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# NON-LINEAR OBSERVER DESIGN METHOD BASED ON DISSIPATION NORMAL FORM[1]

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Observer design is one of large fields investigated in automatic control theory and a lot of articles have already been dedicated to it in technical literature. Non-linear observer design method based on dissipation normal form proposed in the paper represents a new approach to solving the observer design problem for a certain class of non-linear systems. As the theoretical basis of the approach the well known dissipative system theory has been chosen. The main achievement of the contribution consists in the fact that the error dynamics of the observer is *priory* chosen *non-linear*. It provides more flexibility in the sense of specifying error convergence properties to zero in comparison with other techniques. Lyapunov's stability theory is the other basic point of the approach.

*Keywords:* invariance, structure, stability, structural condition, Lyapunov function

*AMS Subject Classification:* 93C10

## 1. INTRODUCTION

At the beginning, known observer design methods for non-linear systems are shortly discussed. Bestle and Zeitz [3] were probably the first to introduce a non-linear canonical form needed for non-linear observer design. However, actual computation of a non-linear transformation into the form remains an unsolved problem. Krener and Isidori [16] explored the problem of transforming a non-linear system without inputs into a linear one by changing state variables and output injection. The observer design problem for non-linear systems with inputs was discussed in the paper written by Krener and Respondek [17]. They separated the system to be observed into two parts, an unforced part and an input-dependent part. Then the unforced part is transformed into an unforced linear observer form. If the transformation can be determined then it has to be checked whether it changes the input-dependent part into a non-linear mapping that only depends on input and output variables. To avoid this restriction, Keller [15] proposed a non-linear observer design method which consists in transforming the whole (undivided) system into a generalized observer canonical form. In comparison with previous forms, it depends on the first

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$n$-time derivatives of input variables. The consequence is that the resulting observer has to be supplied not only with the input and output variables of the system but also with the first $n$-time derivatives of the input variables. Birk and Zeitz [4] developed a method for non-linear observer design of MIMO systems based on extending the Luenberger observer. The question of reducing the dependency of an observer on derivatives of the input was discussed by Proychev and Mishkov [21]. The method of Krener and Respondek [17] was practically implemented by Chiasson and Novotnak [5] for the pm stepper motor. One of recent approaches to non-linear observer design originally proposed by Glumineau, Moog and Plestan is based on input-output injection [12, 18, 20]. In contrast to other methods, the transformation carrying a non-linear system into a proper canonical form is computed algorithmically via a GIOIA procedure.

The characteristic feature of the methods mentioned above is the *linear* error dynamics of the appropriate observer. Guaranteeing error convergence to zero is then performed mostly by the pole assignment technique.

The paper deals with a non-linear observer design problem without any prior assumption about the structure of the observer and/or linearity of its error dynamics. Instead of that two natural conditions are formulated. The first one determining the structure of the observer is an *error invariance condition.* This means that error time evolution has to be independent of the unknown internal state of a system, the state of the observer and external measured (input-output) signals. The second one determining the parametrization of the observer is an *error convergence condition* to zero corresponding to the asymptotical stability of the error dynamics. The approach presented in the paper consists in the *prior* choice of the error dynamics selected in order to fulfill the two conditions mentioned above. The error dynamics is chosen in the so called *dissipation normal form.* Its *non-linear* character provides more flexibility in specifying error convergence properties to zero in comparison with a *linear* one. By means of it we can specify not only required convergence rate but also other of its characteristics. It is possible to implement for example magnitude dependent damping by a non-linear function. Then the observer containing the function has a bigger and more robust damping ability than observers designed in other ways.

## 2. PROBLEM FORMULATION

Consider the representation $R(S)$ of a system $S$ in the form:

$$R(S) : \frac{\mathrm{d}x(t)}{\mathrm{d}t} = f[x(t), u(t)] \tag{1}$$

$$y(t) = h[x(t)] \tag{2}$$

where $x(t) \in X \subset \mathbb{R}^n$ is a state, $u(t) \in U \subset \mathbb{R}^p$ is an input, $y(t) \in \mathbb{R}^1$ is an output, $n, p \in \mathbb{N} \setminus \{0\}$, $f \in \mathbb{C}^n : X \times U \to \mathbb{R}^n$ is a vector function and $h \in \mathbb{C}^n : X \to \mathbb{R}^1$ is a scalar function. Assume that the representation $R(S)$ is observable for any input [9] in the sense that it holds:

$$\forall x(t) \in X, u(t) \in U : \det H_o[x(t), u(t)] \neq 0 \tag{3}$$

where:

$$H_o[x(t), u(t)] = \frac{\partial}{\partial x(t)} \begin{bmatrix} h[x(t)] \\ D_f\{h[x(t)]\} \\ \vdots \\ D_f^{n-1}\{h[x(t)]\} \end{bmatrix} \tag{4}$$

is a generalized observability matrix (for the definition of $D$ see the Appendix).

The aim is to design an observer $R(\hat{S})$:

$$R(\hat{S}) : \frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = \hat{f}[\hat{x}(t), u(t), y(t)] \tag{5}$$

which will produce an asymptotic estimate $\hat{x}(t)$ of the state $x(t)$ using the input $u(t)$ and the output $y(t)$ in such a way that the following two conditions will be fulfilled.

The first one is the *error invariance condition*:

$$R(\tilde{S}) : \frac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t} = \tilde{f}[\tilde{x}(t), x(t), \hat{x}(t), u(t), y(t), t] = \tilde{f}[\tilde{x}(t)] \tag{6}$$

where $\tilde{S}$ is error dynamics and $\tilde{x}(t)$ is an error defined as:

$$\tilde{x}(t) = x(t) - \hat{x}(t). \tag{7}$$

The second one is the *error convergence condition* to zero:

$$\lim_{t \to \infty} \tilde{x}(t) = 0 \tag{8}$$

corresponding to the asymptotical stability of the error dynamics:

$$\begin{aligned} \tilde{V}[\tilde{x}(t)] &> 0 \ \text{ for } \ \tilde{x}(t) \neq \tilde{x}_e && (9) \\ \tilde{V}[\tilde{x}(t)] &= 0 \ \text{ for } \ \tilde{x}(t) = \tilde{x}_e && (10) \\ L_{\tilde{f}}\{\tilde{V}[\tilde{x}(t)]\} &< 0 \ \text{ for } \ \tilde{x}(t) \neq \tilde{x}_e && (11) \\ L_{\tilde{f}}\{\tilde{V}[\tilde{x}(t)]\} &= 0 \ \text{ for } \ \tilde{x}(t) = \tilde{x}_e && (12) \end{aligned}$$

where $\tilde{V}[\tilde{x}(t)]$ is a Lyapunov function related to the representation $R(\tilde{S})$ and $\tilde{x}_e = 0$ is its equilibrium state for which it holds that:

$$\frac{\mathrm{d}\tilde{x}_e}{\mathrm{d}t} = 0. \tag{13}$$

## 3. DISSIPATION NORMAL FORM

**Definition 1.** Consider the representation $R_D(S)$ of a system $S$ in the form:

$$\begin{aligned} R_D(S) : \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= f[x(t)] && (14) \\ y(t) &= h[x(t)] && (15) \end{aligned}$$

where $x(t) \in X \subset \mathbb{R}^n$ is a state, $X$ is a smooth manifold defined on $\mathbb{R}^n$, $n \in \mathbb{N} \setminus \{0\}$, $y(t) \in \mathbb{R}^1$ is an output, $f : X \to \mathbb{R}^n$ is a smooth vector field and $h : X \to \mathbb{R}^1$ is a smooth scalar function. Let $x_e$ be an equilibrium state of the representation $R_D(S)$. Assume that there exists a function $W : Y \to \mathbb{R}^1$ defined on a neighborhood $Y \subset \mathbb{R}^n$ of the equilibrium state $x_e$. The representation $R_D(S)$ will be called the *dissipation normal form* if the function $W$ fulfills the following conditions:

$$\begin{aligned} W[x(t)] &= \|x(t)\|^2 && (16) \\ L_f\{W[x(t)]\} &= \beta[y(t)] \leq 0. && (17) \end{aligned}$$

**Remark 1.** There is an obvious connection between the function $W[x(t)]$ and the Lyapunov function. The function $W[x(t)]$ is also related to the available storage [24] and a non-linear function $\beta[y(t)]$ corresponds to the Rayleigh function [22].

### 3.1. Asymptotical stability and observability

The following theorem will be used later for guaranteing the asymptotical stability of the error dynamics.

**Theorem 1.** Let $k_2, \ldots, k_n \in \mathbb{R}$; $k_2, \ldots, k_n \neq 0$ and $\alpha, \varphi_1 : \mathbb{R}^1 \to \mathbb{R}^1$ are continuous functions satisfying the following conditions: $\forall x(t) \in X : \alpha[x_1(t)]$ is strictly monotonous; $\forall x(t) \in Z, Z \subset Y : \varphi_1[x_1(t)] < 0 \Leftrightarrow x_1(t) \neq 0$. If the representation $R_D(S)$ has the following structure [13]:

$$\begin{aligned} R_D(S) : \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \begin{bmatrix} \varphi_1[x_1(t)] & k_2 & 0 & \cdots & 0 \\ -k_2 & 0 & k_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -k_{n-1} & 0 & k_n \\ 0 & \cdots & 0 & -k_n & 0 \end{bmatrix} x(t) && (18) \\ y(t) &= \alpha[x_1(t)] && (19) \end{aligned}$$

then it is observable in the sense of (3) and the equilibrium state $x_e = 0$, $x_e \in Z$ is asymptotically stable in $Z$. Additionally, the function $W[x(t)]$ fulfills the conditions (16), (17) for any $\alpha[x_1(t)]$, $\varphi_1[x_1(t)]$ and $k_2, \ldots, k_n$ on $Z$ satisfying the premises given at the beginning of the theorem.

Proof. At first, the observability of the representation $R_D(S)$ will be proved and subsequently the proof of the asymptotical stability of its equilibrium state will follow using the second (direct) Lyapunov stability method.

1. It holds that:

$$\det H_o[x(t)] = \det \frac{\partial}{\partial x(t)} \begin{bmatrix} \alpha[x_1(t)] \\ L_f\{\alpha[x_1(t)]\} \\ \vdots \\ L_f^{n-1}\{\alpha[x_1(t)]\} \end{bmatrix} = k_2^{n-1} \cdot k_3^{n-2} \cdot \ldots \cdot k_n \cdot \left\{ \frac{\mathrm{d}\alpha[x_1(t)]}{\mathrm{d}x_1(t)} \right\}^n . \tag{20}$$

It follows from the relation (20) that the representation $R_D(S)$ is observable in the sense of (3) under the assumptions stated at the beginning of the theorem.

2. Assume that the representation $R_D(S)$ has the form (18), (19) and consider the function $W[x(t)] = ||x(t)||^2$ defined on $\mathbb{R}^n$.

   - The relation (18) implies that:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 0 \Leftrightarrow x(t) = x_e = 0. \tag{21}$$

   Hence, $x_e = 0$, $x_e \in Z$ is the equilibrium state of the representation $R_D(S)$.

   - It holds that:

$$\begin{aligned}
W[x(t)] &> 0 \ \text{for} \ x(t) \neq 0 && (22)\\
W[x(t)] &= 0 \ \text{for} \ x(t) = 0 && (23)\\
L_f\{W[x(t)]\} &= 2x_1^2(t)\varphi_1[x_1(t)] = 2\{\alpha^{-1}[y(t)]\}^2\varphi_1\{\alpha^{-1}[y(t)]\} \\
&= \beta[y(t)] < 0 \ \text{for} \ x(t) \notin M \subset Z && (24)\\
L_f\{W[x(t)]\} &= 2x_1^2(t)\varphi_1[x_1(t)] = 2\{\alpha^{-1}[y(t)]\}^2\varphi_1\{\alpha^{-1}[y(t)]\} \\
&= \beta[y(t)] = 0 \ \text{for} \ x(t) \in M && (25)
\end{aligned}$$

   where $M = \{x(t) \in Z, L_f\{W[x(t)]\} = 0\}$ is the largest invariant subset of $Z$. The relations (22), (23), (24), (25) and invoking La Salle's invariance principle [11] imply that the function $W[x(t)]$ is a Lyapunov function on $Z$. Thus, the equilibrium state $x_e = 0$ is asymptotically stable in $Z$. It is also obvious that the function $W[x(t)]$ fulfills the conditions (16), (17) for any $\alpha[x_1(t)]$, $\varphi_1[x_1(t)]$ and $k_2, \ldots, k_n$ on $Z$. □

**Remark 2.** The dissipation normal form is similar to the Schwarz matrix [23] and can be seen as the generalization of a corresponding linear system representation.

**Remark 3.** In linear case, if the coefficients of the form are as follows:

$$\begin{aligned}
R_D(S): \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \omega_0 \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{bmatrix} x(t) && (26)\\
y(t) &= x_1(t) && (27)
\end{aligned}$$

where $\omega_0 \in \mathbb{R}$, $\omega_0 > 0$ then it is optimal with respect to the output signal energy optimality criterion [13, 19]:

$$J = \int_{t_0}^{\infty} ||y(t)||^2 \mathrm{d}t. \tag{28}$$

## 4. NON-LINEAR OBSERVER DESIGN USING DISSIPATION NORMAL FORM

Consider the representation $R(S)$ of a system $S$ in the form:

$$R(S): \frac{\mathrm{d}x(t)}{\mathrm{d}t} = f[x(t), u(t)] \tag{29}$$

$$y(t) = h[x(t)]. \tag{30}$$

In the sequel, the dissipation normal form will be used for non-linear observer design expressing the requirements mentioned in Section 2.

### 4.1. Error dynamics representation

Let us choose the representation of the error dynamics in the *dissipation normal form*:

$$R^*(\tilde{S}): \frac{\mathrm{d}\tilde{x}^*(t)}{\mathrm{d}t} = \omega_0 \begin{bmatrix} \delta_1^*[\tilde{x}_1^*(t)] & \delta_2^* & 0 & \cdots & 0 \\ -\delta_2^* & 0 & \delta_3^* & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\delta_{n-1}^* & 0 & \delta_n^* \\ 0 & \cdots & 0 & -\delta_n^* & 0 \end{bmatrix} \tilde{x}^*(t) \tag{31}$$

where $\delta_1^*[\tilde{x}_1^*(t)]$, $\delta_2^*, \ldots, \delta_n^*$, $\omega_0$ are design parameters. It means that error convergence properties to zero can be specified by their selecting. It holds that:

$$L_{\tilde{f}^*}\{\tilde{V}^*[\tilde{x}^*(t)]\} = L_{\tilde{f}^*}\{\|\tilde{x}^*(t)\|^2\} = 2\omega_0 {\tilde{x}_1^*}^2(t)\delta_1^*[\tilde{x}_1^*(t)] \tag{32}$$

where $\tilde{V}^*[\tilde{x}^*(t)] = \|\tilde{x}^*(t)\|^2$ is a Lyapunov function related to the representation $R^*(\tilde{S})$. The relations (31), (32) imply that both the *error invariance condition* and the *error convergence condition* to zero are fulfilled in case of the design parameters are properly chosen. Assuming that $\omega_0 > 0$ and $\delta_1^*[\tilde{x}_1^*(t)] < 0$ for all $\tilde{x}_1^*(t)$ then the error dynamics is globally asymptotically stable. On condition that $\omega_0 > 0$ and $\delta_1^*[\tilde{x}_1^*(t)] < 0$ only for $\tilde{x}_1^*(t) \in r$, $r \subset \mathbb{R}$ then the error dynamics is locally asymptotically stable over a finite area of the state space induced by $r$. The constant $\omega_0$ represents a time scale transformation and therefore it affects convergence rate. The non-linear function $\delta_1^*[\tilde{x}_1^*(t)]$ describes in what way system energy dissipates and accordingly it specifies convergence mode. It is obvious from the relation (32) that the constants $\delta_2^*, \ldots, \delta_n^* \neq 0$ do not have any effect on rate and/or mode of convergence. From this point of view, they can in principle be chosen in an arbitrary way. It is even possible them to be non-linear functions in general. This implies that the error invariance condition is not necessary to hold. However, it is more comfortable for the problem solution when the condition is valid. Therefore, the non-linear observer design will be performed under the assumption of the condition relevance. Moreover, the complication is not mandatory. It has already been said that the elements do not have any effect on either rate or mode of convergence. From this point of view, they are selected without loss of generality as constants.

**Remark 4.** In fact, supposing that $\omega_0 \to \infty$ ($\frac{1}{\omega_0} \to 0$) then the observer is similar to the high-gain observer [1, 2, 8] in the sense of possible setting the error convergence to zero fast enough in order that the asymptotical stability of a closed-loop system is guaranteed [6].

### 4.2. Observer structure

Consider a class of representations (29), (30) transformable into the following canonical form induced by the error dynamics representation structure (31):

$$R^*(S): \begin{aligned} \frac{\mathrm{d}x^*(t)}{\mathrm{d}t} &= A^* x^*(t) + \psi^*[x_1^*(t), u(t), u_d(t)] && (33)\\ y(t) &= h^*[x_1^*(t)] && (34)\end{aligned}$$

where

$$A^* = \begin{bmatrix} 0 & a_2^* & 0 & \cdots & 0 \\ -a_2^* & 0 & a_3^* & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -a_{n-1}^* & 0 & a_n^* \\ 0 & \cdots & 0 & -a_n^* & 0 \end{bmatrix}; \; a_2^*, \dots, a_n^* \in R; \; a_2^*, \dots, a_n^* \neq 0;$$

$$u_d(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}, \dots, \frac{\mathrm{d}^{n-1}u(t)}{\mathrm{d}t^{n-1}} \quad \text{and} \quad \psi^*[x_1^*(t), u(t), u_d(t)] = \begin{bmatrix} \psi_1^*[x_1^*(t), u(t)] \\ \psi_2^*[x_1^*(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}] \\ \vdots \\ \psi_n^*[x_1^*(t), u(t), u_d(t)] \end{bmatrix}.$$

Further, the inverse:

$$x_1^*(t) = c[y(t)] \tag{35}$$

where $c[y(t)] = h^{*^{-1}}[y(t)]$ is supposed to exist. After derivating the relation (7) with respect to $t$ and an elementary modification we have:

$$\frac{\mathrm{d}\hat{x}^*(t)}{\mathrm{d}t} = \frac{\mathrm{d}x^*(t)}{\mathrm{d}t} - \frac{\mathrm{d}\tilde{x}^*(t)}{\mathrm{d}t}. \tag{36}$$

Substituting $\frac{\mathrm{d}x^*(t)}{\mathrm{d}t}$, $\frac{\mathrm{d}\tilde{x}^*(t)}{\mathrm{d}t}$ from the relations (31), (33) into (36) we get the structure of the observer in the form:

$$R^*(\hat{S}): \frac{\mathrm{d}\hat{x}^*(t)}{\mathrm{d}t} = A^*\hat{x}^*(t) + \psi^*\{c[y(t)], u(t), u_d(t)\} - \omega_0 \begin{bmatrix} \delta_1^*\{c[y(t)] - \hat{x}_1^*(t)\} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \{c[y(t)] - \hat{x}_1^*(t)\} \tag{37}$$

where the equalities:

$$a_2^* = \omega_0 \delta_2^*, \dots, a_n^* = \omega_0 \delta_n^* \tag{38}$$

hold.

### 4.3. Observer parametrization

The observer parametrization means here to determine the unknown functions $c[y(t)]$ and $\psi^*\{c[y(t)], u(t), u_d(t)\}$ in (37) and consequently in (33), (34). It will be performed through the generalized observability canonical form [25]:

$$\bar{R}(S) : \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \bar{x}_1(t) \\ \vdots \\ \bar{x}_{n-1}(t) \\ \bar{x}_n(t) \end{bmatrix} = \begin{bmatrix} \bar{x}_2(t) \\ \vdots \\ \bar{x}_n(t) \\ \bar{\mu}[\bar{x}(t), u(t), u_d(t)] \end{bmatrix} \tag{39}$$

$$y(t) = \bar{x}_1(t). \tag{40}$$

The form can be generated from the representation (29), (30) by the diffeomorphism:

$$\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \vdots \\ \bar{x}_n(t) \end{bmatrix} = \begin{bmatrix} h[x(t)] \\ D_f\{h[x(t)]\} \\ \vdots \\ D_f^{n-1}\{h[x(t)]\} \end{bmatrix} \tag{41}$$

and exists if the observability condition (3) holds. Assume that a diffeomorphism:

$$\bar{x}(t) = T[x^*(t), u(t), u_d(t)] \tag{42}$$

exists. The condition for its existence is:

$$\det \frac{\partial T[x^*(t), u(t), u_d(t)]}{\partial x^*(t)} \neq 0. \tag{43}$$

Then it is determined by the following relation:

$$\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \vdots \\ \bar{x}_n(t) \end{bmatrix} = \begin{bmatrix} h^*[x_1^*(t)] \\ D_{f^*}\{h^*[x_1^*(t)]\} \\ \vdots \\ D_{f^*}^{n-1}\{h^*[x_1^*(t)]\} \end{bmatrix}. \tag{44}$$

**Lemma 1.** The existence of (42) implies that the structural condition:

$$\bar{\mu}[\bar{x}(t), u(t), u_d(t)] = D_{f^*}^n\{h^*[x_1^*(t)]\} \tag{45}$$

is fulfilled for $x^*(t) = T^{-1}[\bar{x}(t), u(t), u_d(t)]$.

Proof. If (42) exists then the equality:

$$D_{\bar{f}}^n[\bar{x}_1(t)] = D_{f^*}^n\{h^*[x_1^*(t)]\} \tag{46}$$

holds. Substituting into the relation (46) from (39) we have:

$$\bar{\mu}[\bar{x}(t), u(t), u_d(t)] = D_{f^*}^n\{h^*[x_1^*(t)]\} \quad \text{for} \quad x^*(t) = T^{-1}[\bar{x}(t), u(t), u_d(t)]. \qquad \square$$

The unknown functions $\psi^*[x_1^*(t), u(t), u_d(t)]$ and $c[y(t)]$ in the observer (37) can be computed from a system of differential equations which is a consequence of validity of the structural condition (45).

4.3.1. Explicit solution for second-order system

For a second-order system, the structural condition (45) has the form:

$$\begin{aligned}\bar{\mu}\left[\bar{x}(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right] &= D_{f^*}^2\{h^*[x_1^*(t)]\} = F_1[\bar{x}_1(t)]\bar{x}_2^2(t) + F_2[\bar{x}_1(t), u(t)]\bar{x}_2(t) \\ &\quad + F_3\left[\bar{x}_1(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right] \end{aligned} \tag{47}$$

for $x^*(t) = T^{-1}[\bar{x}(t), u(t)]$ where the functions $F_3[\bar{x}_1(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}]$, $F_2[\bar{x}_1(t), u(t)]$ and $F_1[\bar{x}_1(t)]$ are known. In case of the structural condition (47) is fulfilled then the unknown functions $\psi_1^*[x_1^*(t), u(t)]$, $\psi_2^*[x_1^*(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}]$ and $h^*[x_1^*(t)]$, $x_1^*(t) = c[y(t)]$, in the observer:

$$\begin{aligned} R^*(\hat{S}) : \frac{\mathrm{d}\hat{x}^*(t)}{\mathrm{d}t} &= \begin{bmatrix} 0 & a_2^* \\ -a_2^* & 0 \end{bmatrix} \hat{x}^*(t) + \begin{bmatrix} \psi_1^*\{c[y(t)], u(t)\} \\ \psi_2^*\{c[y(t)], u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\} \end{bmatrix} \\ &\quad -\omega_0 \begin{bmatrix} \delta_1^*\{c[y(t)] - \hat{x}_1^*(t)\} \\ 0 \end{bmatrix} \{c[y(t)] - \hat{x}_1^*(t)\} \end{aligned} \tag{48}$$

can be computed from the system of the three differential equations:

$$F_1[\bar{x}_1(t)]|_{\bar{x}_1(t)=h^*[x_1^*(t)]} = \frac{\mathrm{d}^2 h^*[x_1^*(t)]}{\mathrm{d}x_1^{*2}(t)} \frac{1}{\left\{\frac{\mathrm{d}h^*[x_1^*(t)]}{\mathrm{d}x_1^*(t)}\right\}^2} \tag{49}$$

$$F_2[\bar{x}_1(t), u(t)]|_{\bar{x}_1(t)=h^*[x_1^*(t)]} = \frac{\partial \psi_1^*[x_1^*(t), u(t)]}{\partial x_1^*(t)} \tag{50}$$

$$\begin{aligned} F_3\left[\bar{x}_1(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right]\Bigg|_{\bar{x}_1(t)=h^*[x_1^*(t)]} &= a_2^* \frac{\mathrm{d}h^*[x_1^*(t)]}{\mathrm{d}x_1^*(t)} \psi_2^*\left[x_1^*(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right] \\ &\quad + \frac{\mathrm{d}h^*[x_1^*(t)]}{\mathrm{d}x_1^*(t)} \frac{\partial \psi_1^*[x_1^*(t), u(t)]}{\partial u(t)} \frac{\mathrm{d}u(t)}{\mathrm{d}t} \\ &\quad - a_2^{*2} x_1^*(t) \frac{\mathrm{d}h^*[x_1^*(t)]}{\mathrm{d}x_1^*(t)}. \end{aligned} \tag{51}$$

## 4.4. Determination of observer in original coordinates

A diffeomorphism $x(t) = T[x^*(t), u(t), u_d(t)]$ exists if the condition:

$$\det \frac{\partial T[x^*(t), u(t), u_d(t)]}{\partial x^*(t)} \neq 0 \tag{52}$$

is fulfilled. Then it is determined by the following relation:

$$\begin{bmatrix} h[x(t)] \\ D_f\{h[x(t)]\} \\ \vdots \\ D_f^{n-1}\{h[x(t)]\} \end{bmatrix} = \begin{bmatrix} h^*[x_1^*(t)] \\ D_{f^*}\{h^*[x_1^*(t)]\} \\ \vdots \\ D_{f^*}^{n-1}\{h^*[x_1^*(t)]\} \end{bmatrix}. \tag{53}$$

It obviously holds that:

$$\hat{x}(t) = T[\hat{x}^*(t), u(t), u_d(t)]. \tag{54}$$

Subsequently, derivating (54) with respect to $t$ we get the observer in the original coordinates:

$$R(\hat{S}) : \frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = D_{\hat{f}^*}\{T[\hat{x}^*(t), u(t), u_d(t)]\} \tag{55}$$

for $\hat{x}^*(t) = T^{-1}[\hat{x}(t), u(t), u_d(t)]$.

## 5. ILLUSTRATIVE EXAMPLE

The non-linear observer design method described above will be illustrated on the following prey-predator model [15]:

$$R(S) : \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = ax_1(t) - bx_1(t)x_2(t) \tag{56}$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = cx_1(t)x_2(t) - dx_2(t) - fx_2(t)u(t) \tag{57}$$

$$y(t) = x_2(t) \tag{58}$$

where $x_1(t)$ and $x_2(t)$ represent prey and predator populations. The predator population is decimated by humans via the input variable $u(t)$. The coefficients $a = 1.5$, $b = 1$, $c = 0.3$, $d = 1$ are constant birth and death rates and $f = 0.5$ is an extermination rate. At first, the representation $R(S)$ is transformed into the generalized observability canonical form:

$$\bar{R}(S) : \frac{\mathrm{d}\bar{x}_1(t)}{\mathrm{d}t} = \bar{x}_2(t) \tag{59}$$

$$\frac{\mathrm{d}\bar{x}_2(t)}{\mathrm{d}t} = \bar{\mu}[\bar{x}(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}] \tag{60}$$

$$y(t) = \bar{x}_1(t) \tag{61}$$

by the transformation:

$$\bar{x}(t) = T[x(t), u(t)] = \begin{bmatrix} x_2(t) \\ D_f[x_2(t)] \end{bmatrix} = \begin{bmatrix} x_2(t) \\ cx_1(t)x_2(t) - dx_2(t) - fx_2(t)u(t) \end{bmatrix}. \tag{62}$$

It holds that:

$$\forall x(t), u(t) : \det \frac{\partial T[x(t), u(t)]}{\partial x(t)} = -cx_2(t) \neq 0 \Leftrightarrow x_2(t) \neq 0. \tag{63}$$

This means that the predator population should not die out. Using the transformation (62) we get:

$$\begin{aligned} \bar{\mu}\left[\bar{x}(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right] &= \frac{1}{\bar{x}_1(t)}\bar{x}_2^2(t) + [a - b\bar{x}_1(t)]\bar{x}_2(t) + [a - b\bar{x}_1(t)][d + fu(t)]\bar{x}_1(t) \\ &\quad - f\bar{x}_1(t)\frac{\mathrm{d}u(t)}{\mathrm{d}t}. \end{aligned} \tag{64}$$

Obviously, the function $\bar{\mu}[\bar{x}(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}]$ fulfills the structural condition (47) with:

$$\begin{aligned} F_1[\bar{x}_1(t)] &= \frac{1}{\bar{x}_1(t)} && (65)\\ F_2[\bar{x}_1(t), u(t)] &= a - b\bar{x}_1(t) && (66)\\ F_3\left[\bar{x}_1(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right] &= [a - b\bar{x}_1(t)][d + fu(t)]\bar{x}_1(t) - f\bar{x}_1(t)\frac{\mathrm{d}u(t)}{\mathrm{d}t}. && (67) \end{aligned}$$

The unknown functions $\psi_1^*[x_1^*(t), u(t)]$, $\psi_2^*[x_1^*(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}]$ and $c[y(t)]$ in the observer (48) are computed from the system of the three differential equations (49), (50), (51). The solution of the differential equation (49) is:

$$h^*[x_1^*(t)] = e^{x_1^*(t)} \Rightarrow x_1^*(t) = \ln[y(t)] = c[y(t)]. \tag{68}$$

The solution of the differential equation (50) leads to:

$$\begin{aligned} \psi_1^*[x_1^*(t), u(t)] &= \int [a - be^{x_1^*(t)}]\,\mathrm{d}x_1^*(t) = ax_1^*(t) - be^{x_1^*(t)} = a\ln[y(t)] - by(t)\\ &= \psi_1^*\{c[y(t)], u(t)\}. \end{aligned} \tag{69}$$

Substituting into the differential equation (51) from (68), (69) we have:

$$\begin{aligned} \psi_2^*\left[x_1^*(t), u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right] &= \frac{1}{a_2^*}[a - be^{x_1^*(t)}][d + fu(t)] + a_2^*x_1^*(t) - \frac{1}{a_2^*}f\frac{\mathrm{d}u(t)}{\mathrm{d}t}\\ &= \frac{1}{a_2^*}[a - by(t)][d + fu(t)] + a_2^*\ln[y(t)] - \frac{1}{a_2^*}f\frac{\mathrm{d}u(t)}{\mathrm{d}t}\\ &= \psi_2^*\left\{c[y(t)], u(t), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right\}. \end{aligned} \tag{70}$$

The appropriate transformation $\bar{x}(t) = T[x^*(t)]$ has the form:

$$\bar{x}(t) = T[x^*(t)] = \begin{bmatrix} e^{x_1^*(t)} \\ a_2^*x_2^*(t)e^{x_1^*(t)} + ax_1^*(t)e^{x_1^*(t)} - be^{2x_1^*(t)} \end{bmatrix}. \tag{71}$$

It holds that:

$$\forall x^*(t) : \det\frac{\partial T[x^*(t)]}{\partial x^*(t)} = a_2^*e^{2x_1^*(t)} \neq 0. \tag{72}$$

Subsequently, the design parameters in the error dynamics representation:

$$R^*(\tilde{S}) : \frac{\mathrm{d}\tilde{x}^*(t)}{\mathrm{d}t} = \omega_0\begin{bmatrix} \delta_1^*[\tilde{x}_1^*(t)] & \delta_2^* \\ -\delta_2^* & 0 \end{bmatrix}\tilde{x}^*(t) \tag{73}$$

are chosen as follows:

1. $\delta_1^*[\tilde{x}_1^*(t)] = \delta_1^* = -1$, $\omega_0 = 4$ and $\delta_2^* = 1$. Let us define the output of the error dynamics as $\tilde{y}(t) = \tilde{x}_1^*(t)$. Then the representation $R^*(\tilde{S})$ is optimal with respect to the output signal energy optimality criterion (see Section 3):

$$\tilde{J} = \int_{t_0}^{\infty} \|\tilde{y}(t)\|^2\mathrm{d}t. \tag{74}$$

2. $\delta_1^*[\tilde{x}_1^*(t)] = -0.3\tilde{x}_1^{*^4}(t) - 2$, $\omega_0 = 4$ and $\delta_2^* = 1$. The constant $\delta_1^*$ was replaced by the non-linear function $\delta_1^*[\tilde{x}_1^*(t)]$. The other design parameters stayed the same.

Substituting (68), (69), (70) and the selected design parameters into (48) we get the observers in the forms:

1.

$$
\begin{aligned}
R^*(\hat{S}): \frac{\mathrm{d}\hat{x}^*(t)}{\mathrm{d}t} &= \begin{bmatrix} 0 & a_2^* \\ -a_2^* & 0 \end{bmatrix} \hat{x}^*(t) \\
&+ \begin{bmatrix} a\ln[y(t)] - by(t) \\ \frac{1}{a_2^*}[a - by(t)][d + fu(t)] + a_2^* \ln[y(t)] - \frac{1}{a_2^*} f \frac{\mathrm{d}u(t)}{\mathrm{d}t} \end{bmatrix} \\
&-\omega_0 \begin{bmatrix} -1 \\ 0 \end{bmatrix} \{\ln[y(t)] - \hat{x}_1^*(t)\}
\end{aligned} \tag{75}
$$

2.

$$
\begin{aligned}
R^*(\hat{S}): \frac{\mathrm{d}\hat{x}^*(t)}{\mathrm{d}t} &= \begin{bmatrix} 0 & a_2^* \\ -a_2^* & 0 \end{bmatrix} \hat{x}^*(t) \\
&+ \begin{bmatrix} a\ln[y(t)] - by(t) \\ \frac{1}{a_2^*}[a - by(t)][d + fu(t)] + a_2^* \ln[y(t)] - \frac{1}{a_2^*} f \frac{\mathrm{d}u(t)}{\mathrm{d}t} \end{bmatrix} \\
&-\omega_0 \begin{bmatrix} -0.3\{\ln[y(t)] - \hat{x}_1^*(t)\}^4 - 2 \\ 0 \end{bmatrix} \{\ln[y(t)] - \hat{x}_1^*(t)\}
\end{aligned} \tag{76}
$$

where $a_2^* = \omega_0\delta_2^*$. Finally, the observers are transformed into the original coordinates using the relation (55) where:

$$
x(t) = T[x^*(t), u(t)] = \begin{bmatrix} \frac{a}{c}x_1^*(t) - \frac{b}{c}e^{x_1^*(t)} + \frac{a_2^*}{c}x_2^*(t) + \frac{d}{c} + \frac{f}{c}u(t) \\ e^{x_1^*(t)} \end{bmatrix}. \tag{77}
$$

It holds that:

$$
\forall x^*(t), u(t) : \det \frac{\partial T[x^*(t), u(t)]}{\partial x^*(t)} = -\frac{a_2^*}{c}e^{x_1^*(t)} \neq 0. \tag{78}
$$

The resulting observers for the given original representation $R(S)$ are the following:

1.

$$
\begin{aligned}
R(\hat{S}): \frac{\mathrm{d}\hat{x}_1(t)}{\mathrm{d}t} &= \underline{a\hat{x}_1(t) - b\hat{x}_1(t)\hat{x}_2(t)} + \left[\frac{a}{c} - \frac{b}{c}\hat{x}_2(t)\right] \{-a\ln[\hat{x}_2(t)] + b\hat{x}_2(t) \\
&+ a\ln[y(t)] - by(t) + \frac{b}{c}[d + fu(t)][\hat{x}_2(t) - y(t)] \\
&+ \frac{a_2^{*2}}{c}\{\ln[y(t)] - \ln[\hat{x}_2(t)]\} + \omega_0\{\ln[y(t)] - \ln[\hat{x}_2(t)]\}\} \\
\frac{\mathrm{d}\hat{x}_2(t)}{\mathrm{d}t} &= \underline{c\hat{x}_1(t)\hat{x}_2(t) - d\hat{x}_2(t) - f\hat{x}_2(t)u(t)} + \hat{x}_2(t)\{-a\ln[\hat{x}_2(t)] \\
&+ b\hat{x}_2(t) + a\ln[y(t)] - by(t) + \omega_0\{\ln[y(t)] - \ln[\hat{x}_2(t)]\}\}
\end{aligned} \tag{79}
$$

2.

$$
\begin{aligned}
R(\hat{S}): \frac{\mathrm{d}\hat{x}_1(t)}{\mathrm{d}t} &= \underline{a\hat{x}_1(t) - b\hat{x}_1(t)\hat{x}_2(t)} + \left[\frac{a}{c} - \frac{b}{c}\hat{x}_2(t)\right] \{-a\ln[\hat{x}_2(t)] + b\hat{x}_2(t) \\
&\quad + a\ln[y(t)] - by(t) + \frac{b}{c}[d + fu(t)][\hat{x}_2(t) - y(t)] \\
&\quad + \frac{a_2^{*2}}{c}\{\ln[y(t)] - \ln[\hat{x}_2(t)]\} + 0.3\omega_0\{\ln[y(t)] - \ln[\hat{x}_2(t)]\}^5 \\
&\quad + 2\omega_0\{\ln[y(t)] - \ln[\hat{x}_2(t)]\}\} \\
\frac{\mathrm{d}\hat{x}_2(t)}{\mathrm{d}t} &= \underline{c\hat{x}_1(t)\hat{x}_2(t) - d\hat{x}_2(t) - f\hat{x}_2(t)u(t)} + \hat{x}_2(t)\{-a\ln[\hat{x}_2(t)] \\
&\quad + b\hat{x}_2(t) + a\ln[y(t)] - by(t) + 0.3\omega_0\{\ln[y(t)] - \ln[\hat{x}_2(t)]\}^5 \\
&\quad + 2\omega_0\{\ln[y(t)] - \ln[\hat{x}_2(t)]\}\}. \qquad (80)
\end{aligned}
$$

The behaviour of the designed observers is shown in Figures 1, 2. The effect of the non-linear function $\delta_1^*[\tilde{x}_1^*(t)]$ on the error trajectory can be seen as implementing magnitude dependent damping and a consequent bigger damping ability in comparison with the other one. Particularly, it is obvious from the right picture of Figure 2.

**Remark 5.** An observer for the given system representation $R(S)$ may also be designed by the method presented in [10] (see Figure 3). The method is based on transforming a given system representation into phase variables where the error dynamics of the observer is made asymptotically stable with optional error convergence rate to zero by a parameter $\theta$. The parametrization of the observer is given by solution of an algebraic equation obtained on behalf of using standard Lyapunov arguments. The equation can also be derived from a modified form for a Grammian observability matrix that connects asymptotic stability and observability properties of a system and its representation [26]. The solution of the equation $S_\infty(\theta)$ is then equivalent to the matrix. The advantage of the method is simplicity of the observer. On the other hand, it only works for affine system representations.

## 6. CONCLUSION

The non-linear observer design method based on the dissipation normal form has successfully been solved. The approach consists in transforming a given system representation into proper new coordinates where the error dynamics of the observer can easily be made homogeneous and asymptotically stable, which is equivalent to the methods presented for example in [3, 12, 15, 16, 17, 18, 20, 21]. Nevertheless, the difference from those methods is that the error dynamics of the observer is chosen *priory* here. The appropriate canonical form of the system representation and the observer is then a consequence of the choice. Further, the method is exact and does not require any linearization in the sense that the non-linear system to be observed is replaced by a linear one. In contrast to linear case the resulting observer has to be

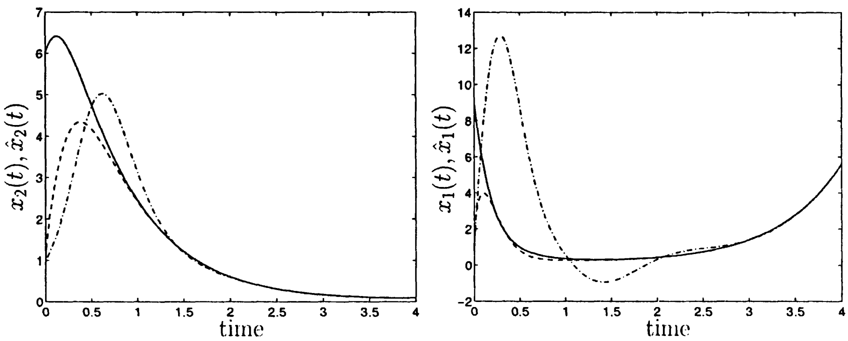


**Fig. 1.** Observing the state: the dashed and dotted line for the observer No. 1, the dashed line for the observer No. 2.

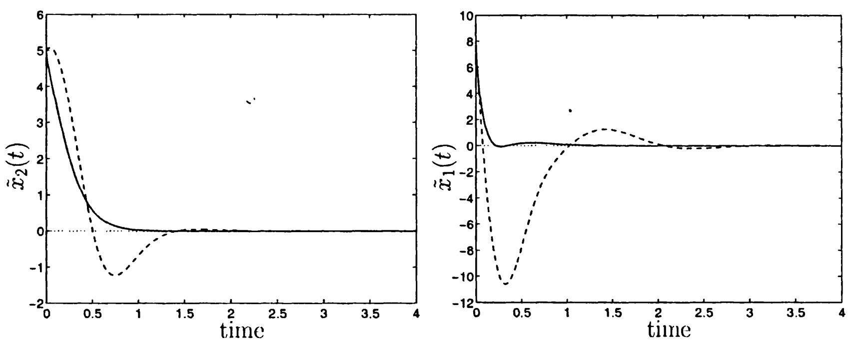


**Fig. 2.** Course of the error: the dashed line for the observer No. 1, the solid line for the observer No. 2.

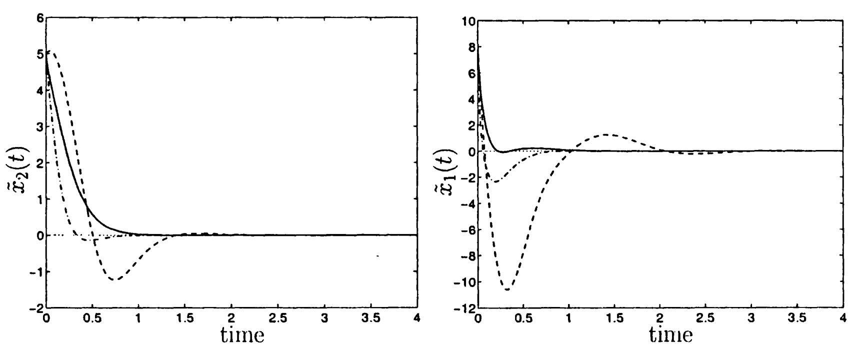


**Fig. 3.** Course of the error: the dashed line for the observer No. 1, the solid line for the observer No. 2, the dashed and dotted line for the observer designed along the method presented in [10] for the design parameter $\theta = 4$.

supplied not only with input and output variables of a given system but also with derivatives of the input variables as for example in [15].

The main achievement of the contribution is the *non-linear* error dynamics of the observer with a properly placed non-linear function. From this point of view, the approach can be seen as a straightforward extension of the methods cited above where the error dynamics is *linear*. The non-linear function provides more flexibility in the choice of error convergence properties to zero than a linear one. By means of it we can specify not only convergence rate but also other of its characteristics. It is possible to implement for example magnitude dependent damping by the function. The observer containing the function has then a bigger (and more robust) damping ability in comparison with observers designed in other ways (see Figure 2). It is also feasible to guarantee not only the global asymptotical stability of the error dynamics but also only a local asymptotical stability over a finite area of the state space by the non-linear function again. Finally, the approach to non-linear observer design presented in the paper has effectively been used in signal filtering, too [7, 14].

## APPENDIX

Let $x(t) \in X \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^p$, $f \in \mathbb{C}^k : X \times U \to \mathbb{R}^n$, $n$, $p$, $k \in N \setminus \{0\}$ be a vector function and $h \in \mathbb{C}^k : X \times U \to \mathbb{R}^1$ be a scalar function. Then $D_f(h)$ is a differential operator for which it holds that:

$$D_f^0\{h[x(t),u(t)]\} = h[x(t),u(t)] \tag{81}$$

$$D_f^1\{h[x(t),u(t)]\} = D_f\{h[x(t),u(t)]\} = \sum_{i=1}^{n} \frac{\partial h[x(t),u(t)]}{\partial x_i(t)} f_i[x(t),u(t)] + \sum_{j=1}^{p} \frac{\partial h[x(t),u(t)]}{\partial u(t)} \frac{\mathrm{d}u_j(t)}{\mathrm{d}t} \tag{82}$$

$$D_f^k\{h[x(t),u(t)]\} = D_f\{D_f^{k-1}\{h[x(t),u(t)]\}\}. \tag{83}$$

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