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TWOFOLD INTEGRAL AND MULTI-STEP CHOQUET INTEGRAL

YASUO NARUKAWA AND VICENÇ TORRA

In this work we study some properties of the twofold integral and, in particular, its relation with the 2-step Choquet integral. First, we prove that the Sugeno integral can be represented as a 2-step Choquet integral. Then, we turn into the twofold integral studying some of its properties, establishing relationships between this integral and the Choquet and Sugeno ones and proving that it can be represented in terms of 2-step Choquet integral.

Keywords: aggregation, Choquet and Sugeno integrals, multi-step integral, twofold integral
AMS Subject Classification: 28E10, 28E05, 03H05, 68T37

1. INTRODUCTION

Choquet and Sugeno integrals are one of the most well known integrals to operate with fuzzy measures. In both cases, the functional calculates the integral of a function with respect to a fuzzy measure.

In 1991, Murofushi and Sugeno [8] proposed the fuzzy t-conorm integral to unify both integrals in a single framework. The generalization is based on the definition of a t-conorm system for integration that generalizes the following pairs of operations: the product and sum (used in the Choquet integral) and the minimum and maximum (used in the Sugeno integral). t-norm-like and t-conorm operators are used for this generalization. A deeper study and overview over this topic can be found in the book [2].

The twofold integral, proposed in [16], is an alternative generalization. Roughly speaking, the generalization process is as follows. Instead of building the new integral in terms of operators generalizing both ($\cdot$ and min) and ($+$ and max), it defines the integral considering all these terms and, additionally, two fuzzy measures (the one used in the Sugeno integral and the one in the Choquet integral).

The rationale of the approach is that the semantics of both measures are different. In particular, the one in the Choquet integral is seen as a "probabilistic-flavor" measure and the one in the Sugeno integral is seen as a "fuzzy-flavor" measure. Due to their semantic difference, the generalization – the twofold integral – considers both. Then, it was proven [16] that with a particular selection of these measures, the twofold integral either reduces to the Choquet integral or to the Sugeno integral.

Since 1995, hierarchical Choquet integral or, in other words, 2-step Choquet integral, has been studied. Sugeno, Fujimoto and Murofushi [9, 15] present the conditions for Choquet integral to be decomposable into an equivalent hierarchical Choquet integral. Mesiar and Vivona [4] present some properties of the 2-step Choquet integral. Benvenuti and Mesiar [1] consider the functional which is monotone, homogeneous and additive homogeneous and provide the hypothesis below, which remains an open problem. "A lower semi continuous functional which is monotone, homogeneous and additively homogeneous can be represented as a many step Choquet integral." Narukawa and Murofushi [7] show that the above hypothesis is not always true.

In this work we further study those integrals. In particular, we show that the twofold integral as well as the Sugeno integral can be represented as a 2-step Choquet integral with constant and then we study some of the properties of the former integral. In particular, we study the integral of a crisp set and some of the relationships with Sugeno and Choquet integrals.

The structure of the paper is as follows. In Section 2, we define the twofold integral and prove its relation with 2-step Choquet integrals. Then, in Section 3, we present some properties about the integral. Finally, in Section 4, we finish with some conclusions.

2. MULTI-STEP CHOQUET INTEGRAL

In this section we give some basic definitions that are needed latter on and we present some results in relation to multi-step Choquet integral. First, we present the definition of fuzzy measures (including an example of the fuzzy measure representing complete ignorance) and, then, Choquet, multi-step Choquet and Sugeno integrals. Finally, we prove that a Sugeno integral can be represented as a 2-step Choquet integral.

In this paper, we assume that the universal set X is a finite set, that is, $X := \{x_1, x_2, \ldots, x_n\}$.

Definition 1. (See [14].) A fuzzy measure μ on $(X, 2^X)$ is a real valued set function,

$$\mu : 2^X \longrightarrow [0,1]$$

with the following properties:

1. $\mu(\emptyset) = 0$, $\mu(X) = 1$
2. $\mu(A) \le \mu(B)$
   whenever $A \subset B$, $A, B \in 2^X$.

Definition 2. The fuzzy measure representing complete ignorance, denoted by μ^*, is defined as follows:

$$\mu^*(A) = \begin{cases} 1 & \text{for all } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}$$

Definition 3. (See [5, 7].) Let μ be a fuzzy measure on $(X, 2^X)$. The Choquet integral $C_\mu(f)$ of $f : X \to R_+$ with respect to μ is defined by

$$C_\mu(f) = \sum_{j=1}^{n} f(x_{s(j)})(\mu(A_{s(j)}) - \mu(A_{s(j+1)}))$$

where $f(x_{s(i)})$ indicates that the indices have been permuted so that

$$0 \leq f(x_{s(1)}) \leq \ldots \leq f(x_{s(n)}) \leq 1, A_{s(i)} = \{x_{s(i)}, \ldots, x_{s(n)}\}, A_{s(n+1)} = \emptyset.$$

The Choquet integral with constant b of $f : X \to R_+$ with respect to μ is defined by

$$C_{\mu,b}(f) := C_\mu(f) + b.$$

Definition 4. Define the Dirac measure δ_x of x for $x \in X$ by

$$\delta_x(E) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if o.w.} \end{cases}$$

for $E \in 2^X$.

Then we have the following property: $C_{\delta_x}(f) = f(x)$.

Definition 5. (See [6, 7, 10].) Let X be a finite set with $|X| = n$ and $\mathcal{I}$ be a functional $\mathcal{I} : R_+^n \to R_+$. A 1-step Choquet integral with constant is defined by

$$\mathcal{I}(x) = C_{\mu,b}(x)$$

for $x \in R_+^n$, and a fuzzy measure μ on 2^X. The functional $\mathcal{I}$ is said to be a k-step Choquet integral with constant if there exist a natural number m_k, $k_j (k_j < k)$ step Choquet integrals with constant $\mathcal{I}_j : R_+^n \to R_+$ for $j = 1, \ldots, m_k$ and a fuzzy measure μ_k on $2^{\{1,\ldots,m_k\}}$ such that $k = \max\{k_j | j = 1, \ldots, m_k\} + 1$ and

$$\mathcal{I}(x) = C_{\mu_{\mu_k, b_k}}(\mathcal{I}_j(x)).$$

Figure 1 gives a graphical representation of 2-step Choquet integrals.

Example 1. Let X be $X = \{1, 2, 3, 4\}$. Consider the function $f : X \to R_+$ such that $f(1) \leq f(2) \leq f(3) \leq f(4)$. Suppose that the Choquet integrals with constant $b_{1i}; i = 1, 2, 3$ with respect to the fuzzy measures μ_{11},μ_{12} and μ_{13}, respectively, satisfy

$$C_{\mu_{\mu_{11}, b_{11}}}(f) \leq C_{\mu_{\mu_{12}, b_{12}}}(f) \leq C_{\mu_{\mu_{13}, b_{13}}}(f).$$

Let $M_2 := \{1, 2, 3\}$ and $A_{21} := \{1, 2, 3\}$,$A_{22} := \{2, 3\}$,$A_{23} := \{3\}$ and $\mu_{2,j}; j = 1, 2$ be a fuzzy measure on 2^{M_2}. The 2-step Choquet integral $TwC_{\mu_{\mu_{2i}, b_{2i}}}(f); i = 1, 2$ with constant b_{2i} is defined by

$$TwC_{\mu_{\mu_{2i}, b_{2i}}}(f) := \sum_{j=1}^{3} C_{\mu_{\mu_{1j}, b_{1j}}}(f)(\mu_{2i}(A_{2j}) - \mu_{2i}(A_{2\ j+1})) + b_{2i}$$

where $i = 1, 2$ and $A_{24} := \emptyset$. Suppose that $TwC_{\mu_{\mu_{21}, b_{21}}}(f) \le TwC_{\mu_{\mu_{22}, b_{22}}}(f)$. Let $M_3 := \{1, 2\}$ and $A_{31} := \{1, 2\}$, $A_{32} := \{2\}$, $A_{33} := \emptyset$ and μ_3 be a fuzzy measure on 2^{M_3}. The 3-step Choquet integral $ThrC_{\mu_{\mu_3, b_3}}(f)$ with constant b_3 is defined by

$$ThrC_{\mu_{\mu_3, b_3}}(f) := \sum_{j=1}^{2} C_{\mu_{\mu_{2j}, b_{2j}}}(f)(\mu_3(A_{3j}) - \mu_3(A_{3\ j+1})) + b_3.$$

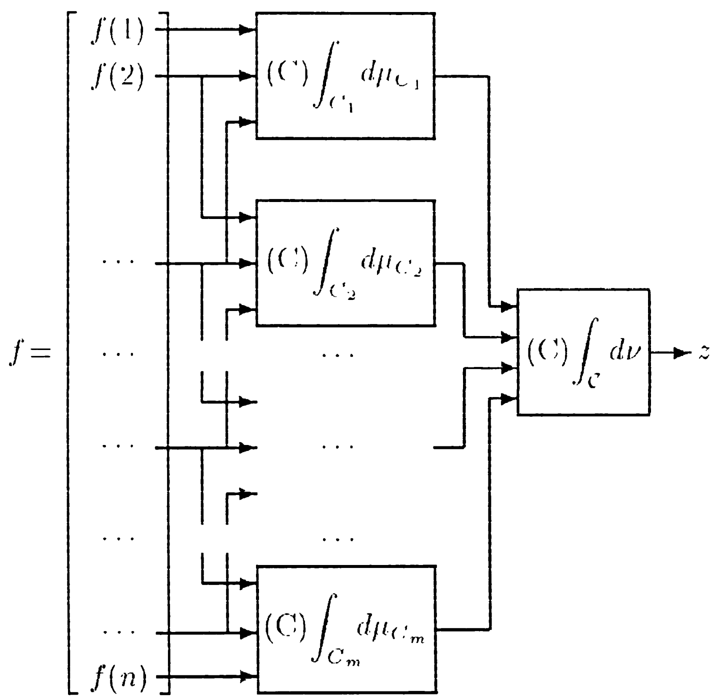

Fig. 1. 2-step Choquet integral.

Definition 6. (See [14].) The Sugeno integral $S_\mu(f)$ of a function $f : X \to [0, 1]$ with respect to μ is defined by

$$S_\mu(f) := \bigvee_{j=1}^{n} f(x_{s(j)}) \wedge \mu(A_{s(j)})$$

where $f(x_{s(i)})$ indicates that the indices have been permuted so that

$$0 \le f(x_{s(1)}) \le \dots \le f(x_{s(n)}) \le 1, A_{s(i)} = \{x_{s(i)}, \dots, x_{s(n)}\}, A_{s(n+1)} = \emptyset.$$

Proposition 1. Sugeno integral can be represented as a 2-step Choquet integral with constant.

Proof. Let $X' := X \cup \{2^X \setminus \{\emptyset\}\}$. Define $0-1$ fuzzy measure on $(X', 2^{X'})$ by

$$\nu(E) := \begin{cases} 1 & \text{if } A \cup \{A\} \subset E \text{ for some } A \in 2^X \setminus \{\emptyset\} \\ 0 & \text{o.w.} \end{cases}$$

for $E \in 2^{X'}$ Then we have

$$C_\nu(F) = \sup_{A \in 2^X \setminus \{\emptyset\}} \inf_{x \in A \cup \{A\}} F(x), \ldots\ldots \qquad (1)$$

where F is a function from X' to $[0,1]$.

Note that in fact, it follows from the definition of ν that

$$C_\nu(F) = \sup_{A \in 2^X \setminus \{\emptyset\}} C_{\mu_A}(F)$$

where μ_A is a 0-1 a possibility measure defined by

$$\mu_A(E) := \begin{cases} 1 & \text{if } A \cup \{A\} \subset E \\ 0 & \text{o.w.} \end{cases}$$

for $E \in 2^{X'}$. Then, since $C_{\mu_A}(F) = \inf_{x \in A \cup \{A\}} F(x)$, we have

$$C_\nu(F) = \sup_{A \in 2^X \setminus \{\emptyset\}} \inf_{x \in A \cup \{A\}} F(x).$$

Next, for each ω in $2^{X'}$ we define a 1st step Choquet integral denoted $C_\omega(f)$. This is a Choquet integral with constant of the function f with respect to the measure ν_ω. The Choquet integrals are defined as follows:

$$C_\omega(f) := \begin{cases} C_{\nu_\omega}(f) & \text{if } \omega = x \in X \\ C_{\nu_\omega}(f) + \mu(A) & \text{if } \omega = A \in 2^X \setminus \{\emptyset\} \end{cases}$$

and where the fuzzy measure ν_ω on $(X, 2^X)$ are defined by

$$\nu_\omega := \begin{cases} \delta_x & \text{if } \omega = x \in X \\ 0 & \text{if } \omega = A \in 2^X \setminus \{\emptyset\} \end{cases}$$

where δ_x is as in Definition 4. Since $C_\omega(f)$ is a function from X' to $[0,1]$ it follows from Equation (1) that

$$C_\nu(C_\omega(f)) = \sup_{A \in 2^X \setminus \{\emptyset\}} \inf_{\omega \in A \cup \{A\}} C_\omega(f).$$

If there exists $x \in A$ such that $f(x) \le \mu(A)$, we have

$$\inf_{\omega \in A} C_\omega(f) = \inf_{x \in A} f(x).$$

If $f(x) \geq \mu(A)$ for all $x \in A$, we have

$$\inf_{\omega \in A} C_\omega(f) = \mu(A).$$

Therefore we have

$$\inf_{\omega \in A \cup \{A\}} C_\omega(f) = (\inf_{x \in A} f(x)) \wedge \mu(A). \ldots\ldots \tag{2}$$

It follows from the definition of the Sugeno integral that

$$C_\nu(C_\omega(f)) \geq S_\mu(f)$$

On the other hand, since X is finite, there exist a positive integer k such that $S_\mu(f) = f(x_{s_k}) \wedge \mu(A_{s_k})$. Suppose that

$$C_\nu(C_\omega(f)) = \sup_{A \in 2^X \setminus \{\emptyset\}} \inf_{\omega \in A \cup \{A\}} C_\omega(f) > S_\mu(f).$$

Then there exist $A \in 2^X$ such that

$$\inf_{\omega \in A \cup \{A\}} C_\omega(f) > f(x_{s_k}) \wedge \mu(A_{s_k}).$$

Since A is a finite set, there exists a positive integer k_0 such that $f(x_{s_{k_0}}) = \inf_{x \in A} f(x)$. Since $A \subset A_{s_{k_0}}$, it follows from the Equation 2 that

$$f(x_{s_{k_0}}) \wedge \mu(A_{s_{k_0}}) \geq f(x_{s_{k_0}}) \wedge \mu(A) > f(x_{s_k}) \wedge \mu(A_{s_k}).$$

This contradicts to the definition of S_μ. Therefore we have

$$C_\nu(C_\omega(f)) \leq S_\mu(f).$$

□

Definition 7. A functional $\mathcal{I} : R_+^n \to R_+$ is said to be piecewise linear (for short PL) if there exists a finite family $\mathcal{D}_j : j \in J$ of closed domains such that

$$\bigcup_{j \in J} \mathcal{D}_j = R^n$$

and the restriction of $\mathcal{I}$ on every $\mathcal{D}_j$ is linear. A unique linear functional $\mathcal{J}_j$ on R_+^n which coincided with $\mathcal{I}$ on a given $\mathcal{D}_j, j \in J$ is said to be a component of $\mathcal{I}$.

It is obvious that the Choquet integral is a PL functional. Ovchinnikov [13] shows that the Sugeno integral is a PL functional. As composition of PL functionals is PL, Proposition 1 also implies this result.

3. TWOFOLD INTEGRAL

This section studies some properties of the twofold integral. We start reviewing its definition.

Definition 8. (See [16].) Let μ_C and μ_S be two fuzzy measures on X, then the *twofold integral* of a function $f : X \to [0,1]$ with respect to the fuzzy measures μ_S and μ_C is defined by:

$$TI_{\mu_S,\mu_C}(f) = \sum_{i=1}^{n} \Big(\big(\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \big) \big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)}) \big) \Big) \tag{3}$$

where $f(x_{s(i)})$ indicates that the indices have been permuted so that

$$0 \leq f(x_{s(1)}) \leq \ldots \leq f(x_{s(n)}) \leq 1, A_{s(i)} = \{x_{s(i)}, \ldots, x_{s(n)}\}, A_{s(n+1)} = \emptyset.$$

Now, we consider the relation between the twofold integral and Choquet and Sugeno integrals.

Proposition 2. (See [16].) The twofold integral satisfies the following properties.

When $\mu_C = \mu^*$, the twofold integral reduces to the Sugeno integral:

$$TI_{\mu_S,\mu_C}(a_1, \ldots, a_n) = SI_{\mu_S}(a_1, \ldots, a_n)$$

When $\mu_S = \mu^*$, the twofold integral reduces to the Choquet integral:

$$TI_{\mu_S,\mu_C}(a_1, \ldots, a_n) = CI_{\mu_C}(a_1, \ldots, a_n)$$

When $\mu_C = \mu_S = \mu^*$, the twofold integral reduces to the maximum:

$$TI_{\mu_S,\mu_C}(a_1, \ldots, a_n) = \bigvee(a_1, \ldots, a_n)$$

Additionally, the twofold integral satisfies the basic properties of aggregation operators. This is, it is monotonic, satisfies unanimity and, therefore, it yields a value between the minimum and the maximum.

Proposition 3. (See [16].) Let X be a finite set and let μ_C and μ_S be two fuzzy measures on X, then

for all functions f_1 and f_2 over X such that $f_2(x) \geq f_1(x)$ for all $x \in X$:

$$TI_{\mu_S,\mu_C}(f_2) \geq TI_{\mu_S,\mu_C}(f_1),$$

for all $\mathbf{a} = (a, \ldots, a)$,

$$TI_{\mu_S,\mu_C}(\mathbf{a}) = a,$$

for all functions f on X,

$$\min_{x\in X} f(x) \leq TI_{\mu_S,\mu_C}(f) \leq \max_{x\in X} f(x).$$

Now, we study some new properties of this integral. We start considering the integration of the characteristic function of a set A and proving that in this case, the integral is the product of the two measures for this set.

Proposition 4. Let A be a subset of X and let 1_A be the characteristic function of A (this is, f is defined for x as one if and only if $x \in A$ and zero otherwise), then the twofold integral of 1_A with respect to the two measures μ_S and μ_C is equal to:

$$TI_{\mu_S,\mu_C}(1_A) = \mu_S(A) \cdot \mu_C(A)$$

.

Proof. First, let us note that $f(x_{s(j)})$ is ordered with respect to s so that $f(x_{s(i)}) = 0$ for $i < |X| - |A| + 1$ and that $f(x_{s(i)}) = 1$ for all $i \geq |X| - |A| + 1$.

Therefore, the terms

$$\bigvee_{j=1}^{i} 1_A(x_{s(j)}) \wedge \mu_S(A_{s(j)})$$

are equal to 0 for $i < |X| - |A| + 1$ and equal to $\mu_S(A)$ for $i \geq |X| - |A| + 1$.

Now, replacing these values in the twofold integral we get:

$$\sum_{i=1}^{|X|-|A|} \Big(0\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big)\Big)$$
$$+ \sum_{i=|X|-|A|+1}^{n} \Big(\mu_S(A)\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big)\Big)$$

In this expression, the first term is zero and the second can be rewritten as:

$$\mu_S(A)\big(\mu_C(A_{s(|X|-|A|+1)}) - \mu_C(A_{s(n+1)})\big)$$

That, being $\mu_C(A_{s(n+1)}) = 0$ because $A_{s(n+1)} = \emptyset$, and being $\mu_C(A_{s(|X|-|A|+1)}) = \mu_C(A)$ because $A_{s(|X|-|A|+1)} = A$, is equivalent to:

$$\mu_S(A)\mu_C(A_{s(|X|-|A|+1)}) = \mu_S(A)\mu_C(A).$$

Therefore, the proposition is proven. □

Proposition 5. For all f, the following inequality holds:

$$TI_{\mu_S,\mu_C}(f) \leq SI_{\mu_S}(f)$$

where SI stands for the Sugeno integral.

Proof. Let us define α as follows: $\alpha := SI_{\mu_S}(f) = \bigvee_{j=1}^{n} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})$. From this definition, it is clear that the following inequality holds:

$$\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq \alpha$$

for all i. From this inequality, we can proof the following:

$$\sum_{i=1}^{n} \Big(\big(\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \big) \big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)}) \big) \Big)$$

$$\leq \sum_{i=1}^{n} \Big(\alpha \big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)}) \big) \Big)$$

As the right hand side of this inequality is equal to α, the proposition is proven. □

Proposition 6. For all f, the following inequality holds:

$$TI_{\mu_S,\mu_C}(f) \leq CI_{\mu_C}(f)$$

where CI stands for the Choquet integral.

Proof. First, we shall prove that $\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq f(x_{s(i)})$. This is so, since

$$f(x_{s(1)}) \leq f(x_{s(2)}) \leq \ldots \leq f(x_{s(i)})$$

and, additionally,

$$f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq f(x_{s(j)}).$$

Therefore, the following holds:

$$\sum_{i=1}^{n} \Big(\big(\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \big) \big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)}) \big) \Big)$$

$$\leq \sum_{i=1}^{n} \Big(f(x_{s(i)}) \big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)}) \big) \Big)$$

As the right hand side of this expression is the Choquet integral of f with respect to the measure μ_C the proposition is proven. □

Since monotone convergence theorem is valid for both Choquet and Sugeno integral with finite universal set X, it is also valid for the twofold integral.

Theorem 1. Let μ_C and μ_S be two fuzzy measures on $(X, 2^X)$. If the monotone increasing sequence $\{f_n\}$ of functions $f_n : X \to [0,1]$ converge to a function f, that is, $f_n \uparrow f$, then

$$TI_{\mu_S,\mu_C}(f_n) \uparrow TI_{\mu_S,\mu_C}(f)$$

as $n \to \infty$.

Proof. Since X is finite, the fuzzy measure μ_S is continuous. So we have

$$\bigvee_{j=1}^{i} f_n(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \uparrow \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})$$

as $n \to \infty$. Since μ_C is also continuous, we have

$$C_{\mu_C}(\bigvee_{j=1}^{i} f_n(x_{s(j)}) \wedge \mu_S(A_{s(j)})) \uparrow C_{\mu_C}(\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}))$$

as $n \to \infty$. □

Calvo et al [3] propose a new construction method for aggregation operators based on composition of aggregation operators, that is one of the generalizations of twofold integral on a finite universal set. They present the generalized proposition of Proposition 4 and Theorem 1.

Narukawa and Torra [11] present a definition of twofold integral generalized to an arbitrary universal set X and show Proposition 5 and 6 generally.

Applying Proposition 1, it is obvious that the twofold integral is represented as 3-step Choquet integral with constant. Moreover since 3-step Choquet integral with constant is piecewise linear, the twofold integral is represented as 2-step integral with constant.

Theorem 2. The twofold integral is represented as the 2-step Choquet integral with constant.

Proof. Let $f \in R^n$ and

$$S_{\mu_{S_i}}(f) := \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}).$$

Suppose that $f \leq g$ for $f, g \in R^n$. Then we have $S_{\mu_{S_i}}(f) \leq S_{\mu_{S_i}}(g)$. Since Choquet integral is monotone, we have

$$C_{\mu_C}(S_{\mu_{S_i}}(f)) \leq C_{\mu_C}(S_{\mu_{S_i}}(g)).$$

That is, twofold integral is monotone. Since Sugeno integral is piecewise linear, there exists a set $\mathcal{D}$ of functions such that

$$S_{\mu_{S_i}}(f+g) = S_{\mu_{S_i}}(f) + S_{\mu_{S_i}}(g)$$

for $f, g \in D$. Suppose that $\mathcal{D}' \subset R^n$ is a set of functions satisfying

$$S_{\mu_{S_i}}(f) < S_{\mu_{S_j}}(f) \Rightarrow S_{\mu_{S_i}}(g) \leq S_{\mu_{S_j}}(g).$$

Then the twofold integral TI is piecewise linear on $\mathcal{D} \cap \mathcal{D}'$. Therefore applying Ovchinnikov's theorem [12] TI has a max-min representation. It is shown similarly to the proof of Proposition 1 that the max-min representation is represented by 2-step Choquet integral with constant. □

Remark. Benvenuti and Mesiar [1] proposed the following hypothesis:

Hypothesis 1. A lower semi-continuous functional $\mathcal{I} : R_+^n \to R_+$ which is monotone, homogeneous and additively homogeneous can be represented as an n-step Choquet integral, where lower semi continuous means that $x_n \uparrow x$ implies $\mathcal{I}(x_n) \uparrow \mathcal{I}(x)$. We say that the functional I on R_+^n is additively homogeneous if $I(f + a) = I(f) + a$ for $f \in R_+^n$ and $a \geq 0$ is constant.

Narukawa and Murofushi [7] presented the counterexample that show that the above hypothesis is not always true. Conversely since the constant a is comonotonic with every function f, it is obvious that the $n-$ step Choquet integral is monotone, homogeneous and additively homogeneous.

The Choquet integral with constant is a one example of monotone and additively homogeneous functional. In fact, monotonicity is obvious and since $\mu(X) = 1$ for all fuzzy measures in this paper, we have

$$C_{\mu,b}(f + a) = C_\mu(f + a) + b = C_\mu(f) + a + b = C_{\mu,b}(f) + a, \tag{4}$$

that is, the Choquet integral with constant is an additively homogeneous functional. On the other hand, an $n-$ step Choquet integral with constant is not always additively homogeneous, since the Sugeno integral, that is 2-step Choquet integral with constant, is not always additively homogeneous.

Theorem 1 shows that the twofold integral is lower semi-continuous in the Benvenuti sense. Twofold integral is also one of the examples of 2-step Choquet integrals with constant that are lower semincontinuous but not additively homogeneous.

4. CONCLUSIONS

In this work we have studied the 2-step Choquet integral. We have shown that the Sugeno integral can be represented in terms of the 2-step Choquet integral. Then we have revised the TI and given some new results. In particular, we have established some relations with the Sugeno and Choquet integrals. Moreover, we have also proven that the twofold integral can be represented in terms of the 2-step Choquet integral.

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