, B)$ (using (DL**)):

$$\begin{aligned} &b\diamond x \leq (\triangle_{i=1}^{N}c)\diamond x = (\triangle_{i=1}^{qn+t}c)\diamond x \\ &= [\amalg_{i=1}^{q}(\triangle_{i=1}^{n}c)\diamond x)]\amalg([\triangle_{i=1}^{r}c]\diamond x) \leq [\amalg_{i=1}^{q}(a\diamond x)]\amalg(a\diamond x) = \amalg_{i=1}^{s}(a\diamond x). \end{aligned}$$

(c) If $\triangle|_{[a_m^\triangle,b_m^\triangle]^2}$ is not strict then $k_m(b_m^\triangle) < \infty$ so that $s' := \lceil\frac{k_m(b_m^\triangle)}{k_m(a)}\rceil + 1 \in \mathbb{N}$.

Now let $b \in (a, b_m^\triangle)$ and let $x \in (A, B)$ be arbitrary. Choose $n, N, q, \in \mathbb{N}$ like in (b). Then we obtain $q < \frac{N}{n} = \frac{1}{n}\lceil\frac{nk_m(b)}{k_m(a)}\rceil \leq \frac{1}{n}(\frac{nk_m(b)}{k_m(a)} + 1) \leq \frac{k_m(b_m^\triangle)}{k_m(a)} + 1 \leq s'$.
Like in (b) we get (using the last inequality) $b\diamond x \leq \ldots \leq [\amalg_{i=1}^{q}(a\diamond x)]\amalg(a\diamond x) \leq \amalg_{i=1}^{s'}(a\diamond x)$.

(d) W.l.o.g let $a \in (a_m^\triangle, b_m^\triangle)$ (otherwise (d) is trivially satisfied). By (b) we have $\bigwedge_{b\in(a,b_m^\triangle)}\bigvee_{s\in\mathbb{N}}\bigwedge_{x\in(A,B)} b\diamond x \leq \amalg_{i=1}^{s}(a\diamond x) \leq \amalg_{i=1}^{\infty}(a\diamond x)$.

In case I we consider $x \in (A, B) : b_m^\triangle \diamond x = \lim_{b\to b_m^\triangle-}(b\diamond x) \leq \amalg_{i=1}^{\infty}(a\diamond x)$.

In case II, where we treat $x = B$, we use case I and monotonicity of $\diamond$: $b_m^\triangle \diamond B = \lim_{b\to B-}(b_m^\triangle \diamond x) \leq \lim_{b\to B-}\amalg_{i=1}^{\infty}(a\diamond x) \leq \amalg_{i=1}^{\infty}(a\diamond B)$.

(e) Again, we assume w.l.o.g. that $a \in (a_m^\triangle, b_m^\triangle)$.

Case I $x \in (A, B)$. We get, using (c): $b_m^\triangle \diamond x = \lim_{b \to b_m^\triangle -}(b \diamond x) \leq \amalg_{i=1}^s (a \diamond x)$.

Case II, $x = B$: $b_m^\triangle \diamond B = \lim_{b \to B-}(b_m^\triangle \diamond x) \leq \lim_{b \to B-} \amalg_{i=1}^s (a \diamond x) \leq \amalg_{i=1}^s (a \diamond B)$.

Thus Lemma 2 is proven. □

Let us now present a structure theorem for pseudo-additions if the pseudo-multiplication satisfies (DL*) (compare with Theorem 3).

Theorem 4. Let $\triangle$ and $\amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (DL*), (RU) and (LU).

Then there are 3 possibilities:

(a) $\triangle = \amalg = \vee$, e is idempotent with respect to $\triangle$, $e \notin D_\triangle$.

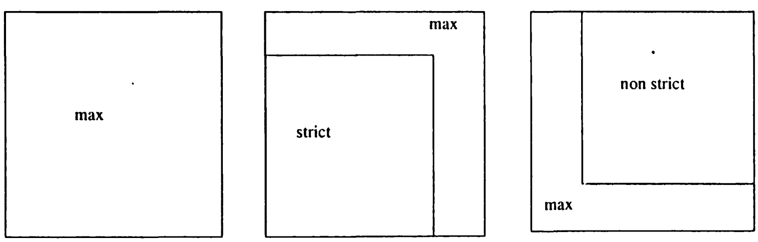
(b) $\triangle = \amalg$, $|K| = 1$, $a_1 = A$, $e \in (A, b_1]$, $a \diamond x \in [A, b_1]$ for all $a, x \in [A, b_1]$ (multiplication is compatible with the structure of $\triangle$).

If $b_1 < B$ then $\triangle]_{[A,b_1]^2}$ is strict.

If $\diamond$ satisfies (Z) and (CLZ) then $b_1 = B$.

(c) $\triangle = \amalg, |K| = 1, b_1 = B, e \in (a_1, B], a \diamond x \in [a_1, B]$ for all $a, x \in [a_1, B]$ (multiplication is compatible with the structure of $\triangle$).

If $a_1 > A$ and if $\diamond$ satisfies (CRB) then $\triangle]_{[a_1,B]}$ is non strict.



We see that also weak distributivity laws and unit elements restrict the structure of the pseudo-additions: We get equality of the pseudo-additions under consideration. Moreover the corresponding ordinal sum consists of at most one Archimedean t-conorm.

Let us now prove Theorem 4.

P r o o f o f T h e o r e m 4. We proceed like in the proof of Theorem 3 and prove at first a Lemma, to present a clearly structured proof of Theorem 3.

Lemma 3. Let $\triangle, \perp$ and $\amalg$ be pseudo-additions and let $\diamond$ be a pseudo-multiplication satisfying (DL*) and (RU). Then we have:

(a) $\bigwedge_{a,x\in[A,B]} [(a \notin D_\triangle \vee (a < B)) \wedge a\ \triangle-\text{idempotent} \Rightarrow a\diamond x\ \amalg-\text{idempotent}],$

$a \in (A,B)\ \triangle-$idempotent, $x_0 \in (A,B]$, $\bigvee_{l\in K_\amalg} a\diamond x_0 = a_l^\amalg \Rightarrow \bigwedge_{x\in[x_0,B]} a\diamond x_0 = a_l^\amalg,$

$a \in (A,B)\ \triangle-$idempotent, $x_0 \in (A,B]$, $\bigvee_{l\in K_\amalg} a\diamond x_0 = b_l^\amalg \Rightarrow \bigwedge_{x\in(A,x_0]} a\diamond x_0 = b_l^\amalg.$

(b) $\triangle = \amalg$ (so that $K_\triangle = K_\amalg = K$, $K = \mathbb{N}$ or $K = \{1,2,\dots,M\}$, and we put $b_0 = A$ and (if K is finite) $a_{M+1} = B$; $a\ \triangle-$idempotent $\Leftrightarrow a\ \amalg-$idempotent $\Leftrightarrow a$ idempotent)

(c) $\bigwedge_{x\in(A,B]}\bigwedge_{l\in K}[[x \in (A,e] \wedge b_l < B] \vee [x \in [e,B] \wedge (a_l > A] \vee [a_l = A \wedge (\mathrm{Z})] \Rightarrow a_l \diamond x = a_l]$

(d) $\bigwedge_{x\in(A,B]}\bigwedge_{l\in K}[[x \in (A,e] \wedge b_l < B] \vee [x \in [e,B] \wedge (a_l > A) \vee [(\mathrm{Z}) \wedge (\mathrm{CLZ})]])] \Rightarrow b_l \diamond x = b_l]$

(e) $\bigwedge_{x\in(A,B]}\bigwedge_{l\in K}\bigwedge_{a\in[a_l,b_l]} [x \in [e,B] \vee (b_l < B) \Rightarrow a\diamond x \in [a_l, a_{l+1}]].$

(f) $\bigwedge_{x\in(A,B]}\bigwedge_{l\in K}\bigwedge_{a\in[a_l,b_l]} [x \in (A,e] \vee (a_l > A) \vee ((\mathrm{Z}) \wedge (\mathrm{CLZ})) \Rightarrow a\diamond x \in [b_{l-1}, b_l]].$

(g) $\bigwedge_{x\in(A,B]}\bigwedge_{l\in K}\bigwedge_{a\in[a_l,b_l]} a\diamond x \in [b_{l-1}, a_{l+1}].$

(h) $\bigwedge_{x\in(A,B]}\bigwedge_{l\in K}\bigwedge_{a\in[b_l,a_{l+1}]} [b_l < B \Rightarrow a\diamond x \in [b_l, a_{l+1}].$

Proof of Lemma 3. Again, we remark that for the proof of the 3 statements of (a) the assumption (RU) is not needed.

(a) First case: Let $x \in (A,B)$.

Subcase (a1). If $a \notin D_\triangle$ then (DL*) implies $(a\diamond x)\amalg(a\diamond x) = (a\triangle a)\diamond x = a\diamond x$.

Subcase (a2). We may assume that there is $m \in K_\triangle$ such that $a = b_m^\triangle < B$. Since $\triangle$ is continuous and because of $b_{m+1}^\triangle > b_m^\triangle = b_m^\triangle \triangle b_m^\triangle$ there is a sequence $(a_n) \subset (b_m^\triangle, B]$ satisfying $\lim_{n\to\infty} a_n = b_m^\triangle$ and $a_n\triangle a_n \in (b_m^\triangle, b_{m+1}^\triangle)$ for all $n \in \mathbb{N}$. Thus we obtain (using (DL*)
$[b_m^\triangle\diamond x]\amalg[b_m^\triangle\diamond x] = [\lim_{n\to\infty}(a_n\diamond x)]\amalg[\lim_{n\to\infty}(a_n\diamond x)] = \lim_{n\to\infty}[(a_n\diamond x)\amalg(a_n\diamond x)] = \lim_{n\to\infty}[(a_n\triangle a_n)\diamond x] = [\lim_{n\to\infty}(a_n\triangle a_n)]\diamond x = (b_m^\triangle\triangle b_m^\triangle)\diamond x = b_m^\triangle\diamond x.$

Second case: $x = B$. We use the first case:

$[a \diamond B] \amalg [a \diamond B] = [\lim_{x \to B-}(a \diamond x)] \amalg [\lim_{x \to B-}(a \diamond x)] = \lim_{x \to B-}[a \diamond x] \amalg [a \diamond x] = \lim_{x \to B-}(a \diamond x) = a \diamond B$. Thus the first statement of (a) is shown.

The second and third statement of (a) are proven exactly like the second and third statement of (a) in Lemma 1. (No use of (DL), (DL*), (RU) or (CRB) is made.) Note that the case '$B \diamond x$' is not $\amalg$-idempotent can occur: In the example before Lemma 2 we have $B \diamond x = x \cdot B$ is not $\amalg$-idempotent.

(b) We first show $\bigwedge_{a \in [A,B]}(a\ \triangle-\text{idempotent} \Leftrightarrow a\ \amalg-\text{idempotent})$.

W.l.o.g. we may assume, that $a \in (A, B)$ (since $A \triangle A = A = A \amalg A, B \triangle B = B = B \amalg B$). If a $\triangle$–idempotent, then the first statement of (a) implies that $a = a \diamond e$ is $\amalg$–idempotent.

To prove the converse we assume that a is $\amalg$–idempotent, but not $\triangle$–idempotent. Then there is $m \in K_\triangle$ such that $a \in (a_m^\triangle, b_m^\triangle)$. Thus we get the contradiction (using Lemma 2 (d) and (RU)) $b_m^\triangle = b_m^\triangle \diamond e \le \amalg_{i=1}^\infty (a \diamond e) = \lim_{n \to \infty}(\amalg_{i=1}^n a) < b_m^\triangle$.

Thus the theorem on ordinal sums yields: $K_\triangle = K_\amalg =: K$, $[a_m^\triangle, b_m^\triangle] = [a_l^\amalg, b_l^\amalg] =: [a_l, b_l]$, and it suffices to show $\bigwedge_{l \in K} \bigwedge_{(a,b) \in (a_l, b_l)^2} a \triangle b = a \amalg b$.

Let $l \in K$ and let $a \in (a_l, b_l)$ be arbitrary, but fixed. We define $M := \{b \in [a_l, b_l] | a \triangle b < b_l\}$ and $b_a := \sup M$. Then $M \neq \emptyset$ (since $a \triangle a_l = a \vee a_l = a < b_l$) and $a \triangle b_a = b_l$ (If otherwise $a \triangle b_a < b_l$ then – because of $b_a < b_l$ – there is $b \in (b_a, b_l)$ satisfying $a \triangle b < b_l$ which contradicts $b_a := \sup M$.

Case 1: If $b \in (a_l, b_a)$ then $a \triangle b \in (a_l, b_l)$ and using (RU) and (DL*) we obtain $a \triangle b = (a \triangle b) \diamond e = (a \diamond e) \amalg (b \diamond e) = a \amalg b$.

Case 2: If $b = b_a \in (a_l, b_l)$ then there is a sequence $(b_n) \subset (a_l, b_a)$ with $\lim_{n \to \infty} b_n = b_a$ so that case 1 implies
$a \triangle b_a = \lim_{n \to \infty}(a \triangle b_n) = \lim_{n \to \infty}(a \amalg b_n) = a \amalg b_a$.

Case 3: If $b \in (b_a, b_l)$ then we get from the monotonicity of $\amalg$ and $\triangle$ $b_l = b_l \amalg b_l \ge a \amalg b \ge a \amalg b_a, b_l = b_l \triangle b_l \ge a \triangle b \ge a \triangle b_a$. Using $a \triangle b_a = b_l$ and case 2 we arrive at $a \amalg b_a = a \triangle b_a = b_l$ so that finally $a \triangle b = a \amalg b = b_l$. This proves (b).

To prove (b) and (c) we prove at first-like in the proof of Lemma 1 – 4 statements (I) – (IV).

(I) $\bigwedge_{x \in [e,B]} \bigwedge_{l \in K}[(a_l > A) \vee (Z) \Rightarrow a_l \diamond x = a_l]$.

Proof of (I): Case 1. If $(a_l > A)$ then we use the second statement of (a) with $a := a_l, x_0 = e$. Case 2. The case $(a_l = A) \wedge (Z)$ is trivial.

(II) $\bigwedge_{x \in (A,e]} \bigwedge_{l \in K}[(b_l < B) \Rightarrow b_l \diamond x = b_l]$.

To prove (II) we use the third statement of (a) with $a := b_l, x_0 := e$.

(III) $\bigwedge_{x\in[e,B]}\bigwedge_{l\in K}[(a_l > A) \vee ((\mathrm{Z}) \wedge (\mathrm{CLZ})) \Rightarrow b_l \diamond x = b_l]$.

Proof of (III). Case 1. Let $x < B$. Then we have by (I): $b_l \diamond x \geq b_l \diamond e = b_l > a_l = a_l \diamond x$. Applying (CLZ) for $a_l = A$ and the intermediate value theorem we get $a \in (a_l, b_l]$ satisfying $a \diamond x = b_l$. Using this and Lemma 2 (d) we get $b_l \leq b_l \diamond x \leq \amalg_{i=1}^{\infty}(a \diamond x) = \lim_{n\to\infty}(\amalg_{i=1}^{n} b_l) = b_l$.par Note that in comparison with Lemma 1 we have used Lemma 2 (d) instead of (DL).

Case 2. If $x = B$ then case 1 leads to $b_l \diamond B = b_l \diamond \sup_{x<B} x = \sup_{x<B}(b_l \diamond x) = b_l$.

(IV) $\bigwedge_{x\in(A,e]}\bigwedge_{l\in K}[b_l < B \Rightarrow a_l \diamond x = a_l]$.

Like in the proof for Lemma 1 (IV) we may assume that $a_l > A$. Assume that (IV) is not true, then we have $\bigvee_{x\in(A,e)}\bigvee_{l\in K} a_l \diamond x \neq a_l$.

Using (II) we get $a_l \diamond x < a_l \diamond e = a_l < b_l = b_l \diamond x$, and again there exists $a \in (a_l, b_l)$ with $a \diamond x = a_l$. Now we again apply Lemma 2 (d) to get the contradiction $a_l < b_l \diamond x \leq \amalg_{i=1}^{\infty}(a \diamond x) = \lim_{n\to\infty}(\amalg_{i=1}^{n} a_l) = a_l$.

Thus (I) and (IV) prove (c) and (II) and (III) prove (d). To show (e) – (h) we first prove the statements (V) – (VIII).

(V) $\bigwedge_{x\in(A,B]}\bigwedge_{l\in K}\bigwedge_{a\in[a_l,b_l]}[x \in [e,B] \vee (b_l < B) \Rightarrow a \diamond x \geq a_l]$.

We distinguish 2 cases. If $x \in [e, B]$ then we have $a \diamond x \geq a_l \diamond e = a_l$. If $x \in (A, e] \wedge (b_l < B)$ then again we get (using (IV)) $a \diamond x \geq a_l \diamond x = a_l$.

(VI) $\bigwedge_{x\in(A,B]}\bigwedge_{l\in K}\bigwedge_{a\in[a_l,b_l]}[x \in (A,e] \vee (a_l > A) \vee ((\mathrm{Z}) \wedge (\mathrm{CLZ})) \Rightarrow a \diamond x \leq b_l]$.

To prove (VI) consider first the case that $x \in (A, e]$: Then $a \diamond x \leq b_l \diamond x = b_l$. If now $[x \in [e, B] \wedge [(a_l > A) \vee ((\mathrm{Z}) \wedge (\mathrm{CLZ}))]$ then again (by (III)): $a \diamond x \leq b_l \diamond x = b_l$.

(VII) $\bigwedge_{x\in(A,B]}\bigwedge_{l\in K}\bigwedge_{a\in[b_{l-1},b_l]} a \diamond x \geq b_{l-1}$.

If $x \in [e, B]$ then $a \diamond x \geq b_{l-1} \diamond e = b_{l-1}$, and if $x \in (A, e]$ then (II) implies $a \diamond x \geq b_{l-1} \diamond x \geq b_{l-1}$ (also if $b_{l-1} = A$).

(VIII) $\bigwedge_{x\in(A,B]}\bigwedge_{l\in K}\bigwedge_{a\in[a_l,a_{l+1}]} a \diamond x \leq a_{l+1}$.

We prove (VIII). Let $x \in (A, e]$. Then $a \diamond x \leq a_{l+1} \diamond e = a_{l+1}$. If $x \in (e, B]$ then (I) yields $a \diamond x \leq a_{l+1} \diamond x \leq a_{l+1}$ (also if $a_{l+1} = B$).

Now we use (V) and (VIII) to get (e), and (VI) and (VII) give (f). Further, (VII) and (VIII) lead to (g) and (h) (in the last case the assumption $b_l < B$ is needed (see (VII)).

Thus Lemma 3 is proven and we see, that the proof for Lemma 3 is still more technically in comparison with Lemma 1. The reason is, that now Lemma 2 replaces in some respect the assumption (DL) of Lemma 1. □

Now we prove Theorem 4. We make use of Lemma 3 (b) and get $\triangle = \amalg$. Note that by hypothesis $\tilde{e} = e$. Now we distinguish two cases:

Case 1: (e idempotent) $\wedge (e \notin D_\triangle)$ and Case 2: (e is not idempotent) $\vee (e \in D_\triangle)$.

In Case 1 we show that a is $\amalg-$idempotent for all $a \in [A, B]$. Indeed, first $A \amalg A = A$, but we have also for all $a \in (A, B]$ (using (DL*)) : $a \amalg a = (e \diamond a) \amalg (e \diamond a) = (e \triangle e) \diamond a = e \diamond a = a$. Thus $\amalg = \vee$, and (a) is proven.

Now we treat Case 2. Then there exists $L \in K$ such that $e \in (a_l, b_l]$, and Lemma 3 (g) implies $x = e \diamond x \in [b_{L-1}, a_{L+1}]$ for all $x \in (A, B]$. Thus we obtain $(b_{L-1} = A) \wedge (a_{L+1} = B)$ which leads to $K = \{L\}$. Therefore we have shown that $|K| = 1$ and $e \in (a_1, b_1]$.

To prove that $a \diamond x \in [a_1, b_1]$ for all $a, x \in [a_1, b_1]$ we show: $a_1 \diamond a_1 \geq a_1$ and $b_1 \diamond b_1 \leq b_1$. Assume that $a_1 \diamond a_1 < a_1$. Then there exists $a \in (a_1, b_1)$ with $a \diamond a_1 < a_1$. But then $a \diamond a_1$ is idempotent, and Lemma 2 (d) yields the contradiction $a_1 = e \diamond a_1 \leq b_1 \diamond a_1 \leq \amalg_{i=1}^{\infty}(a \diamond a_1) = a \diamond a_1 < a_1$. Thus $a_1 \diamond a_1 \geq a_1$, but Lemma 2 (d) also implies $b_1 \diamond b_1 \leq \amalg_{i=1}^{\infty}(e \diamond b_1) = \lim_{n\to\infty}(\amalg_{i=1}^{n} b_1) = b_1$. Finally we get for all $a, x \in [a_1, b_1]$: $a_1 \leq a_1 \diamond a_1 \leq a \diamond x \leq b_1 \diamond b_1 \leq b_1$.

Now let $b_1 < B$, but we assume that $\triangle|_{[a_1,b_1]^2}$ is not strict. Then Lemma 2 (e) implies $\bigvee_{s\in\mathbb{N}} \bigwedge_{x\in(A,B]} b_1 \diamond x \leq \amalg_{i=1}^{s}(e \diamond x) = \amalg_{i=1}^{s} x$.
Using this, $b_1 < B$ and Lemma 2 (b) we obtain $\bigwedge_{x\in(a_1,e]} b_1 \diamond x = b_1$ and get the contradiction $b_1 = \lim_{x\to a_1+}(b_1 \diamond x) \leq \lim_{x\to a_1+}(\amalg_{i=1}^{s} x) = \amalg_{i=1}^{s} a_1 = a_1$ (here we have used that $\amalg_{i=1}^{s}$ is continuous on $[A, B]^s$).

To prove the remaining statements in (b) and (c) of Theorem 4 we first show: $a_1 > A \vee ((\mathrm{Z}) \wedge (\mathrm{CLZ})) \Rightarrow b_1 = B$. Indeed, Lemma 3 (f) yields $\bigwedge_{x\in(A,B]} x = e \diamond x \in [A, b_1]$ so that $b_1 = B$.

Finally, let $a_1 > A \wedge (\mathrm{CRB})$, but assume that $\triangle|_{[a_1,b_1]^2}$ is strict. Then (54) and (77) imply for all $a \in (a_1, b_1]$: $b_1 \diamond a = \lim_{n\to\infty}[(\triangle_{i=1}^{n} b_1) \diamond a] = \lim_{n\to\infty}[\amalg_{i=1}^{n}(b_1 \diamond a)] \geq \lim_{n\to\infty}[\amalg_{i=1}^{n}(e \diamond a)] = \amalg_{i=1}^{\infty} a = b_1$. On the one hand this gives $b_1 \diamond a_1 \geq b_1$ (because ofq (CRB)). On the other hand Lemma 2 (d) leads to $b_1 \diamond a_1 \leq \amalg_{i=1}^{\infty}(e \diamond a_1) = \lim_{n\to\infty}(\amalg_{i=1}^{n} a_1) = a_1 < b_1$. This contradiction proves Theorem 4. □

6. SUMMARY

Assuming the distributivity law, introduced in Definition 3, we get an unexpected structure result for pseudo-additions: The ordinal sum consists of at most two summands, and no nonstrict Archimedean t-conorm can occur. A similar result is true in the case of a weak distributivity law, but here nonstrict Archimedean summands can occur.

Although the proofs are rather technically, the methods used in our investigations are of elementary character.

(Received October 27, 2004.)

REFERENCES

[1] J. Aczél: Lectures on Functional Equations and Their Applications. Academic Press, New York – London 1966.

[2] P. Benvenuti, R. Mesiar, and D. Vivona: Monotone set-functions-based integrals. In: Handbook of Measure Theory, Vol. II (E. Pap, ed.), Elsevier, Amsterdam 2002, pp. 1329–1379.
[3] P. Benvenuti, D. Vivona, and M. Divari: The Cauchy equation on I-semigroups. Aequationes Math. *63* (2002), 220–230.
[4] C. Bertoluzza and D. Cariolaro: On the measure of a fuzzy set based on continuous t-conorms. Fuzzy Sets and Systems *88* (1997), 355–362.
[5] L. M. deCampos and M. J. Bolaños: Characterization and comparison of Sugeno and Choquet integral. Fuzzy Sets and Systems *52* (1992), 61–67.
[6] D. Denneberg: Non-additive Measure and Integral. Kluwer Academic Publishers, Dordrecht 1994.
[7] J. Fodor and M. Roubens: Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer Academic Publishers, Dordrecht 1994.
[8] M. Grabisch, T. Murofushi, and M. Sugeno (eds.): Fuzzy Measures and Integrals. Theory and Applications. Physica–Verlag, Heidelberg 2000.
[9] M. Grabisch, H. T. Nguyen, and E. A. Walker: Fundamentals of Uncertainty Calculi with Applications to Fuzzy Inference. Kluwer Academic Publishers, Dordrecht 1995.
[10] R. L. Kruse and J. J. Deeley: Joint continuity of monotonic functions. Amer. Math. Soc. *76* (1969), 74–76.
[11] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. (Trends in Logic, Volume 8.) Kluwer Academic Publishers, Dordrecht 2000.
[12] C. H. Ling: Representations of associative functions. Publ. Math. Debrecen *12* (1965), 189–212.
[13] R. Mesiar: Choquet-like Integrals. J. Math. Anal. Appl. *194* (1995), 477–488.
[14] P. S. Mostert and A. L. Shields: On the structure of semigroups on a compact manifold with boundary. Ann. of Math. *65* (1957), 117–143.
[15] T. Murofushi and M. Sugeno: Fuzzy t-conorm integral with respect to fuzzy measures: Generalization of Sugeno integral and Choquet integral. Fuzzy Sets and Systems *42* (1991), 57–71.
[16] E. Pap: Null-Additive Set Functions. Kluwer Academic Publishers. Dordrecht 1995.
[17] W. Sander: Associative aggregation operators. In: Aggregation Operators. New Trends and Applications (T. Calvo, G. Mayor, and R. Mesiar, eds.), Physica–Verlag, Heidelberg – New York 2002, pp. 124–158.
[18] N. Shilkret: Maxitive measures and integration. Indag. Math. *33* (1971), 109–116.
[19] J. Siedekum: Multiplikation und t-Conorm Integral. Ph.D. Thesis. Braunschweig 2002.
[20] M. Sugeno and T. Murofushi: Pseudo-additive measures and integrals. J. Math. Anal. Appl. *122* (1987), 197–222.
[21] Z. Wang and G. J. Klir: Fuzzy Measure Theory. Plenum Press, New York – London 1992.
[22] S. Weber: $\perp$-decomposable measures and integrals for Archimedean t-conorms $\perp$. J. Math. Anal. Appl. *101* (1984), 114–138.

Wolfgang Sander, Institute Computational Mathematics, Technical University of Braunschweig, Pockelsstr. 14, 38106 Braunschweig. Germany.
e-mail: w.sander@tu-bs.de

Jens Siedekum, Breitenbachstr. 19, 65187 Wiesbaden. Germany.
e-mail: j.siedekum@web.de