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T–EQUIVALENCES GENERATED BY SHAPE FUNCTION ON THE REAL LINE

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This paper is devoted to give a new method of generating T-equivalence using shape function and finding the exact calculation formulas of T-equivalence induced by shape function on the real line. Some illustrative examples are given.

Keywords: fuzzy number, fuzzy relation, T-norm, T-equivalence, shape function
AMS Subject Classification: 26A21, 03E02

1. INTRODUCTION

For the fuzzy set-theoretical modelling of verbal quantities and computing with these quantities, it appears useful to part the class of real numbers into fuzzy equivalence classes. Jacas and Recasens [8] considered the idea of generating fuzzy numbers as equivalence classes of a T-indistinguishability operator based on a scale function. The theoretical approach suggested in [10] and further developed in [11] indicates that partitions based on the concept of a shape function can be especially significant. De Baets et al [2] and Marková [12] characterized that the shapes by means of which T-equivalences can be generated, are based on the knowledge of idempotents of the T-addition of fuzzy numbers.

In this paper, we give a new method of generating T-equivalence using shape function and finding the exact calculation formulas of T-equivalence induced by shape function on the real line. Some illustrative examples are given.

2. PRELIMINARIES

Definition 1. (Jacas and Recasens [8]) A fuzzy number is a mapping $A : R \to [0,1]$ such that there exists $a \in R$ with $A(a) = 1$ and A is increasing on $(-\infty, a]$ and A is decreasing on $[a, \infty)$.

Definition 2. (De Baets and Mesiar [3]) Consider a t-norm T. A binary fuzzy relation E on an universe X is called a T-equivalence on X if and only if it is reflexive, symmetric and T-transitive, i.e. if and only if for any (x, y, z) in X^3:

(i) $E(x, x) = 1$;

(ii) $E(x,y) = E(y,x)$;

(iii) $T(E(x,y),\ E(y,z)) \le E(x,z)$.

Definition 3. (Jacas and Recasens [8]) A scale is a continuous non-decreasing surjective monotonic mapping $S : R \to R$.

Definition 4. A shape is a non-increasing mapping $\phi : R^+ \to [0,1]$ such that $\phi(0) = 1$.

Definition 5. A mapping $d : X^2 \to [0,\infty]$ is called a pseudo-metric on X if and only if for any (x,y,z) in X^3

(i) $d(x,x) = 0$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \le d(x,y) + d(y,z)$.

It is called a metric if it moreover satisfies, for any $(x,y) \in X^2$

(iv) $d(x,y) = 0 \Leftrightarrow x = y$.

Consider a scale s, then the mapping $d_s : R^2 \to R^+$ defined by

$$d_s(x,y) = |s(x) - s(y)|$$

is a pseudo-metric on R. Now consider a shape ϕ, then we construct the binary fuzzy relation $E_{s,\phi}$ as follows:

$$E_{s,\phi}(x,y) = \phi(|s(x) - s(y)|).$$

Definition 6. A generator (or source of vagueness) g is a scale such that $g(0) = 0$.

A function $T : [0,1] \times [0,1] \to [0,1]$ is said to be a triangular norm [9, 14] (t-norm for short) iff T is symmetric, associative, non-decreasing in each argument, and $T(x,1) = x$ for all $x \in [0,1]$, and, in general, $T(x_1,\cdots,x_n) = T(T(\ldots T(T(x_1,x_2),x_3),\ldots,x_{n-1}),x_n)$. Some well-known continuous t-norms are the minimum operator T_M, the algebraic product T_P and the Lukasiewicz t-norm T_L defined by $T_L(x,y) = \max(x+y-1,0)$. The minimum operator T_M is the strongest (greatest) t-norm. The weakest (smallest) t-norm T_W is defined by

$$T_W(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0, & \text{elsewhere.} \end{cases}$$

We will call t-norm T is Archimedean if and only if T is continuous and $T(x,x) < x$ for all $x \in (0,1)$. Every Archimedean t-norm T is representable by a continuous and decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ and

$$T(x_1,\cdots,x_n) = f^{[-1]}(f(x_1) + \cdots + f(x_n))$$

for all $x_i \in [0,1]$, $1 \le i \le n$, where $f^{[-1]}$ is the pseudo-inverse of f, defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)], \\ 0 & \text{if } y \in [f(0), \infty]. \end{cases}$$

The function f is the additive generator of T. If $T = T_P$, then $f(x) = \log x^{-1}$ and if $T = T_L$, then $f(x) = 1 - x$.

For arbitrary fuzzy numbers A_i, $i = 1, \cdots, n$, $n \in N$, on the real line, their T-sum is defined by means of the extension principle as follows:

$$A_1 \oplus_T \cdots \oplus_T A_n(z) = \sup_{x_1+\cdots+x_n=z} T(A_1(x_1), \ldots, A_n(x_n)), \ z \in R.$$

Definition 7. Let J be a finite or countable set. Let $\{T_i | i \in J\}$ be a collection of t-norms and $\{(a_i, b_i) | i \in J\}$ a collection of disjoint intervals in $[0,1]$. We call ordinal sum of t-norms $\{T_i | i \in J\}$ to the following t-norm :

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) T_i \left(\dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i} \right) & \text{whenever } (x,y) \in (a_i, b_i)^2 \\ & \qquad = (a_i, b_i) \times (a_i, b_i), \\ \min(x,y) & \text{otherwise}, \end{cases}$$

which is denoted by $T = (\langle a_i, b_i, T_i \rangle | i \in J)$, and only if all T_i are generated, then equivalently it can be used $T = (\langle a_i, b_i, f_i \rangle | i \in J)$ where f_i is the additive generator of T_i.

The following theorem gives a general classification of continuous t-norms [9].

Theorem 1. (Ling [9]) Let T be a continuous t-norm. Then T is Archimedean or T-min or T is an ordinal sum of Archimedean t-norms.

3. T-EQUIVALENCE GENERATED BY SHAPES

Consider a generator g and a shape ϕ, and the fuzzy relation $E_{g,\phi}$, which is always reflexive and symmetric. Let T be a t-norm and $\phi_n = \phi \oplus_T \cdots \oplus_T \phi$ (n-fold T-sum of ϕ). Then $\phi_n(x) \le \phi_{n+1}(x)$ for any $x \in R$ and for $n \in N$, the natural numbers. Hence the limit always exists. Let $\lim_{n\to\infty} \phi_n \equiv \phi^*$. We also note that if we define $|\phi| : R \to [0,1]$ such that $|\phi|(z) = \phi(|z|)$ and $|\phi|_n = |\phi| \oplus_T \cdots \oplus_T |\phi|$, then $\lim_{n\to\infty} |\phi|_n \equiv |\phi|^* = |\phi^*|$.

Theorem 2. For a continuous t-norm T, a generator g and a shape ϕ, the fuzzy relation E_{f,ϕ^*} is a T-equivalence on R.

Proof. We only need to show that for any $a, b, y \in R$

$$T(E_{g,\phi^*}(a,y), E_{g,\phi^*}(y,b)) \le E_{g,\phi^*}(a,b),$$

or equivalently

$$T(|\phi|^*(g(y)-g(a)),|\phi|^*(g(b)-g(y)))\le|\phi|^*(g(b)-g(a)). \tag{1}$$

By the continuity of the t-norm T, we have

$$\begin{aligned}&T(|\phi|^*(g(y)-g(a)),|\phi|^*(g(b)-g(y)))\\&=\lim_{n\to\infty}T(|\phi|_n(g(y)-g(a)),|\phi|_n(g(b)-g(y))\end{aligned}$$

and

$$\begin{aligned}&T(|\phi|_n(g(y)-g(a)),|\phi|_n(g(b)-g(y))\\&= T\Bigg(\sup_{x_1+\cdots+x_n=g(y)-g(a)}T(|\phi|(x_1),\cdots,|\phi|(x_n)),\\&\qquad\sup_{x_{n+1}+\cdots+x_{2n}=g(b)-g(y)}T(|\phi|(x_{n+1}),\cdots,|\phi|(x_{2n}))\Bigg)\\&=\sup_{\substack{x_1+\cdots+x_n=g(y)-g(a)\\x_{n+1}+\cdots+x_{2n}=g(b)-g(y)}}T(T(|\phi|(x_1),\cdots,|\phi|(x_n)),T(|\phi|(x_{n+1}),\cdots,|\phi|(x_{2n})))\\&\le\sup_{x_1+\cdots+x_{2n}=g(b)-g(a)}T(|\phi|(x_1),\cdots,|\phi|(x_{2n}))\\&=|\phi|_{2n}(g(b)-g(a))\end{aligned}$$

where the second equality comes from the continuity of T and the inequality comes from non-decreasing property of T, hence equation (1) is proved since $\lim_{n\to\infty}|\phi|_{2n}(g(b)-g(a))=|\phi|^*(g(b)-g(a))$. □

The following theorem is due to B. De Baets et al [2]. Here, we give a new proof using the idea of Theorem 2.

Theorem 3. (De Baets et al [2]) Consider a t-norm T, a generator g and a shape ϕ. Let $H=\{|g(u)-g(v)||(u,v)\in R^2\}$. If for any $x\in H$, $\phi\oplus_T\phi(x)=\phi(x)$, then the fuzzy relation $E_{g,\phi}$ is a T-equivalence on R.

Proof. Define ϕ_0 as follows :

$$\phi_0(x)=\begin{cases}\phi(x) & \text{if } x\in H,\\ \inf\{\phi(w)|w<x,\ w\in H\} & \text{if } x\notin H.\end{cases}$$

Then ϕ_0 is a shape with $E_{g,\phi}(x,y)=E_{g,\phi_0}(x,y)$ for $(x,y)\in R^2$. We can also show that for any $x\in R$, $\phi_0\oplus_T\phi_0(x)=\phi_0(x)$. It is because $\phi_0\oplus_T\phi_0(x)\ge\phi_0(x)$ is always true and for $x\notin H$, $w\in H$ and $w<x$,

$$\begin{aligned}\phi_0\oplus_T\phi_0(x)&\le\phi_0\oplus\phi_0(w)\\&=\phi(w)\end{aligned}$$

and hence

$$\begin{aligned}\phi_0 \oplus_T \phi_0(x) &\leq \inf\{\phi(w) | w < x,\, w \in H\} \\ &= \phi_0(x).\end{aligned}$$

We now note that $\phi_0 = \phi_0^*$ and can prove that E_{g,ϕ_0} is a T-equivalence on R according to the exactly same method as Theorem 1 without the assumption of continuity of T using $\phi_0 \oplus_T \phi_0 = \phi_0$. This completes the proof. □

Recently, many authors [5, 6, 7, 13] studied facts about T-sums of shape function and their limits.

Theorem 4. (Hong and Hwang [6], Hong and Ro [7], Mesiar [11]) Consider a continuous Archimedean t-norm T with additive generator f and a shape ϕ. If $f \circ \phi$ is convex, then

$$\phi_n(x) = f^{[-1]}\left(nf \circ \phi\left(\frac{x}{n}\right)\right).$$

Theorem 5. (Hong and Hwang [5]) Consider a continuous Archimedean t-norm T with additive generator f and a shape ϕ. If $f \circ \phi$ is convex, then $\phi^*(0) = 1$ and for $x > 0$,

$$\lim_{n\to\infty} \phi_n(x) = \phi^*(x) = f^{[-1]}(xf'_-(1)\phi'_+(0)).$$

Definition 8. Consider $(a, b) \in R$, $a \neq b$, then $\phi_{(a,b)}$ is the linear transformation defined by

$$\phi_{(a,b)}(x) = \frac{x-a}{b-a}$$

Note that the inverse mapping $\phi^{-1}_{(a,b)}$ of $\phi_{(a,b)}$ is given by $\phi^{-1}_{(a,b)}(x) = a + (b-a)x$.

Definition 9. Consider a fuzzy quantity A and $(a, b) \in [0, 1]^2$, $a < b$.

(i) The fuzzy quantity $A^{[a,b]}$ is defined as $A^{[a,b]} = \text{tr} \circ \phi_{(a,b)} \circ A$, i.e. $A^{[a,b]}(x) = \text{tr}((A(x) - a)/(b - a))$, where $\text{tr} : R \to [0, 1]$ is defined by

$$\text{tr}(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

(ii) The fuzzy quantity $A_{[a,b]}$ is defined by

$$A_{[a,b]}(x) = \begin{cases} \phi^{-1}_{(a,b)}(A(x)), & \text{if } A(x) > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

We need the following result to generalize Theorem 5 to arbitrary continuous t-norm.

Theorem 6. (De Baets and Marková [1]) Consider an ordinal sum of continuous t-norm $T = (\langle a_i, b_i, f_i\rangle | i \in I)$ written in such a way that $\bigcup_{\lambda \in I}[a_i, b_i] = [0,1]$ and a shape ϕ. If $f_i \circ \phi^{[a_i,b_i]}$ is convex for all $i \in I$, then

$$\phi_n(x) = \sup_{i \in I} \left\{ (\phi_n^{T_i,[a_i,b_i]})_{[a_i,b_i]}(x) \right\}$$

where $\phi_n^{T_i,[a_i,b_i]}(x) = f_i^{[-1]}\left(nf_i \circ \phi^{[a_i,b_i]}\left(\frac{x}{n}\right)\right)$.

Theorem 5 can be easily generalized to arbitrary ordinal sums of continuous t-norm T.

Theorem 7. Consider an ordinal sums of continuous t-norm $T = (\langle a_i, b_i, f_i\rangle | i \in I)$ written in such a way that $\bigcup_{\lambda \in I}[a_i, b_i] = [0,1]$ and a shape ϕ. If $f_i \circ \phi^{[a_i,b_i]}$ is convex for all $i \in I$, then

$$\begin{aligned} \phi^*(x) &= \lim_{n \to \infty} \phi_n(x) \\ &= \sup_{i \in I} \left\{ (\phi^{T_i,[a_i,b_i]})_{[a_i,b_i]}(x) \right\}, \end{aligned}$$

where $\phi^{T_i,[a_i,b_i]}(x) = \lim_{n\to\infty} \phi_n^{T_i,[a_i,b_i]}(x) = f_i^{[-1]}(x(f_i)'_-(1)\left(\phi^{[a_i,b_i]})'_+(0)\right)$.

4. EXAMPLES

Example 1. Consider the product t-norm T_P with additive generator $f(x) = \log x^{-1}$, and a generator g and a shape function ϕ defined by $\phi(x) = \max\{1-x, 0\}$. Then, by Theorem 5 (or see [5]), $\phi^*(x) = e^{-x}$, and hence $E_{g,\phi^*}(x,y) = e^{-|g(x)-g(y)|}$ is a T-equivalence on R.

Example 2. Consider the Lukasiewicz t-norm T_L with additive generator $f(x) = 1-x$, and generator g and a shape function ϕ defined by $\phi(x) = \max\{1-x, 0\}$. Then, by Theorem 5 (or see [5]), $\phi^*(x) = \phi(x)$, and hence $E_{g,\phi^*}(x,y) = \max\{1 - |g(x) - g(y)|, 0\}$ is a T-equivalence on R.

Example 3. Consider the ordinal sums $T = (\langle 0, \frac{1}{3}, \log x^{-1}\rangle, \langle \frac{1}{3}, 1, 1-x\rangle)$, a generator g and a shape function ϕ defined by $\phi(x) = \max\{1-x, 0\}$. Then, by Theorem 7, $\phi^*(x) = \max\left\{1-x, \frac{1}{3}\right\}$, and hence $E_{g,\phi^*}(x,y) = \max\left\{1 - |g(x)-g(y)|, \frac{1}{3}\right\}$ is a T-equivalence on R.

Example 4. Consider the ordinal sums $T = (\langle 0, \frac{1}{3}, 1-x\rangle, \langle \frac{1}{3}, 1, \log x^{-1}\rangle)$, a generator g and a shape function ϕ defined by $\phi(x) = \max\{1-x, 0\}$. Then, by Theorem 7, $\phi^*(x) = \frac{1}{3} + \frac{2}{3}e^{-\frac{3}{2}x}$ since $f^{T_P,[\frac{1}{3},1]}(x) = e^{-\frac{3}{2}x}$ and $f^{T_L,[0,\frac{1}{3}]}(x) = 1$. Hence $E_{g,\phi^*}(x,y) = \frac{1}{3} + \frac{2}{3}e^{-\frac{3}{2}|g(x)-g(y)|}$ is a T-equivalence on R.

Example 5. Consider the ordinal sums $T = (\langle 0, \frac{1}{3}, \log x^{-1}\rangle, \langle \frac{1}{3}, 1, 1-x\rangle)$, a generator g and a shape function ϕ defined by

$$\phi(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3}(1-x) & \text{if } 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Theorem 7,

$$\phi^*(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3}e^{-x} & \text{otherwise,} \end{cases}$$

since

$$f^{T_L,[\frac{1}{3},1]}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $f^{T_P,[0,\frac{1}{3}]}(x) = e^{-x}$. Hence

$$E_{g,\phi^*}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ \frac{1}{3}e^{-|g(x)-g(y)|} & \text{otherwise,} \end{cases}$$

is a T-equivalence on R.

Example 6. Consider the ordinal sums $T = (\langle 0, \frac{1}{3}, 1-x\rangle, \langle \frac{1}{3}, 1, \log x^{-1}\rangle)$, a generator g and a shape function ϕ defined by

$$\phi(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3}(1-x) & \text{if } 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Theorem 7,

$$\phi^*(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3}(1-x) & \text{if } |x| \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

since

$$f^{T_P,[\frac{1}{3},1]}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $f^{T_L,[0,\frac{1}{3}]}(x) = 1 - x$. Hence

$$\phi_{g,\phi^*}(x,y) = \begin{cases} 1 & \text{if } g(x) = g(y), \\ \frac{1}{3}(1-|g(x)-g(y)|) & \text{if } |g(x)-g(y)| \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

is a T-equivalence on R.

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