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TRANSITIVE DECOMPOSITION OF FUZZY PREFERENCE RELATIONS: THE CASE OF NILPOTENT MINIMUM

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Transitivity is a fundamental notion in preference modelling. In this work we study this property in the framework of additive fuzzy preference structures. In particular, we depart from a large preference relation that is transitive w.r.t. the nilpotent minimum t-norm and decompose it into an indifference and strict preference relation by means of generators based on t-norms, i. e. using a Frank t-norm as indifference generator. We identify the strongest type of transitivity these indifference and strict preference components show, both in general and for the important class of weakly complete large preference relations.

Keywords: fuzzy relation, indifference, nilpotent minimum, strict preference, transitivity
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1. INTRODUCTION

In the context of preference modelling, the concept of transitivity arises as a natural property many relations must satisfy. In the classical setting, i. e. when working with crisp relations, the transitivity of a large preference relation R can be characterized by the transitivity of the corresponding indifference relation I and strict preference relation P and two additional relational inequalities involving P and I [18]. In case the relation R is complete, its transitivity is completely characterized by the transitivity of P and I only.

The above-mentioned characterization has also been studied in the fuzzy case, i. e. when working with fuzzy relations. In the well-defined context of additive fuzzy preference structures [4], a characterization of the transitivity of a large preference relation R has been obtained when R is strongly complete [6]. Other studies require less restrictive completeness conditions (such as weak completeness) or no completeness condition at all [2, 3, 19]. However, in none of these studies a full characterization has been obtained.

In this paper, we focus on the propagation of the T-transitivity of a large preference relation R to the corresponding indifference relation I and strict preference relation P, when using as indifference generator a Frank t-norm $T_\lambda^{\mathbf{F}}$. Furthermore, we restrict ourselves to Fodor's nilpotent minimum $T_{\mathbf{nM}}$ [9], as it is the most famous member of the class of rotation-invariant t-norms. Rotation-invariant t-norms are witnessing a growing interest [12, 13, 14] and are of particular importance to fuzzy preference modelling (see e. g. [8]).

Our paper is organised as follows. In Section 2, we give a brief introduction to crisp and fuzzy preference modelling. In particular, we explain how additive fuzzy preference structures can be constructed by means of an indifference generator. Section 3 features a brief review of known results on the transitivity of decompositions of transitive large preference relations. In Section 4, we characterize the strongest type of transitivity shown by the generated indifference relation. The same is done in Section 5 for the strict preference relation. This study is repeated in Section 6 for the case of a weakly complete large preference relation. A summarizing conclusion is provided.

2. ADDITIVE FUZZY PREFERENCE STRUCTURES

We briefly recall two equivalent relational representations of preferential information [18]. On the one hand, one can consider a *large preference* relation R, i. e. a reflexive (binary) relation on the set of alternatives A, with the following interpretation:

$$aRb \textit{ if and only if } a \textit{ is at least as good as } b\,.$$

On the other hand, R can be decomposed into disjoint parts: an irreflexive and asymmetric *strict preference* component P, a reflexive and symmetric *indifference* component I and an irreflexive and symmetric *incomparability* component J such that $P \cup P^t \cup I \cup J = A^2$, $R = P \cup I$ and $R^c = P^t \cup J$ (where $\cdot^t$ denotes the transpose of a relation and $\cdot^c$ denotes the complement of a relation). These components can be obtained by considering various intersections: $P = R \cap R^d$, $I = R \cap R^t$ and $J = R^c \cap R^d$ (where $\cdot^d$ denotes the dual of a relation, i. e. the complement of its transpose).

In fuzzy preference modelling, a reflexive fuzzy relation R on A can also be decomposed into what is called an *additive fuzzy preference structure*, by means of an *(indifference) generator* i, which is defined as a symmetric (commutative) $[0,1]^2 \to [0,1]$ mapping located between the Łukasiewicz t-norm $T_{\mathbf{L}}$ (i. e. $T_{\mathbf{L}}(x,y) = \max(x + y - 1, 0)$) and the minimum operator $T_{\mathbf{M}}$, i. e. $T_{\mathbf{L}} \le i \le T_{\mathbf{M}}$. More specifically, the *strict preference* relation P, the *indifference* relation I and the *incomparability* relation J are obtained as follows [5].

$$\begin{aligned}
P(a,b) &= p(R(a,b),R(b,a)) = R(a,b) - i(R(a,b),R(b,a))\\
I(a,b) &= i(R(a,b),R(b,a))\\
J(a,b) &= j(R(a,b),R(b,a)) = i(R(a,b),R(b,a)) - (R(a,b) + R(b,a) - 1)\,.
\end{aligned}$$

An additive fuzzy preference structure (AFPS) (P, I, J) on A is then characterized as a triplet of fuzzy relations on A such that I is reflexive and symmetric and

$$P(a,b) + P(b,a) + I(a,b) + J(a,b) = 1\,,$$

whence the adjective 'additive'. The corresponding large preference relation R is then given by $R(a,b) = P(a,b) + I(a,b)$.

Most of the studies on additive fuzzy preference structures consider t-norm generators only [10, 11], meaning that not only the generator $i(x,y)$, but also $p(x, 1-y)$ and $j(1-x, 1-y)$ are t-norms. However, this is exactly the same as requiring that i is a *Frank t-norm* [5]. The Frank family is a parametric family of continuous t-norms, usually denoted as $T_\lambda^{\mathbf{F}}$ with $\lambda \in [0,\infty]$. For $\lambda \in]0,1[\cup]1,\infty[$, it holds that

$$T_\lambda^{\mathbf{F}}(x,y) = \log_\lambda\left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{(\lambda - 1)}\right)\,,$$

while $T_0^{\mathbf{F}} = T_{\mathbf{M}}$, $T_1^{\mathbf{F}} = T_{\mathbf{P}}$ (the algebraic product) and $T_\infty^{\mathbf{F}} = T_{\mathbf{L}}$ are obtained via a limit procedure. Any Frank t-norm $T_\lambda^{\mathbf{F}}$ with $\lambda \in]0,\infty[$ is a *strict t-norm*, which means that it can be written as a transformation of the algebraic product $T_{\mathbf{P}}$ by means of a $[0,1]$-automorphism ϕ_λ (also called *multiplicative generator*). More explicitly, for any $x \in [0,1]$ it holds that $\phi_1(x) = x$ and

$$\phi_\lambda(x) = \frac{\lambda^x - 1}{\lambda - 1}\,,$$

for any $\lambda \in]0,1[\cup]1,\infty[$, and $T_\lambda^{\mathbf{F}}(x,y) = \phi_\lambda^{-1}(\phi_\lambda(x)\cdot\phi_\lambda(y))$. An important property of the Frank t-norm family is the following [11]:

$$T_{1/\lambda}^{\mathbf{F}}(x,y) = x - T_\lambda^{\mathbf{F}}(x, 1-y)\,,$$

for any $x, y \in [0,1]$ and any $\lambda \in [0,\infty]$.

3. TRANSITIVITY OF LARGE PREFERENCE RELATIONS

A relation Q on A is said to be *transitive* if

$$(\forall (a,b,c) \in A^3)((aQb \wedge bQc) \Rightarrow aQc)\,.$$

Transitivity can be stated equivalently as a relational inequality: $Q \circ Q \subseteq Q$. Using the latter notation, the characterization of the transitivity of a large preference relation R can be written as follows [18]:

Proposition 1. For any reflexive relation R with corresponding preference structure (P, I, J) it holds that

$$R \circ R \subseteq R \quad \Leftrightarrow \quad \begin{cases} P \circ P \subseteq P \\ I \circ I \subseteq I \\ P \circ I \subseteq P \\ I \circ P \subseteq P\,. \end{cases}$$

In case R is *complete* (i.e. aRb or bRa for any $a, b \in A$), the following simpler characterization holds. Note that in this case $J = \emptyset$.

Proposition 2. For any complete relation R with corresponding preference structure $(P, I, \emptyset)$ it holds that

$$R \circ R \subseteq R \quad \Leftrightarrow \quad \begin{cases} P \circ P \subseteq P \\ I \circ I \subseteq I . \end{cases}$$

The most popular type of transitivity of fuzzy relations is *T-transitivity*, with T a t-norm [15]. For reasons that will become clear further on, we consider here the more general definition of *f-transitivity*, with f a *conjunctor* (i.e. an increasing $[0,1]^2 \to [0,1]$ mapping that coincides on $\{0,1\}^2$ with the Boolean conjunction). A fuzzy relation Q on A is called f-transitive if it holds that

$$(\forall (a,b,c) \in A^3)(f(Q(a,b), Q(b,c)) \leq Q(a,c)) .$$

Obviously, if $f \geq g$, then f-transitivity implies g-transitivity. The *sup-f composition* of two fuzzy relations U and V on A is the fuzzy relation $U \circ_f V$ on A defined by

$$U \circ_f V(x,z) = \sup_{y \in A} f(U(x,y), V(y,z)) .$$

Trivially, f-transitivity can then be expressed equivalently as a relational inequality: $Q \circ_f Q \subseteq Q$.

As far as we know, the only generalization of Proposition 2 has been obtained in the case of a *strongly complete* large preference relation R (i.e. $\max(R(a,b), R(b,a)) = 1$ for any $a, b \in A$). Note that in that case any generator i (not only the Frank t-norms) leads to the same AFPS and that again $J = \emptyset$.

Proposition 3. (See [6].) Consider a strongly complete fuzzy relation R with corresponding fuzzy preference structure $(P, I, \emptyset)$. For any t-norm $T \geq T_{\mathbf{L}}$ it holds that:

$$R \circ_T R \subseteq R \quad \Leftrightarrow \quad \begin{cases} P \circ_{T_{\mathbf{M}}} P \subseteq P \\ I \circ_T I \subseteq I \\ P \circ_{T_{\mathbf{L}}} I \subseteq P \\ I \circ_{T_{\mathbf{L}}} P \subseteq P . \end{cases}$$

Note that Proposition 3 really only is a generalization of Proposition 2, due to the completeness condition, although it formally looks like Proposition 1.

For the *nilpotent minimum* $T_{\mathbf{nM}}$ it holds that $T_{\mathbf{nM}} \geq T_{\mathbf{L}}$. Recall that the nilpotent minimum is the t-norm defined by [9, 16, 17]:

$$T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & \text{if } x + y \leq 1 , \\ \min(x,y), & \text{otherwise} . \end{cases}$$

The nilpotent minimum is a very important member of the class of *rotation-invariant* t-norms (w.r.t. the standard negator $N(x) = 1 - x$) [12, 13, 14] which satisfy

$$(\forall (x,y,z) \in [0,1]^3)(T(x,y) \leq z \Leftrightarrow T(y, 1-z) \leq 1-x)\,.$$

According to the above proposition, for any strongly complete reflexive fuzzy relation R with corresponding preference structure $(P, I, \emptyset)$, the following equivalence holds:

$$R \circ_{T_{\mathbf{nM}}} R \subseteq R \quad \Leftrightarrow \quad \begin{cases} P \circ_{T_{\mathbf{M}}} P \subseteq P \\ I \circ_{T_{\mathbf{nM}}} I \subseteq I \\ P \circ_{T_{\mathbf{L}}} I \subseteq P \\ I \circ_{T_{\mathbf{L}}} P \subseteq P\,. \end{cases}$$

In this paper, we focus on the implication from left to right in the above equivalence, and try to relax the strong completeness condition. Also, we state the transitivity explicitly and do not make use of the sup-f composition notation.

4. FROM LARGE PREFERENCE TO INDIFFERENCE RELATIONS

In this section, we characterize the transitivity of the indifference relation I generated from a $T_{\mathbf{nM}}$-transitive large preference relation R. As generators we consider the members of the Frank t-norm family, i. e.

$$I(a,b) = T^{\mathbf{F}}_{\lambda}(R(a,b), R(b,a))\,,$$

with $\lambda \in [0, \infty]$.

For this study, we can partially rely on earlier results [7] that are briefly summarized and adopted to the present context hereafter. Consider a $T_{\mathbf{nM}}$-transitive large preference relation R and a generator i belonging to the Frank t-norm family, then it holds that:

(i) in general, the indifference relation I can neither be 'more' transitive than $T_{\mathbf{nM}}$-transitive nor 'more' transitive than i-transitive;

(ii) in the following two cases, maximal transitivity of I is achieved:

  (a) if $i \leq T_{\mathbf{nM}}$, then I is $i = \min(i, T_{\mathbf{nM}})$-transitive (this result mainly depends on the fact that a t-norm is bisymmetric);

  (b) if i *dominates* $T_{\mathbf{nM}}$, denoted $T_{\mathbf{nM}} \ll i$, where $f \ll g$ means that

$$(\forall (x,y,z,t) \in [0,1]^4)(g(f(x,y), f(z,t)) \geq f(g(x,z), g(y,t)))\,,$$

  then I is $T_{\mathbf{nM}} = \min(i, T_{\mathbf{nM}})$-transitive.

As a consequence of these general results, as $T_{\mathbf{L}} \leq T_{\mathbf{nM}}$ and $T_{\mathbf{nM}} \ll T_{\mathbf{M}}$, given the $T_{\mathbf{nM}}$-transitivity of R, we conclude that the following are the strongest results possible:

(i) if $i = T_{\mathbf{L}}$, then I is $T_{\mathbf{L}}$-transitive;

(ii) if $i = T_{\mathbf{M}}$, then I is $T_{\mathbf{nM}}$-transitive.

However, for $i = T^{\mathbf{F}}_\lambda$, $\lambda \in]0,\infty[$, it neither holds that $T^{\mathbf{F}}_\lambda \leq T_{\mathbf{nM}}$ nor $T_{\mathbf{nM}} \ll T^{\mathbf{F}}_\lambda$ (as even $T_{\mathbf{nM}} \leq T^{\mathbf{F}}_\lambda$ does not hold); hence, we can only apply result (i) above. We conclude that these results are far from satisfactory, and a tailor-made theorem is necessary.

Such a theorem will be presented next and involves a family of conjunctors that are obtained by annihilating the Frank t-norms in the same way as the nilpotent minimum is obtained from the minimum. More explicitly, for $\lambda \in [0,\infty]$, we define

$$f_\lambda(x,y) = \begin{cases} 0 & \text{, if } x+y \leq 1\,, \\ T^{\mathbf{F}}_\lambda(x,y) & \text{, otherwise}\,. \end{cases}$$

Note that $f_\lambda = \min(T^{\mathbf{F}}_\lambda, T_{\mathbf{nM}})$. Of course, $f_0 = T_{\mathbf{nM}}$ and $f_\infty = T_{\mathbf{L}}$. For $\lambda \in]0,\infty[$, the conjunctor f_λ is not a t-norm as it is not associative.

Theorem 1. For any reflexive fuzzy relation R with corresponding indifference relation I generated by means of $i = T^{\mathbf{F}}_\lambda$, $\lambda \in [0,\infty]$, the following implication holds:

$$R \text{ is } T_{\mathbf{nM}}\text{-transitive} \quad \Rightarrow \quad I \text{ is } f_\lambda\text{-transitive}\,.$$

Moreover, this is the strongest result possible.

Proof. In view of the definition of f_λ, it is sufficient to consider the case $I(a,b) + I(b,c) > 1$, whence also $R(a,b) + R(b,c) > 1$ and $R(b,a) + R(c,b) > 1$. Since R is $T_{\mathbf{nM}}$-transitive, it then follows that

$$\begin{aligned} I(a,c) &= T^{\mathbf{F}}_\lambda(R(a,c), R(c,a)) \\ &\geq T^{\mathbf{F}}_\lambda(\min(R(a,b), R(b,c)), \min(R(c,b), R(b,a))) \\ &\geq T^{\mathbf{F}}_\lambda(T^{\mathbf{F}}_\lambda(R(a,b), R(b,c)), T^{\mathbf{F}}_\lambda(R(c,b), R(b,a))) \\ &= T^{\mathbf{F}}_\lambda(T^{\mathbf{F}}_\lambda(R(a,b), R(b,a)), T^{\mathbf{F}}_\lambda(R(b,c), R(c,b))) \\ &= T^{\mathbf{F}}_\lambda(I(a,b), I(b,c)) = f_\lambda(I(a,b), I(b,c))\,. \end{aligned}$$

Moreover, since $f_\lambda = \min(i, T_{\mathbf{nM}}) = \min(T^{\mathbf{F}}_\lambda, T_{\mathbf{nM}})$ and according to the discussion above, this is clearly the strongest result possible. □

Note that since f_λ is not a t-norm in general, it has proven very useful to generalize the notion of T-transitivity to f-transitivity, with f a conjunctor, in order to be able to characterize the maximal transitivity of I.

5. FROM LARGE PREFERENCE TO STRICT PREFERENCE RELATIONS

As for the indifference relation I, in this section we discuss the transitivity of the strict preference relation P generated from a $T_{\mathbf{nM}}$-transitive large preference relation R. As generators we consider again the members of the Frank t-norm family, i. e.

$$P(a,b) = R(a,b) - T^{\mathbf{F}}_\lambda(R(a,b), R(b,a)) = T^{\mathbf{F}}_{1/\lambda}(R(a,b), 1 - R(b,a))\,,$$

with $\lambda \in [0, \infty]$. The following theorem shows that the transitivity of P is in some sense reciprocal to the transitivity of I obtained in Theorem 1.

Theorem 2. For any reflexive fuzzy relation R with corresponding strict preference relation P generated by means of $i = T^{\mathbf{F}}_{\lambda}$, $\lambda \in [0, \infty]$, the following implication holds:

$$R \text{ is } T_{\mathbf{nM}}\text{-transitive} \quad \Rightarrow \quad P \text{ is } f_{1/\lambda}\text{-transitive}.$$

Moreover, this is the strongest result possible.

Proof. In view of the definition of $f_{1/\lambda}$, it is sufficient to consider the case $P(a,b) + P(b,c) > 1$, whence also $R(a,b) + R(b,c) > 1$. The $T_{\mathbf{nM}}$-transitivity of R then implies that

$$T_{\mathbf{nM}}(R(a,b), R(b,c)) = \min(R(a,b), R(b,c)) \leq R(a,c).$$

Without loss of generality, we can assume that $R(a,b) \leq R(b,c)$, and hence $R(a,b) \leq R(a,c)$. We distinguish two cases:

(i) *The case* $R(b,c) + R(c,a) \leq 1$. It holds that

$$R(c,a) \leq 1 - \max(R(a,b), R(b,c)) \leq 1 - \max(P(a,b), P(b,c)).$$

It then easily follows that

$$\begin{aligned} f_{1/\lambda}(P(a,b), P(b,c)) &= T^{\mathbf{F}}_{1/\lambda}(\min(P(a,b), P(b,c)), \max(P(a,b), P(b,c))) \\ &\leq T^{\mathbf{F}}_{1/\lambda}(\min(P(a,b), P(b,c)), 1 - R(c,a)). \end{aligned}$$

Since $R(a,b) \leq R(a,c)$, it surely holds that

$$R(a,c) \geq P(a,b) \geq \min(P(a,b), P(b,c)).$$

Hence, it follows that

$$\begin{aligned} P(a,c) &= T^{\mathbf{F}}_{1/\lambda}(R(a,c), 1 - R(c,a)) \\ &\geq T^{\mathbf{F}}_{1/\lambda}(\min(P(a,b), P(b,c)), 1 - R(c,a)) \\ &\geq f_{1/\lambda}(P(a,b), P(b,c)). \end{aligned}$$

(ii) *The case* $R(b,c) + R(c,a) > 1$. We will first show that the assumption $R(c,a) > R(b,a)$ leads to a contradiction. The $T_{\mathbf{nM}}$-transitivity of R implies that

$$\begin{aligned} R(c,a) > R(b,a) &\geq T_{\mathbf{nM}}(R(b,c), R(c,a)) \\ &= \min(R(b,c), R(c,a)) = R(b,c) \geq R(a,b). \end{aligned}$$

Furthermore, since $R(a,b) + R(c,a) > R(a,b) + R(b,c) > 1$, it holds that

$$R(c,b) \geq T_{\mathbf{nM}}(R(c,a), R(a,b)) = \min(R(c,a), R(a,b)) = R(a,b),$$

and similarly, since $R(c,a) + R(b,c) > R(a,b) + R(b,c) > 1$, that $R(b,a) \geq R(b,c)$. Combining these results then leads to

$$\begin{aligned} P(a,b) &= T^{\mathbf{F}}_{1/\lambda}(R(a,b), 1 - R(b,a)) \leq T^{\mathbf{F}}_{1/\lambda}(R(a,b), 1 - R(b,c)),\\ P(b,c) &= T^{\mathbf{F}}_{1/\lambda}(R(b,c), 1 - R(c,b)) \leq T^{\mathbf{F}}_{1/\lambda}(R(b,c), 1 - R(a,b)). \end{aligned}$$

Since $T^{\mathbf{F}}_{1/\lambda} \leq T_{\mathbf{M}}$ it follows that

$$\begin{aligned} P(a,b) + P(b,c) &\leq \min(R(a,b), 1 - R(b,c)) + \min(R(b,c), 1 - R(a,b))\\ &= 2 - R(a,b) - R(b,c) \leq 2 - P(a,b) - P(b,c). \end{aligned}$$

However, this implies that $P(a,b) + P(b,c) \leq 1$, a contradiction.
It therefore holds that $R(c,a) \leq R(b,a)$. Recall that also $R(a,b) \leq R(a,c)$. It then holds that

$$\begin{aligned} P(a,c) &= T^{\mathbf{F}}_{1/\lambda}(R(a,c), 1 - R(c,a)) \geq T^{\mathbf{F}}_{1/\lambda}(R(a,b), 1 - R(c,a))\\ &\geq T^{\mathbf{F}}_{1/\lambda}(R(a,b), 1 - R(b,a)) = P(a,b)\\ &\geq \min(P(a,b), P(b,c)) \geq f_{1/\lambda}(P(a,b), P(b,c)). \end{aligned}$$

Next, consider a conjunctor g such that $g(x,y) > f_{1/\lambda}(x,y)$ in some point $(x,y) \in]0,1[^2$. Consider the following fuzzy relation R on $A = \{a,b,c\}$:

R	a	b	c
a	1	x	$T_{\mathbf{nM}}(x,y)$
b	0	1	y
c	$T_{\mathbf{M}}(1-x, 1-y)$	0	1

Then R is $T_{\mathbf{nM}}$-transitive, but the strict preference relation P generated by means of $i = T_\lambda$,

P	a	b	c
a	0	x	$f_{1/\lambda}(x,y)$
b	0	0	y
c	$T^{\mathbf{F}}_{1/\lambda}(T_{\mathbf{M}}(1-x, 1-y), 1 - T_{\mathbf{nM}}(x,y))$	0	0

is not g-transitive, since $P(a,c) = f_{1/\lambda}(x,y) < g(x,y) = g(P(a,b), P(b,c))$. □

Remark 1.

(i) Since the Frank t-norm family is strictly increasing with decreasing parameter values (see e.g. [1]), it follows from Theorem 2 that the transitivity of P becomes weaker with decreasing λ, while Theorem 1 shows that the transitivity of I becomes stronger.

(ii) Also note that for $\lambda > 1$, the transitivity of P is stronger than that of I, while for $\lambda < 1$, this is just the opposite. For $\lambda = 1$, both types obviously coincide.

6. THE WEAKLY COMPLETE CASE

In the foregoing sections, we have identified the strongest type of transitivity the strict preference and indifference relations generated from a $T_{\mathbf{nM}}$-transitive large preference relation R exhibit in general. In Proposition 2, it was already indicated that in case of a strongly complete large preference relation R, stronger results can be obtained: P is min-transitive and I is $T_{\mathbf{nM}}$-transitive. In this section, we consider the more interesting and more general case of a *weakly complete* large preference relation R: $R(a,b) + R(b,a) \geq 1$ for any $a, b \in A$.

6.1. Indifference relations

Generating the indifference relation I by means of $i = T_\lambda^{\mathbf{F}}$, we already know that I is f_λ-transitive. The following proposition shows that when restricting our attention to the class of weakly complete large preference relations, no stronger result can be obtained.

Theorem 3. For any weakly complete reflexive fuzzy relation R with corresponding indifference relation I generated by means of $i = T_\lambda^{\mathbf{F}}$, $\lambda \in [0, \infty]$, the following implication holds:

$$R \text{ is } T_{\mathbf{nM}}\text{-transitive} \quad \Rightarrow \quad I \text{ is } f_\lambda\text{-transitive}.$$

Moreover, this is the strongest result possible.

Proof. In view of Theorem 1, we only need to show that no stronger result can be obtained for weakly complete R. Consider a conjunctor g such that $g(x,y) > f_\lambda(x,y)$ in some point $(x,y) \in]0,1[^2$. Define the set

$$B = \{(u,v) \in [0,1]^2 \mid u + v > 1\}.$$

Consider the following fuzzy relation R on $A = \{a, b, c\}$:

R	a	b	c
a	1	x	$x \cdot \chi_B(x,y)$
b	1	1	$\chi_B(x,y) + y \cdot \chi_{B^c}(x,y)$
c	$y \cdot \chi_B(x,y) + \chi_{B^c}(x,y)$	$y \cdot \chi_B(x,y) + \chi_{B^c}(x,y)$	1

where χ_B (resp. χ_{B^c}) is the characteristic mapping of B (resp. B^c). Then R is weakly complete and $T_{\mathbf{nM}}$-transitive but the indifference relation I generated by means of $i = T_\lambda$,

I	a	b	c
a	1	x	$f_\lambda(x,y)$
b	x	1	y
c	$f_\lambda(x,y)$	y	1

is not g-transitive since $I(a,c) = f_\lambda(x,y) < g(x,y) = g(I(a,b), I(b,c))$. □

6.2. Strict preference relations

Generating the strict preference relation P by means of $i = T^{\mathbf{F}}_{\lambda}$, we already know that P is $f_{1/\lambda}$-transitive. In this section, we will show that when restricting our attention to the class of weakly complete large preference relations, stronger results can be obtained, except in the case $\lambda = \infty$.

Theorem 4. For any weakly complete reflexive fuzzy relation R with corresponding strict preference relation P generated by means of $i = T^{\mathbf{F}}_{\infty} = T_{\mathbf{L}}$, the following implication holds:

$$R \text{ is } T_{\mathbf{nM}}\text{-transitive} \quad \Rightarrow \quad P \text{ is } T_{\mathbf{nM}}\text{-transitive}.$$

Moreover, this is the strongest result possible.

Proof. In view of Theorem 2 ($f_0 = T_{\mathbf{nM}}$), we only need to show that no stronger result can be obtained for weakly complete R. Consider a conjunctor g such that $g(x,y) > T_{\mathbf{nM}}(x,y)$ in some point $(x,y) \in]0,1[^2$. Consider the following fuzzy relation R on $A = \{a,b,c\}$:

R	a	b	c
a	1	x	$T_{\mathbf{nM}}(x,y)$
b	$1-x$	1	y
c	$1-T_{\mathbf{nM}}(x,y)$	$1-y$	1

Then R is weakly complete and $T_{\mathbf{nM}}$-transitive but the strict preference relation P generated by means of $i = T_{\mathbf{L}}$,

P	a	b	c
a	0	x	$T_{\mathbf{nM}}(x,y)$
b	$1-x$	0	y
c	$1-T_{\mathbf{nM}}(x,y)$	$1-y$	0

is not g-transitive, since $P(a,c) = T_{\mathbf{nM}}(x,y) < g(x,y) = g(P(a,b),P(b,c))$. □

Next we prove an inequality involved in the proof of Theorem 5.

Lemma 1. Consider $\lambda \in]0,\infty[$. For the multiplicative generator ϕ_λ of the Frank t-norm $T^{\mathbf{F}}_{\lambda}$ it holds that

$$\phi_\lambda^{-1}\left(\sqrt{\phi_\lambda(x)\,\phi_\lambda(1-y)}\right) + \phi_\lambda^{-1}\left(\sqrt{\phi_\lambda(1-x)\,\phi_\lambda(y)}\right) \leq 1$$

for any $(x,y) \in [0,1]^2$.

Proof. For $\lambda = 1$ it holds that $\phi_1(x) = x$ and we have to prove that

$$\sqrt{x(1-y)} + \sqrt{(1-x)y} \leq 1 .$$

First observe that $x(1-y) \leq \left(1 - \sqrt{(1-x)y}\right)^2$ is equivalent to

$$(1-x+y)^2 \geq \left(2\sqrt{(1-x)y}\right)^2 .$$

A simple verification shows that

$$(1-x+y)^2 - \left(2\sqrt{(1-x)y}\right)^2 = (1-(x+y))^2 \geq 0 ,$$

which completes the proof for $\lambda = 1$.

Now let $\lambda \in]0,1[\cup]1,\infty[$. The inequality is trivially fulfilled when at least one of x and y is either 0 or 1. We therefore consider $(x,y) \in]0,1[^2$. Using the explicit expression of ϕ_λ, the desired inequality is equivalent to $h_y(x) \leq h_y(1-y)$, where h_y is the function on $[0,1]$ defined by

$$\begin{aligned} h_y(x) &= \frac{|\lambda-1|}{\lambda-1} + \sqrt{(\lambda^y-1)(\lambda^{1-x}-1)} + \sqrt{(\lambda^x-1)(\lambda^{1-y}-1)} \\ &\quad + \frac{|\lambda-1|}{\lambda-1}\sqrt{(\lambda^y-1)(\lambda^{1-x}-1)(\lambda^x-1)(\lambda^{1-y}-1)} . \end{aligned}$$

We first compute the derivative of h_y on $]0,1[$:

$$\begin{aligned} h'_y(x) &= \frac{\ln\lambda}{2}\left[-\frac{\lambda^{1-x}(\lambda^y-1)}{\sqrt{(\lambda^y-1)(\lambda^{1-x}-1)}} + \frac{\lambda^x(\lambda^{1-y}-1)}{\sqrt{(\lambda^{1-y}-1)(\lambda^x-1)}}\right. \\ &\quad \left. + \frac{|\lambda-1|}{\lambda-1}\frac{(\lambda^{1-y}-1)(\lambda^y-1)(\lambda^{1-x}-\lambda^x)}{\sqrt{(\lambda^{1-y}-1)(\lambda^x-1)(\lambda^y-1)(\lambda^{1-x}-1)}}\right] \\ &= \frac{\ln\lambda}{2}\left[\frac{\lambda^x(\lambda^{1-y}-1)}{\sqrt{(\lambda^{1-y}-1)(\lambda^x-1)}}\left(1-\sqrt{\frac{\lambda^y-1}{\lambda^{1-x}-1}}\right)\right. \\ &\quad \left. + \frac{\lambda^{1-x}(\lambda^y-1)}{\sqrt{(\lambda^y-1)(\lambda^{1-x}-1)}}\left(\sqrt{\frac{\lambda^{1-y}-1}{\lambda^x-1}}-1\right)\right] . \end{aligned}$$

We distinguish two cases:

(i) *The case* $x+y \geq 1$. It holds that

$$1 - \sqrt{\frac{\lambda^y-1}{\lambda^{1-x}-1}} \leq 0 \quad \text{and} \quad \sqrt{\frac{\lambda^{1-y}-1}{\lambda^x-1}} - 1 \leq 0 ,$$

and therefore $h'_y(x) \leq 0$ for any $x \in [1-y, 1[$, i.e. h_y is decreasing on this interval and $h_y(1-y) \geq h_y(x)$.

(ii) *The case* $x + y \leq 1$. It holds that

$$1 - \sqrt{\frac{\lambda^y - 1}{\lambda^{1-x} - 1}} \geq 0 \quad \text{and} \quad \sqrt{\frac{\lambda^{1-y} - 1}{\lambda^x - 1}} - 1 \geq 0\,,$$

and therefore $h'_y(x) \geq 0$ for any $x \in]0, 1-y]$, i.e. h_y is increasing on this interval and $h_y(x) \leq h_y(1-y)$.

This completes the proof. □

This lemma will be invoked in the following theorem, which characterizes the transitivity of P generated from a weakly complete $T_{\mathbf{nM}}$-transitive large preference relation R by means of a Frank t-norm $T_\lambda^{\mathbf{F}}$ with $\lambda \in]0, \infty[$. The transitivity of P will be expressed by means of a t-norm as well: a φ-transform of the nilpotent minimum. For any $[0,1]$-automorphism φ, the φ-nilpotent minimum is the t-norm given by

$$T_{\mathbf{nM}}^{\varphi}(x,y) = \begin{cases} 0, & \text{if } \varphi(x) + \varphi(y) \leq 1\,, \\ \min(x,y), & \text{otherwise}\,. \end{cases}$$

Theorem 5. For any weakly complete reflexive fuzzy relation R with corresponding strict preference relation P generated by means of $i = T_\lambda^{\mathbf{F}}$, $\lambda \in]0, \infty[$, the following implication holds:

$$R \text{ is } T_{\mathbf{nM}}\text{-transitive} \quad \Rightarrow \quad P \text{ is } T_{\mathbf{nM}}^{\varphi_{1/\lambda}}\text{-transitive}\,,$$

with φ_λ defined by $\varphi_\lambda(x) = \phi_\lambda^{-1}\left(\sqrt{\phi_\lambda(x)}\right)$. Moreover, this is the strongest result possible.

Before proving this theorem, we first notice that φ_λ, as composition of three $[0,1]$-automorphisms, is itself a $[0,1]$-automorphism. It therefore makes sense to consider the φ_λ-nilpotent minimum. The automorphism φ_λ, $\lambda \in]0,1[\cup]1,\infty[$, is given explicitly by

$$\varphi_\lambda(x) = \log_\lambda \left(\sqrt{\frac{\lambda^x - 1}{\lambda - 1}}\,(\lambda - 1) + 1 \right)\,.$$

For $\lambda = 1$, we obtain $\varphi_1(x) = \sqrt{x}$. Note that it holds that $\lim_{\lambda \to 1} \varphi_\lambda = \varphi_1$. In that case, the t-norm $T_{\mathbf{nM}}^{\varphi_1}$ reads

$$T_{\mathbf{nM}}^{\varphi_1}(x,y) = \begin{cases} 0 & , \text{ if } \sqrt{x} + \sqrt{y} \leq 1\,, \\ \min(x,y) & , \text{ otherwise}\,. \end{cases}$$

Proof. In view of the definition of $T_{\mathbf{nM}}^{\varphi_{1/\lambda}}$, it is sufficient to prove that

$$\min(P(a,b), P(b,c)) \leq P(a,c)$$

whenever $\varphi_{1/\lambda}(P(a,b)) + \varphi_{1/\lambda}(P(b,c)) > 1$. By definition of P and taking into account the weak completeness of R, it follows that

$$\begin{aligned} P(a,b) &= T^{\mathbf{F}}_{1/\lambda}(R(a,b), 1 - R(b,a)) \\ &\leq T^{\mathbf{F}}_{1/\lambda}(R(a,b), R(a,b)) = \phi^{-1}_{1/\lambda}\left(\phi^2_{1/\lambda}(R(a,b))\right), \end{aligned}$$

or equivalently, $\varphi_{1/\lambda}(P(a,b)) \leq R(a,b)$. It then also holds that $\varphi_{1/\lambda}(P(b,c)) \leq R(b,c)$ and hence

$$R(a,b) + R(b,c) \geq \varphi_{1/\lambda}(P(a,b)) + \varphi_{1/\lambda}(P(b,c)) > 1\,.$$

Since R is $T_{\mathbf{nM}}$-transitive, it then holds that

$$T_{\mathbf{nM}}(R(a,b), R(b,c)) = \min(R(a,b), R(b,c)) \leq R(a,c)\,.$$

Without loss of generality, we can assume that $R(a,b) \leq R(b,c)$.

Now suppose that $R(c,a) > R(b,a)$, then the weak completeness of R implies that $1 \leq R(a,b) + R(b,a) < R(b,c) + R(c,a)$. Since R is $T_{\mathbf{nM}}$-transitive, it holds that

$$T_{\mathbf{nM}}(R(b,c), R(c,a)) = \min(R(b,c), R(c,a)) = R(b,c) \leq R(b,a)\,.$$

Moreover, the weak completeness of R implies that $R(c,a) + R(a,b) > 1$, whence again

$$T_{\mathbf{nM}}(R(c,a), R(a,b)) = \min(R(c,a), R(a,b)) = R(a,b) \leq R(c,b)\,.$$

Using these two inequalities, it follows with Lemma 1 that

$$\begin{aligned} &\varphi_{1/\lambda}(P(a,b)) + \varphi_{1/\lambda}(P(b,c)) \\ &= \phi^{-1}_{1/\lambda}\left(\sqrt{\phi_{1/\lambda}(R(a,b))\phi_{1/\lambda}(1 - R(b,a))}\right) + \phi^{-1}_{1/\lambda}\left(\sqrt{\phi_{1/\lambda}(R(b,c))\phi_{1/\lambda}(1 - R(c,b))}\right) \\ &\leq \phi^{-1}_{1/\lambda}\left(\sqrt{\phi_{1/\lambda}(R(a,b))\phi_{1/\lambda}(1 - R(b,c))}\right) + \phi^{-1}_{1/\lambda}\left(\sqrt{\phi_{1/\lambda}(R(b,c))\phi_{1/\lambda}(1 - R(a,b))}\right) \\ &\leq 1\,. \end{aligned}$$

This is clearly a contradiction. Therefore, it must hold that $R(c,a) \leq R(b,a)$. But then we have that

$$P(a,c) = T^{\mathbf{F}}_{1/\lambda}(R(a,c), 1 - R(c,a)) \geq T^{\mathbf{F}}_{1/\lambda}(R(a,b), 1 - R(b,a)) = P(a,b)\,,$$

which completes the proof of the implication.

It remains to be shown that no stronger result can be obtained. Consider a conjunctor g such that $g(x,y) > T_{\mathbf{nM}}^{\varphi_{1/\lambda}}(x,y)$ in some point $(x,y) \in]0,1[^2$. Suppose that $x \leq y$ (the case $y \leq x$ is completely similar). Define the set

$$C = \{(u,v) \in [0,1]^2 \mid \varphi_{1/\lambda}(u) + \varphi_{1/\lambda}(v) > 1\}\,.$$

Consider the following fuzzy relation R on $A = \{a,b,c\}$:

R	a	b	c
a	1	$\varphi_{1/\lambda}(x)$	$\varphi_{1/\lambda}(x) \cdot \chi_C(x,y)$
b	$1 - \varphi_{1/\lambda}(x)$	1	$\varphi_{1/\lambda}(y)$
c	$1 - \varphi_{1/\lambda}(x) \cdot \chi_C(x,y)$	$1 - \varphi_{1/\lambda}(y)$	1

where χ_C is the characteristic mapping of C. Then R is weakly complete and $T_{\mathbf{nM}}$-transitive but the strict preference relation P generated by means of $i = T_{\lambda}^{\mathbf{F}}$,

P	a	b	c
a	0	x	$T_{\mathbf{nM}}^{\varphi_{1/\lambda}}(x,y)$
b	$1 - 2\varphi_{1/\lambda}(x) + x$	0	y
c	$(1 - 2\varphi_{1/\lambda}(x) + x)\chi_C(x,y) + \chi_{C^c}(x,y)$	$1 - 2\varphi_{1/\lambda}(y) + y$	0

is not g-transitive, since $P(a,c) = T_{\mathbf{nM}}^{\varphi_{1/\lambda}}(x,y) < g(x,y) = g(P(a,b), P(b,c))$. □

Remark 2. It is easy to prove that $T_{\mathbf{nM}}^{\varphi_{1/\lambda}}$ is greater than $f_{1/\lambda}$, the strongest conjunctor in the general case. As illustration, consider for instance $\lambda = 4$, then we find:

$$T_{\mathbf{nM}}^{\varphi_{1/4}}(x,1/2) = \begin{cases} 0, & \text{if } x \leq \log_4(11 - 4\sqrt{6})\,, \\ \min(x,1/2), & \text{otherwise}, \end{cases}$$

and

$$f_{1/4}(x,1/2) = \begin{cases} 0, & \text{if } x \leq 1/2\,, \\ T_{1/4}^{\mathbf{F}}(x,1/2), & \text{otherwise}. \end{cases}$$

Since $\log_4(11 - 4\sqrt{6}) = 0.13 < 1/2$ and $\min \geq T_{1/4}^{\mathbf{F}}$ it holds that $T_{\mathbf{nM}}^{\varphi_{1/4}} > f_{1/4}$.

The final theorem of this paper concludes our study and shows that when the strict preference relation is generated by means of the minimum operator, the strongest result in the weakly complete case is much stronger than in the general case ($T_{\mathbf{L}}$-transitivity). In fact, the result obtained here is the strongest type of transitivity described by a conjunctor: min-transitivity.

Theorem 6. For any weakly complete reflexive fuzzy relation R with corresponding strict preference relation P generated by means of $i = T_{\mathbf{M}}$, the following implication holds:

$$R \text{ is } T_{\mathbf{nM}}\text{-transitive} \quad \Rightarrow \quad P \text{ is } T_{\mathbf{M}}\text{-transitive}\,.$$

Moreover, this is the strongest result possible.

Proof. Obviously, we only need to consider the case $\min(P(a,b),P(b,c)) > 0$. In that case, it holds that $P(a,b) = R(a,b) - R(b,a)$ with $R(a,b) > R(b,a)$, and $P(b,c) = R(b,c) - R(c,b)$ with $R(b,c) > R(c,b)$. Since R is weakly complete, it then follows that

$$R(a,b) + R(b,c) > R(b,a) + R(c,b) \geq 1 - R(a,b) + 1 - R(b,c)\,,$$

and hence $R(a,b) + R(b,c) > 1$. Since R is $T_{\mathbf{nM}}$-transitive, it follows that

$$R(a,c) \geq T_{\mathbf{nM}}(R(a,b),R(b,c)) = \min(R(a,b),R(b,c))\,.$$

Without loss of generality, we can assume that $R(a,b) \leq R(b,c)$ and hence $R(a,b) \leq R(a,c)$. We now distinguish two cases:

(i) *The case* $R(b,c) + R(c,a) > 1$. Since R is $T_{\mathbf{nM}}$-transitive, it holds that

$$R(b,a) \geq T_{\mathbf{nM}}(R(b,c),R(c,a)) = \min(R(b,c),R(c,a))\,.$$

Since $R(b,c) \geq R(a,b) > R(b,a)$ it must hold that $\min(R(b,c),R(c,a)) = R(c,a)$ and $R(b,a) \geq R(c,a)$.

(ii) *The case* $R(b,c) + R(c,a) \leq 1$. Since R is weakly complete, it then holds that

$$R(c,a) \leq 1 - R(b,c) \leq 1 - R(a,b) \leq R(b,a)\,.$$

In both cases, we obtain $R(c,a) \leq R(b,a)$. Together with $R(a,b) \leq R(a,c)$ it finally follows that

$$\begin{aligned}\min(P(a,b),P(b,c)) &\leq P(a,b) = R(a,b) - R(b,a)\\ &\leq R(a,c) - R(c,a)\\ &= \max(R(a,c) - R(c,a),0) = P(a,c)\,.\end{aligned}$$

This completes the proof. □

It would be desirable to be able to combine Theorems 4–6 into a single theorem. We therefore consider the limits of the $[0,1]$-automorphisms φ_λ for $\lambda \to 0$ and $\lambda \to \infty$. It is easily verified that

$$\lim_{\lambda\to 0} \varphi_\lambda(x) = x$$

and hence $\lim_{\lambda\to 0} T^{\varphi_\lambda}_{\mathbf{nM}} = T_{\mathbf{nM}}$. On the other hand, it holds that

$$\lim_{\lambda\to\infty} \varphi_\lambda(x) = m(x) = \begin{cases} \frac{1+x}{2}\,, & \text{if } x \in]0,1]\,,\\ 0\,, & \text{if } x = 0\,.\end{cases}$$

Although m is clearly not a $[0,1]$-automorphism, it holds that

$$\lim_{\lambda\to\infty} T^{\varphi_\lambda}_{\mathbf{nM}} = T_{\mathbf{M}}\,,$$

i. e.

$$T_{\mathbf{M}}(x,y) = \begin{cases} 0, & \text{if } m(x)+m(y) \leq 1\,, \\ \min(x,y), & \text{otherwise}. \end{cases}$$

Introducing the notations $\varphi_0(x) = x$ and $\varphi_\infty(x) = m(x)$ (stressing once more that the latter is not a $[0,1]$-automorphism) we can write $T_{\mathbf{nM}}^{\varphi_0} = T_{\mathbf{nM}}$ and $T_{\mathbf{nM}}^{\varphi_\infty} = T_{\mathbf{M}}$.

Summarizing Theorems 4 – 6, we can write

Corollary 1. For any weakly complete reflexive fuzzy relation R with corresponding strict preference relation P generated by means of $i = T_\lambda^{\mathbf{F}}$, $\lambda \in [0,\infty]$, the following implication holds:

$$R \text{ is } T_{\mathbf{nM}}\text{-transitive} \quad \Rightarrow \quad P \text{ is } T_{\mathbf{nM}}^{\varphi_{1/\lambda}}\text{-transitive}.$$

Moreover, this is the strongest result possible.

7. CONCLUSION

In this paper, we have studied the propagation of the $T_{\mathbf{nM}}$-transitivity of large preference relations to indifference and strict preference relations generated by means of t-norm generators, i. e. by means of Frank t-norms. The strongest types of transitivity that can be assured both in general and in the case of weakly complete large preference relations have been identified. In particular, we have shown that the transitivity of the indifference relation is not influenced by the weak completeness of the large preference relation, while for the transitivity of the strict preference relation, except for a limit case, there is a considerable improvement. These results are summarized in Table 1 which contains the conjunctors characterizing the transitivity of the indifference relation I and strict preference relation P generated from a $T_{\mathbf{nM}}$-transitive large preference relation R by means of $i = T_\lambda^{\mathbf{F}}$. We hope to extend these results in the future to a more general class of rotation-invariant t-norms.

Table 1. Transitivity of I and P generated from a $T_{\mathbf{nM}}$-transitive R by means of $i = T_\lambda^{\mathbf{F}}$.

	General R	Weakly Complete R	Strongly Complete R
Transitivity of I	f_λ	f_λ	$T_{\mathbf{nM}}$
Transitivity of P	$f_{1/\lambda}$	$T_{\mathbf{nM}}^{\varphi_{1/\lambda}}$	$T_{\mathbf{M}}$

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