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THE PRINCIPLE OF THE LARGEST TERMS AND QUANTUM LARGE DEVIATIONS

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We give an approach to large deviation type asymptotic problems without evident probabilistic representation behind. An example provided by the mean field models of quantum statistical mechanics is considered.

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1. INTRODUCTION

The idempotent measures appear naturally as asymptotic solution for some problems in different areas such as differential equations and optimization theory. This approach provides the technic of semi-classical analysis for nonlinear problems. Another motivation for the study of idempotent measures comes from the large deviation theory (LD).

Let $\{P_n\}$ be the sequence of probability measures on a Polish space X and let I be a function on X with compact level sets. $\{P_n\}$ obeys the large deviation principle with a rate function I if and only if

$$\lim_{n\to\infty}\left[\int_X (g(x))^n P_n(\mathrm{d}x)\right]^{1/n} = \sup_{x\in X} g(x)\, e^{-I},$$

for all bounded continuous nonnegative functions g on X [3, 9].

We may say that in this sense $\{P_n\}$ converges to an idempotent measure $\exp\{-I\}$. The r.h.s. of the last display is called a *sup-integral* or *idempotent integral* with respect to the idempotent measure.

Some asymptotic problems of quantum mechanics leads to more general setting for large deviations. Let consider a simple example of non-commutative analogue of large deviation problem [8]. In this case the random variables become self-adjoint operators in an operator algebra.

Let $\mathbf{M}$ be the algebra of all complex $m\times m$ matrices and $\mathcal{A} = \bigotimes_{i\in\mathrm{N}} \mathbf{M}^i$, where $\mathbf{M}^i$ is a copy of $\mathbf{M}$. Let $x = x^* \in \mathbf{M}$ and $\tilde{x}_n = (x_1 + x_2 + \cdots + x_n)/n$, where x_i is a copy of x in $\mathbf{M}^i$. Let ϱ be a faithful state of $\mathbf{M}$, and ω_ϱ the associated infinite product-state of $\mathcal{A}$.

The sequence $x_1, x_2, \cdots$ and the state ω_ϱ play the role of independent random variables. Let D_n be the density of ω_ϱ restricted to $\bigotimes_{i=1}^n \mathbf{M}^i$ and f be a continuous real-valued function on the interval $[-|x|, |x|]$. The problem is to compute the limit

$$\lim_{n\to\infty} [\operatorname{Tr} \exp(\log D_n + nf(\tilde{x}_n))]^{1/n}. \tag{1.1}$$

The classical LD principle is formulated in fact in terms of convergence of a sequence of normalized positive linear functionals on a cone of positive bounded continuous functions to idempotent integral. We however have to study more general class of functionals. Our aim is to extract such properties of positive not necessary linear functionals on a cone which guarantee that properly normalized sequence of such functionals converges to an idempotent integral.

One can easily see that $\int_X g(x)P_n(\mathrm{d}x)$ is the monotone homogeneous functional on a cone of positive bounded continuous functions which possesses the following strong sub-additive property

$$\int_X g_1(x) \vee g_2(x) P_n(\mathrm{d}x) \le \int_X g_1(x)P_n(\mathrm{d}x) + \int_X g_2(x)P_n(\mathrm{d}x).$$

These properties lead to the remarkable consequence that the limit

$$\lim_{n\to\infty} \left[\int_X (g(x))^n P_n(\mathrm{d}x)\right]^{1/n} = G(g),$$

if exists, is the *sup-functional*, i. e.

$$G(g_1 \vee g_2) = G(g_1) \vee G(g_2).$$

In Section 2, following Choquet's ideas [4], we introduce convex cone of nonlinear *alternating of order 2* functionals that generalizes the above properties of the linear functionals. We show that the limits of convergent sequences of properly normalized functionals from the cone are the sup-functionals (*the principle of the largest terms*). The sup-functionals are the *extremal elements* of a cone of the *totally alternating* functionals. In Section 3 we prove a generalization of the Choquet theorem provides us by the sup-integral or the idempotent integral representation of the sup-functional (the analogue of the Riesz theorem) which is in some sense unique. In Section 4 we discuss some properties of the Fenchel–Moreau transform. Based on these results, in Section 5 we prove under Gärtner-type condition the Varadhan-type variational principle, i. e. that properly normalized sequence of such functionals converges to an idempotent integral.

These tools make it possible to handle non-linear (non-commutative) asymptotic problems as one would handle classical LD. In Section 6 we apply this approach to an example of non-commutative LD, which is based on an analysis of mean-field quantum crystal model.

2. CONES OF ALTERNATING MAPS AND EXTREMAL ELEMENTS

2.1. Alternating functions on a vector lattice

Let E be a vector lattice that is a vector space with a cone E^+ which defines the order structure $\geq$ on E such that for each pair x and y of elements of E there exists a supremum $x \top y$ in the space E. Consider a map $g : (E, \top) \to R^+$. The *successive differences* of g with respect to parameters $a_1, a_2, \ldots$ in E are defined as follows:

$$\nabla_1 g(x; a_1) := g(x) - g(x \top a_1),$$
$$\nabla_{n+1} g(x; a_1, \ldots, a_n, a_{n+1}) := \nabla_n g(x; a_1, \ldots, a_n) - \nabla_n g(x \top a_{n+1}; a_1, \ldots, a_n).$$

Since $\nabla_n g(x; a_1, \ldots, a_n)$ is a symmetric function over a_i we denote it by $\nabla_n g(x; \{a_i\})$, $i \in I$, for a given set I.

Definition 2.1. [4] The map $g : E \to R^+$ is called *alternating map* of the order n, $n \geq 1$, if $\nabla_p f(x; \{a_i\}) \leq 0$ for each $p \leq n$ and for every finite family $\{a_i\}$, $0 \leq a_i$. The map g is called *totally alternating map*, if it is alternating of all orders $n \geq 1$. Denote by $\mathbf{A}_n(E, \top)$ and $\mathbf{A}_\infty(E, \top)$ the cones of *positive positively homogeneous* functions g ($g(\lambda x) = \lambda g(x)$, for $\lambda \geq 0$) *alternating* of order n and *totally alternating* respectively.

Theorem 2.2. (*The principle of the largest terms*) Let $(D, \geq)$ be a directed set and $\{g_d\}_{d \in D}$ be a net of the functions $g_d \in \mathbf{A}_2(E, \top)$. Let $\{t_d\}_{d \in D}$ be normalizing net of real numbers tending to 0. If the limit g of the net $\{g_d^{t_d}\}_{d \in D}$ exists for all $x \in E$:

$$\lim_{d \in D} g_d^{t_d}(x) = g(x),$$

then $g \in \mathbf{A}_2$ and for $x_1, x_2 \in E$

$$g(x_1 \top x_2) = g(x_1) \vee g(x_2).$$

Proof. If $g_d \in \mathbf{A}_2$, then $g_d^{t_d}$ is the alternative map of the order 2 [4]. Hence

$$\nabla_1 g_d^{t_d}(x_1; x_2) \leq 0$$

and so

$$g_d^{t_d}(x_1 \top x_2) \geq g_d^{t_d}(x_1) \vee g_d^{t_d}(x_2). \tag{2.1}$$

On the other hand, $\nabla_2 g(0; x_1, x_2) \leq 0$.
Taking into account that by homogeneity $g(0) = 0$, we have

$$g_d^{t_d}(x_1 \top x_2) \leq [g_d(x_1) + g_d(x_2)]^{t_d} \leq 2^{t_d}[g_d(x_1)^{t_d} \vee g_d(x_2)^{t_d}]. \tag{2.2}$$

Passing to the limit in (2.1) and (2.2), we obtain

$$g(x_1 \top x_2) = g(x_1) \vee g(x_2).$$

□

Remark 2.3. Note that (2.2) holds if the first inequality in (2.2) holds to within normalized by $\{t_d\}$ terms vanishing as normalizing net $\{t_d\}$ tends to 0. This fact we shall use in non-commutative example below.

Definition 2.4. An element $g \in \mathbf{A}_2(\mathrm{E}, \top)$ is called the *sup-functional* if and only if for $x_1, x_2 \in \mathrm{E}$

$$g(x_1 \top x_2) = g(x_1) \vee g(x_2).$$

Lemma 2.5. [4] A homogeneous functional f on E is the sup–functional if and only if it belongs to $\mathbf{A}_2(\mathrm{E}, \top)$ and satisfies the following property:

$$f(x) = f(y) \Rightarrow f(x \top y) = f(x) = f(y).$$

Definition 2.6. [4] Let $\mathcal{C}$ be a convex cone in a vector space X. An element $a \in \mathcal{C}$ is called the *extremal element* of the cone $\mathcal{C}$ if and only if the equation $a = a_1 + a_2$ with a_1 and $a_2 \in \mathcal{C}$, implies $a_1 = \lambda_1 a$ and $a_2 = \lambda_2 a$.

Theorem 2.7. The sup–functionals are the *extremal elements* of the cone $\mathbf{A}_\infty(\mathrm{E}, \top)$

Proof. It follows immediately from definition by induction that sup-functionals are elements of $\mathbf{A}_\infty$. Let f be a sup-functional and $f = g + g_1$ where $g,\ g_1 \in \mathbf{A}_\infty$. Fix $x, y \in E^+$ such that $f(x) = f(y) \neq 0$. Then by Lemma 2.5

$$g(x \top y) + g_1(x \top y) = f(x \top y) = f(x) = f(y).$$

Thus

$$0 \leq g(x \top y) - g(x) = g_1(x) - g_1(x \top y) \leq 0,$$

and so $g(x \top y) = g(x)$. Analogously $g(x \top y) = g(y)$. Then

$$\frac{g(x)}{f(x)} = \frac{g(y)}{f(y)} = \alpha.$$

In the case $f(x) = f(y) = 0$ functions g and g_1 also vanish at these points and $f(x) = \alpha g(x)$ and $f(x) = \alpha g(x)$ for all α. Fix arbitrary points x and y in E^+ such that $f(x) > f(y) > 0$. Then $\exists \alpha$ such that $f(x) = f(\alpha y)$ and in view of the results proved above, $g(x) = g(\alpha y) = \alpha g(y)$. Thus

$$\frac{g(x)}{f(x)} = \frac{g(y)}{f(y)} = \lambda,$$

i.e. $g(x) = \lambda f(x)$ for all x. □

Remark 2.8. In [4] the more strong result is claimed without proof for the case when E^+ is a cone of non–negative continuous functions of compact support on locally compact space: the extremal elements of $\mathbf{A}_\infty(\mathrm{E}, \top)$ are the sup-functionals. Unfortunately I do not know how to prove that.

2.2. Alternating functionals on a cone of positive continuous functions

In this section we consider the cone $\mathcal{A}_2(C_b^+)$ of alternating functionals J defined on the cone $C_b^+ = C_b^+(X)$ of positive bounded continuous functions on a Hausdorff topological space X. Following the classical extension procedure of the measure theory [1] and the theory of capacities [4] we are able to define $\mathcal{A}_2(C_b^+)$ in such a way that for $J \in \mathcal{A}_2(C_b^+)$ an analog of the *inner regular property* of Choquet's capacities is preserved and the *monotonė convergence theorem* is valid [2].

Note that for alternating functionals the property of being in this sense continuous are determined by the order structure of the cone.

Theorem 2.9. *(monotone convergence)* [2] If the increasing sequence of non–negative functions $\{f_n\}_{n\geq 1}, f_n \in C_b^+$ converges to a function $f \in C_b^+$, then $J(f_n)$ converges to $J(f)$.

Theorem 2.10. Let $(D, \geq)$ be a directed set. Assume that a net $\{J_d\}_{d\in D}$, $J_d \in \mathcal{A}_2(C_b^+)$ converges to a functional $J \in \mathcal{A}_2(C_b^+)$, for all $g \in C_b^+(X)$. Let(not necessary bounded) continuous function $f \in C^+(X)$ be *uniformly J-integrable* with respect to the net $\{J_d\}_{d\in D}$, i. e.

$$\lim_{N\to\infty} \limsup_{d\in D} J_d\left((f-N)\vee 0\right) = 0$$

and there exits $\lim_{N\to\infty} J(f \wedge N) = J(f)$. Then

$$\lim_{d\in D} J_d(f) = J(f).$$

Proof. By the property of the cone $\mathcal{A}_2(C_b^+)$

$$J_d(f) - J_d(f\wedge N) \leq J_d((f-N)\vee 0).$$

The statement follows from the Theorem 2.9 and the inequality

$$|J_d(f) - J(f)| \leq |J_d(f) - J_d(f\wedge N)| + |J_d(f\wedge N) - J(f\wedge N)| + |J(f\wedge N) - J(f)|.$$

□

3. SUP-FUNCTIONALS AND SUP-INTEGRALS

In this section we show that a sup-functional J admits the representation $J(g) = \sup_x g(x)V(x)$. Note that the upper semi-continuous regularization $\overline{V}$ (see Definition 3.5) of V induces for any set A the set function $I(A) := \sup_{x\in A} \overline{V}(x)$. The set function $I(A)$ is the *inner regular* Choquet capacity [4] with the property

$$I(A_1 \cup A_2) = I(A_1) \vee I(A_2).$$

3.1. The sup-integral representation

Let T be a locally compact topological space and $\mathcal{K}^+ = \mathcal{K}^+(T)$ be a cone of non-negative continuous functions of compact support on T.

Theorem 3.1. Let $J(\cdot)$ be a sup-functional on the cone $\mathcal{K}^+(T)$. Then there exists a positive function $V: T \to \mathbf{R}^+$ such that for all $\phi \in \mathcal{K}^+$

$$J(\phi) = \sup_{x \in T} \phi(x) V(x). \tag{3.1}$$

Remark 3.2. This result was first formulated without proof in [4], section 53.1 and than rediscovered and generalized by several authors [2, 3, 6, 9].

Let X be a Hausdorff totally regular (Tichonoff) space. It is well known that the Tichonoff space is homeomorphic to a sub-space of certain compact cube. Denote by $e : X \to$ "cube" and by $e[X]$ a homeomorphic map and an image of X under this map respectively. The set $\tilde{X} := \overline{e[X]}$ is called the Stone–Čech compactification of the space X. Let $\tilde{C}^+(\tilde{X})$ be a cone of non-negative continuous functions on the compact space $\tilde{X}$. For any $\tilde{g} \in \tilde{C}^+(\tilde{X})$ define a functional $\tilde{J}$ induced by the sup-functional J on $C_b^+(X)$ as follows:

$$\tilde{J}(\tilde{g}) := J(\tilde{g}\big|_{e[X]} \circ), \tag{3.2}$$

where $\tilde{g}\big|_{e[X]}$ is the restriction of $\tilde{g}$ on $e[X]$, and $\circ$ denotes the composition of functions. It is easy to see that $\tilde{J}$ is the sup-functional.

Definition 3.3. A sup-functional J on C_b^+ is called *tight* if for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset\subset X$ such that for all $g \in C_b^+(X)$, $0 \le g \le 1$, with $g(x) = 0$ for all $x \in K_\varepsilon$, the inequality $J(g) \le \varepsilon$ takes place.

Theorem 3.4. [2] Let J be the *tight* sup-functional on $C_b^+(X)$. Then there exists a function $V: X \to \mathbf{R}^+$ such that for all $g \in C_b^+(X)$

$$J(g) = \sup_{x \in X} g(x) V(x). \tag{3.3}$$

P r o o f. In view of Theorem 3.1, functional $\tilde{J}(\cdot)$ defined by (3.2) has the representation

$$\tilde{J}(\tilde{g}) = \sup_{\tilde{x} \in \tilde{X}} \tilde{g}(\tilde{x}) \tilde{V}(\tilde{x})$$

with a certain $\tilde{V}(\cdot)$. Fix $\varepsilon > 0$ and $\tilde{x}_0 \notin e[X]$. Then $\tilde{x}_0 \notin e[K_\varepsilon]$ where K_ε is compact set, chosen according to the tight property. Since e is the homeomorphic map, $e[K_\varepsilon]$ is compact set in $\tilde{X}$ and hence closed. Since $\tilde{X}$ is the normal space there exists a continuous function $\tilde{g}_\varepsilon$, $0 \le \tilde{g}_\varepsilon \le 1$, such that $\tilde{g}_\varepsilon(\tilde{x}_0) = 1$ and $\tilde{g}_\varepsilon(\tilde{x}) = 0$, for $\tilde{x} \in e[K_\varepsilon]$. Then by the tightness condition

$$0 \le \tilde{V}(\tilde{x}_0) \le \tilde{J}(\tilde{g}_\varepsilon) = J(\tilde{g}_\varepsilon|_{e[X]} \circ e) \le \varepsilon.$$

Thus, by arbitrariness of ε for all $\tilde{x} \notin e[X]$ we get $\tilde{V}(\tilde{x}) = 0$. By the Stone–Čech theorem for all $g \in C_b^+(X)$ there exists a continuous extension $\tilde{g}$ of g to $\tilde{X}$. Hence

$$J(\tilde{g}) = \tilde{J}(\tilde{g}) = \sup_{\tilde{x}\in\tilde{X}} \tilde{g}(\tilde{x})\tilde{V}(\tilde{x}) = \sup_{x\in X} g(x)V(x),$$

with $V(x) := \tilde{V}(e(x))$. □

Definition 3.5. The function $\overline{V}$ defined by

$$\overline{V}(x) := \limsup_{y\to x} V(y),$$

is called the *upper semi-continuous regularization* of V (USC-regularization).

Next theorem gives a necessary and sufficient condition of *tightness*.

Theorem 3.6. [2] The USC-regularization $\overline{V}$ of V has the compact level set $\{x \in X : \overline{V}(x) \geq \varepsilon\}$, $\varepsilon > 0$, if and only if the functional J is tight.

3.2. Uniqueness of the representation

Theorem 3.7. [2] Let X be a topological space. Let $\overline{V}$ be the USC-regularization of $V : X \to \mathsf{R}^+$. Then for all continuous functions $g : X \to \mathsf{R}^+$

$$\sup_{x\in X} g(x)V(x) = \sup_{x\in X} g(x)\overline{V}(x). \tag{3.4}$$

Let X be a totally regular space.

Theorem 3.8. [2] Let bounded sup-functional J admits the representation (3.3). Then the representation is unique in the USC-class.

Definition 3.9. The functional J which admits the representation

$$J(g) = \sup_{x\in T} g(x)V(x), \tag{3.5}$$

with the USC function V is called *the sup-integral* with respect to the *density* V.

4. CONVEX DUALITY

It is well known that two measures on a locally convex space coincide provided characteristic functions are equal. It turns out that in some sense similar result takes place for sup-integrals with the Fenchel–Moreau transform playing the role of the Fourier transform.

Let E, E' be a dual pair of locally convex vector spaces. Let J_1 and J_2 be sup-integrals with respect to V_1 and V_2 correspondingly defined by (3.5) and let a function $h(\cdot)$ coincides with the Fenchel–Moreau transforms f_1^* and f_2^* of the functions $f_1 = -\ln V_1$ and $f_2 = -\ln V_2$:

$$h(x') = f_1^*(x') := \sup_{x \in X}[\langle x, x' \rangle - f_1(x)],$$

$$h(x') = f_2^*(x') := \sup_{x \in X}[\langle x, x' \rangle - f_2(x)].$$

We show that if $h(\cdot)$ meets the smoothness conditions of Theorem 4.7 then

$$h^* = f_1 = f_2,$$

i. e. functions V_1 and V_2 and hence the sup-integrals J_1 and J_2 coincide.

Remark 4.1. At first sight this statement looks like the Young–Fenchel theorem which states that a convex lower semicontinuous function f coincides with the bipolar f^{**}. However, in our case conditions are imposed on the polar f^* instead of f and so the arguments of the proof are completely different. We shall use this result to prove the Gärtner type Theorem 5.3 with h being the limit function in Gärtner's hypothesis (see Definition 5.2).

We start with the case $E = \mathsf{R}^m$.

Definition 4.2. A concave function $f\colon \mathsf{R}^m \to \mathsf{R}$ is called *essentially smooth* if it satisfies the following conditions:

1. $C := \operatorname{int}(\operatorname{dom} f) \neq \emptyset$,
2. f is differentiable for all $x \in C$,
3. for any convergent sequence $\{x_n\}_{n \geq 1}$, $x_n \in C$ such that the limit $x \notin C$,

$$\lim_{i \to \infty} |\nabla f(x_i)| = +\infty.$$

Theorem 4.3. Let $f\colon \mathsf{R}^m \to (-\infty\,;\infty]$ be the lower semicontinuous (LSC) function and let there exists a convex essentially smooth function $h\colon \mathsf{R}^m \to (-\infty\,;\infty]$ such that

$$f^*(x') = h(x'),\ \forall\, x' \in \operatorname{int}(\operatorname{dom} h). \tag{4.1}$$

Then $f = f^{**} = h^*$.

Remark 4.4. This theorem is a modification of lemma 3.2 of [10].

Assume now that E is a *locally convex space.*

Denote by $\mathcal{F}(E)$ a class of closed vector subspaces of E with the finite codimension partially ordered by $\supset$. Subspace V of E belongs to $\mathcal{F}(E)$ (i. e. the quotient space

E/V of E is the finite dimensional space) if and only if there exists a finite number of elements $x'_1, \ldots, x'_n$ in E' such that

$$V = \{x \in E : \langle x, x'_i \rangle = 0, \mathrm{i} = 1, \ldots, n\}.$$

For all closed subspaces $V \in \mathcal{F}(E)$ denote by p_V the quotient map of E onto E/V, i. e.

$$p_V(x) = x + V,$$

where $x \in E$ and $V \in \mathcal{F}(E)$. p_V is *linear continuous open* map for the quotient topology. A topological dual $(E/V)'$ of E/V is isomorphic to the annihilator of V, i. e.

$$(E/V)' = V^\perp := \{x' \in E' : \langle x, x' \rangle = 0,\ x \in V\}.$$

The dual map $p'_V : V^\perp \to \mathsf{E}'$ of the quotient map p_V is the canonical map: $p'_V(x') = \iota_V(x') := x'$ where $x' \in V^\perp$. Let $f: E \to \overline{\mathsf{R}}$ and $h: E' \to \overline{\mathsf{R}}$ be arbitrary real functions. Denote by $p_V f: E/V \to \overline{\mathsf{R}}$ the *image* of the function f under the map p_V and by $h\,p'_V : V^\perp \to \overline{\mathsf{R}}$ *the inverse image* of the function h under the map p'_V:

$$p_V f(p_V(x)) := \inf_{y - x \in V} f(y);$$

$$h\,p'_V(x') := h(p'_V(x')) = h\big|_{V^\perp}(x') := h(x').$$

Definition 4.5. A subset of elements $x \in A$ is called *c–interior* of A if for all $y \in E$ there exists $\varepsilon > 0$ such that $x + \varepsilon y \in A$. Denote this subset by $\mathrm{c} - \mathrm{int}\, A$.

Remark 4.6. If A is the absorbing set then by definition $\mathrm{c} - \mathrm{int}\, A \neq \emptyset$ and $0 \in \mathrm{c} - \mathrm{int}\, A$.

Theorem 4.7. [2] Let $f: E \to (-\infty, \infty]$ be the LSC function for the weak topology $\sigma(E, E')$ and let $p_V f$ be the LSC function for all V. Assume that there exists a function $h: E' \to (-\infty; \infty]$ such that

1. $\mathrm{dom}\, h$ is the absorbing set;
2. $h\,p'_V$ is the essentially smooth function for all $V \in \mathcal{F}(E)$;
3. $f^*(x') = h(x')$ for all $x' \in \mathrm{c} - \mathrm{int}\,(\mathrm{dom}\, h)$.

Then $f = f^{**} = h^*$.

5. THE VARADHAN–TYPE VARIATIONAL PRINCIPLE

In this chapter we tie together the results of previous sections to prove the Varadhan-type variational principle using an analog of the *characteristic functions method* of the week convergence theory.

Definition 5.1. A net of functionals $\{J_\alpha\}_{\alpha\in D}$, $J_\alpha \in \mathcal{A}_2(C_b^+)$, is called *exponentially tight* with a net $\{n_\alpha\}_{\alpha\in D}$, $n_\alpha \to \infty$, if for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset\subset X$ such that

$$\limsup_{\alpha\in D} \left(J_\alpha(g^{n_\alpha})\right)^{1/n_\alpha} \le \varepsilon \tag{5.1}$$

for all $g \in C_b^+(X)$, $0 \le g \le 1$, with $g(x) = 0$ for all $x \in K_\varepsilon$.

Definition 5.2. We say that the *Gärtner condition* for the net $\{J_\alpha\}_{\alpha\in D}$, $J_\alpha \in \mathcal{A}_2(C_b^+)$ is valid if there exists a function $h: E' \to \overline{\mathsf{R}}$ such that

$$\lim_{\alpha\in D} \left(J_\alpha(e^{n_\alpha\langle\cdot,x'\rangle})\right)^{1/n_\alpha} = e^{h(x')}, \tag{5.2}$$

and the following hypothesizes are fulfilled:

1. $\operatorname{dom} h := \{x \in E\colon\ h(x) < \infty\}$ is the absorbing set;
2. $\forall V \in \mathcal{F}(E)$ the restriction h to the finite subspace $V^\perp \subset E'$ is *essentially smooth*.

Theorem 5.3. [2] Let a net $\{J_\alpha\}_{\alpha\in D}$ of bounded normalized functionals $J_\alpha \in \mathcal{A}_2(C_b^+)$ be exponentially tight and the Gärtner condition is fulfilled. Then for all $g \in C_b^+$

$$\lim_{\alpha\in D} \left(J_\alpha(g^{n_\alpha})\right)^{1/n_\alpha} = \sup_{x\in X}[g(x)e^{-h^*(x)}], \tag{5.3}$$

where

$$h^*(x) = \sup_{x'\in X'} [\langle x, x'\rangle - h(x')]$$

has compact level sets.

P r o o f. Since J_α belongs to $\mathcal{A}_2(C_b^+)$ then [4] the functionals $\left(J_\alpha((\cdot)^t)\right)^{1/t}$ belong to $\mathcal{A}_2(C_b^+)$ for all $t \ge 1$. Consider a topology of pointwise convergence on the cone $\mathcal{A}_2(C_b^+)$. The set

$$K = \{J \in \mathcal{A}_2\colon\ J\}\,(\mathbf{1}) = 1$$

is the *compact* (by Tichonoff's theorem) base of the cone. Note that

$$\left(J_\alpha((\cdot)^{n_\alpha})\right)^{1/n_\alpha} \in K.$$

Then for each subnet D' there exists a subsubnet D'' which converges to a functional $J'' \in \mathcal{A}_2$:

$$\lim_{\alpha\in D''} \left(J_\alpha(g^{n_\alpha})\right)^{1/n_\alpha} = J''(g), \quad g \in C_b^+. \tag{5.4}$$

In view of Theorem 2.2 the functional J'' possesses the sup-property:

$$J''(g_1 \vee g_2) = J''(g_1) \vee J''(g_2).$$

Since the net $\{J_\alpha\}_{\alpha\in D}$ is exponentially tight the functional J'' is *tight*. Since the vector space is totally regular the sup-functional J'' has by Theorem 3.4 the sup-integral representation with the unique (see Theorem 3.8) USC function V'' on E which by Theorem 3.6 has compact level sets. Thus from (5.4) we get for all $g \in C_b^+$:

$$\lim_{\alpha\in D''} (J_\alpha(g^{n_\alpha}))^{1/n_\alpha} = \sup_{x\in E} g(x)V''(x). \tag{5.5}$$

We next show that for $x' \in \mathrm{c-int}\,(\mathrm{dom}\, h)$

$$\lim_{\alpha\in D''} \left(J_\alpha(e^{n_\alpha\langle\cdot,x'\rangle})\right)^{1/n_\alpha} = \sup_{x\in E}[e^{\langle x,x'\rangle}V''(x)]. \tag{5.6}$$

Indeed, for any $x' \in \mathrm{c-int}\,(\mathrm{dom}\, h)$ there exists $\varepsilon > 0$ such that $h((1+\varepsilon)x') < \infty$. (Note that by the hypotheses (1) of Definition 5.2, $\mathrm{c-int}\,(\mathrm{dom}\, h) \neq \emptyset$.) By the *monotone property* of the functionals J_α an analogue of the *Chebyshev inequality* is valid and so

$$\left(J_\alpha((e^{\langle\cdot,x'\rangle} - N)\vee 0)^{n_\alpha})\right)^{1/n_\alpha} \leq \frac{\left(J_\alpha(e^{n_\alpha\langle\cdot,(1+\varepsilon)x'\rangle})\right)^{1/n_\alpha}}{N^\varepsilon}.$$

Hence, passing to the limit, in view of (5.2) with $h((1+\varepsilon)x') < \infty$, we get:

$$\lim_{N\to\infty} \limsup_{\alpha\in D} \left(J_\alpha((e^{\langle\cdot,x'\rangle} - N)\vee 0)^{n_\alpha})\right)^{1/n_\alpha} = 0.$$

(A function with this property is called *uniformly exponentially J-integrable* w.r.t the net J_α.) Then the conditions of Theorem 2.10 are fulfilled, (5.6) is valid and hence

$$e^{h(x')} = \sup_{x\in E}[e^{\langle x,x'\rangle}V''(x)],$$

for all $x' \in \mathrm{c-int}\,(\mathrm{dom}\, h)$. By Theorem 4.7 it follows that $V'' = e^{-h^*}$ and hence all functions V'' coincide for all subsubnets D''. □

Let $E = \mathrm{R}^n$. The main goal of the next theorem is to show that in this case *the exponential tightness* follows from the Gärtner condition.

Theorem 5.4. [2] Let a net $\{J_\alpha\}_{\alpha\in D}$ meets Gärtner's condition. Then the statement of Theorem 5.3 is valid.

6. NON–COMMUTATIVE LARGE DEVIATIONS

The problem is to describe the thermodynamic limit of the free energy density for a class of model of quantum anharmonic crystal [7]. The crystal is made of a large number of anharmonic oscillators with mean-field type coupling.

Each anharmonic oscillators individually is described by a quantum mechanical Hamiltonian which is a Schrödinger operator of the form

$$H_0 = -\frac{1}{2}\partial_x^2 + V(x) \tag{6.1}$$

acting in $L^2(\mathrm{R}, \mathrm{d}x)$, where $V(x) \approx |x|^{2s}$ for large $|x|$. The quantum crystal itself consists of some large number N of oscillators. For each k the corresponding oscillator coordinate x_k ranges over R. The configuration of all the oscillators is a point x in R^N and the corresponding Hilbert space describing the quantum states is $L^2(\mathrm{R}^N, \mathrm{d}^N x)$.

The Hamiltonian is the sum of two parts. The first is

$$H_0^N = \sum_{k=1}^{N} H_0^{(i)}, \tag{6.2}$$

where the operator $H_0^{(i)}$ is the oscillator Hamiltonian depending on the kth coordinate. The other part of the Hamiltonian is a multiplication operator. It is specified by a continuous bounded function f on R. It is

$$W^N = f\left(N^{-1}\sum_{i=1}^{N} x_k\right). \tag{6.3}$$

The total Hamiltonian of interest is

$$H^N = H_0^N + W^N. \tag{6.4}$$

This is a Schrödinger operator in a high dimensional space. Since $V(x) \approx |x|^{2s}$, by known properties of such operator [5], the Hamiltonian H^N generates a Tr-class semigroup $\exp\{-tH^N\}$.

The problem is to compute the limit

$$\lim_{n\to\infty}\left[\operatorname{Tr}\exp\left\{-t\left(H_0^N + Nf\left(N^{-1}\sum_{k=1}^{N} x_k\right)\right)\right\}\right]^{1/N}. \tag{6.5}$$

We first note that in the linear case $Nf(N^{-1}\sum_{k=1}^N x_k) = \sum_{i=1}^N x_i$, we simply have

$$\operatorname{Tr}\left[\exp\left\{-t\left(H_0^N + \sum_{i=1}^{N} x_i\right)\right\}\right] = \operatorname{Tr}\left[\exp\{-t(H_0 + x_1)\}\right],$$

and so the first part (5.2) of the Gärtner condition is fulfilled. Let I be the Legendre transform of the function $G(t) := \log\operatorname{Tr}\exp\{-t(H_0 + x_1)\}$. As it follows from the preceding sections, we may expect that a limit in (6.5) (if exists) has the form

$$\sup_{u\in\mathrm{R}} e^{f(u)}e^{-I(u)}.$$

Thus, in order to prove large deviations with the help of Theorem 5.3, we have to show that the functional

$$J_N^{1/N}(Nf) = \left(\mathrm{Tr}\left[\exp\left\{-t\left(H_0^N + Nf\left(N^{-1}\sum_{k=1}^{n} x_k\right)\right)\right\}\right]\right)^{1/N}$$

possesses some properties of the class $\mathcal{A}_2(C_b^+)$.

We first note that $J_N^{1/N}(Nf)$ is of course monotone and positively homogeneous functional. So we have to check that it is alternative functional of the order 2. This property is equivalent [4] to the strong sub-additivity of the functional in the sense

$$J_N^{1/N}(Nf_1 \vee f_2) \leq J_N^{1/N}(Nf_1) + J_N^{1/N}(Nf_2).$$

The main difference between classical large deviations and quantum large deviations is that this inequality is not correct in the case of consideration. Nevertheless, we will prove that this is valid asymptotically.

To this end we use the Feynmann–Kac formula to reduce the quantum mechanics problem to a probability problem. This is just the standard reduction of quantum statistical mechanics to classical statistical mechanics in higher dimension.

We start with the quantum oscillator Hamiltonian H_0. It is a self-adjoint operator and has eigenvector $\psi_0 > 0$ with eigenvalue λ_0. Consider the operator $\hat{H}_0$ given by

$$\psi_0^{-1}(H_0 - \lambda_0)\psi_0 = -\left(\frac{1}{2}\partial_x^2 + u(x)\partial_x\right), \tag{6.6}$$

where $u(x) = \partial_x \psi_0 / \psi_0$. The operator$\hat{H}_0$ is a self-adjoint operator acting in a new Hilbert space $L^2(\mathsf{R}, \mathrm{d}\nu^0)$, where $\mathrm{d}\nu^0(x) = \psi_0^2(x)\,\mathrm{d}x$. It has eigenvector 1 with eigenvalue 0. By the definition $\hat{H}_0$ is unitary equivalent to the operator $H_0 - \lambda_0$ acting in the original space.

The one-parameter semigroup of operators associated with the generator

$$\exp\{-t\hat{H}_0\}$$

is positivity preserving with $\exp\{-t\hat{H}_0\}1 = 1$. Thus the semigroup determines a Markov process according to a standard construction. Since it is self-adjoint, the process may be taken to be a stationary time-reversible process with ν^0 as the invariant measure. We assume that $0 \leq t \leq T$. The relation between the process and semigroup is

$$\exp\{-t\hat{H}_0\}g(x) = \int g(y)p_0(x,y,t)\,\mathrm{d}\nu^0(y), \tag{6.7}$$

where $p_0(x,y,t) = p_0(y,x,t)$ is the density of transition probability function with respect to the measure ν^0. A Markov process with a generator of this type is a diffusion process with continuous sample path defined by a probability measure μ_x

on a space of continuous function $\Omega_T = C([0,T], \mathsf{R})$ as follows: if φ is a function of the form $\varphi(\omega) = F(\omega(t_1), \dots, \omega(t_m))$, $0 \le t_1 \le \dots \le t_m \le T$, then

$$\mu_x(\varphi) = \int_{\mathsf{R}} \dots \int_{\mathsf{R}} F(x_1, \dots, x_m) p_0(x, x_1, t_1) p_0(x_1, x_2, t_2 - t_1) \dots p_0(x_{m-1}, x_m, t_m - t_{m-1})\, \mathrm{d}\nu^0(x_1) \dots \mathrm{d}\nu^0(x_m). \tag{6.8}$$

If B is a Borel subset, $\mu_x(B)$ is the probability that a particle starting at x at time zero shall follow one of the trajectories in B.

Since we assumed that $V(x) \sim |x|^{2s}$ for some $s > 1$ then $\log \psi_0(x) \sim |x|^{s+1}$ [5]. This says that when the potential V grows faster than quadratic, the probability density $|\psi_0(x)|^2$ is more concentrated than Gaussian.

The corresponding semigroup is *intrinsically ultracontractive* [5] and so there exist c_1 and c_2 such that

$$c_1 \le p_0(x, y, t) \le c_2. \tag{6.9}$$

The intuition is that the motion is a combination of symmetric diffusion and systematic drift given by the $u(x)$. Since the drift is growing faster than linear the evolution of the process from a starting point x to the stationary distribution is extraordinarily rapid and *uniform*.

Since each quantum oscillator corresponds to a diffusion process, the entire quantum crystal may be represented as a collection of independent diffusion processes $\omega = (\omega_1, \dots, \omega_N)$ with initial conditions $x = (x_1, \dots, x_N)$

$$\mathrm{d}\omega_k = -u(\omega_k)\, \mathrm{d}t + \mathrm{d}W_k, \quad k = 1, \dots, N, \tag{6.10}$$

where W_k are independent Wiener processes.

The Hilbert space for the quantum crystal is $L^2(\mathsf{R}^N, \mathrm{d}^N\nu^0)$. The independent quantum harmonic oscillators in the crystal have Hamiltonian

$$H_0^N = \sum_{k=1}^{N} H_0^{(k)} = \sum_{k=1}^{N} - \left(\frac{1}{2} \partial^2_{x_k} + u(x) \partial_{x_k} \right). \tag{6.11}$$

The ground state for the system of independent oscillators is given by the wave function ψ_0^N with

$$H_0^N \psi_0^N = \lambda_0^N \psi_0^N. \tag{6.12}$$

Here $\psi_0^N(x)$ is the product of the $\psi_0(x_k)$ over k, and $\lambda_0^N = N\lambda_0$.

The corresponding diffusion generator is defined by

$$\hat{H}_0^N = \frac{1}{\psi_0^N} (H_0^N - \lambda_0^N) \psi_0^N = \Sigma_{k=1}^N \hat{H}_0^{(k)}. \tag{6.13}$$

This acts in $L^2(\mathsf{R}^N, d^N\nu^0)$, where $d^N\nu^0 = \Pi_{i=1}^N d\nu_i^0$ is the invariant measure of the diffusion process. The stationary process with this generator consists of independent diffusion processes ω_k for $k = 1, \dots, N$. We now use ω to denote the system of all these processes taken together.

The process is defined by a probability measure $\mu_x^N = \Pi_{i=1}^N \mu_{x_i}$ on a space of continuous function $\Omega_T^N = C([0,T], \mathrm{R}^N)$ as follows: if φ is a function of the form $\varphi(\omega) = F(\omega(t_1), \dots, \omega(t_m))$, $0 \le t_1 \le \dots \le t_m \le T$, then

$$\mu_x^N(\varphi) = \int_{R^N} \cdots \int_{R^N} F(x_1, \dots, x_m) p(x, x_1, t_1)\, p(x_1, x_2, t_2 - t_1) \\ \dots p(x_{m-1}, x_m, t_m - t_{m-1})\, d^N\nu^0(x_1) \dots \mathrm{d}^N\nu^0(x_m), \qquad (6.14)$$

where $p(x, x_1, t)$ is a density of transition probability function with respect to the measure $\mathrm{d}^N\nu^0$.

Our next purpose is to represent the quantum crystal with interaction. Let $H^N = H_0^N + W^N$ be the total Hamiltonian for the crystal. If we transform it to the independent diffusion process setting we get an operator

$$\hat{H}^N = \frac{1}{\psi_0^N}(H^N - \lambda_0^N)\psi_0^N = \hat{H}_0^N + W^N. \qquad (6.15)$$

The Feynman–Kac formula for the associated semigroup is

$$\exp\{-t\hat{H}^N\} = \int_{\Omega_T^N} \mathrm{d}\mu_x^N \exp\left\{ N \int_0^T f(S_\tau^N(\omega))\, \mathrm{d}\tau \right\}, \qquad (6.16)$$

where $S_\tau^N(\omega) = N^{-1}\Sigma_{i=1}^N \omega_i(\tau)$.

Thus the problem is to study the asymptotics of the sequence of functionals

$$\left[\int_{R^N} \mathrm{d}^N\nu^0(x) \int_{\Omega_T^N} \mathrm{d}\mu_x^N \exp\left\{ N \int_0^T f(S_\tau^N(\omega))\, \mathrm{d}\tau \right\} \right]^{1/N}. \qquad (6.17)$$

So we consider

$$J_N(Nf_1 \vee f_2) = \int_{R^N} \mathrm{d}^N\nu^0(x) \int_{\Omega_T^N} \mathrm{d}\mu_x^N \exp\left(N \int_0^T f_1 \vee f_2(S_\tau^N(\omega))\, \mathrm{d}\tau \right) \qquad (6.18)$$

and our aim is to show that functional is asymptotically strongly sub-additive.

Let $\chi_A(\omega)$ be the indicator of the set

$$A = \{\omega \in \Omega_T^N : f_1(S_\tau^N(\omega)) > f_2(S_\tau^N(\omega)), \quad 0 \le \tau \le T\}$$

and $\chi_B(\omega)$ be the indicator of

$$B = \{\omega \in \Omega_T^N : f_2(S_\tau^N(\omega)) > f_1(S_\tau^N(\omega)), \quad 0 \le \tau \le T\},$$

then

$$J_N(Nf_1 \vee f_2) \quad = \quad \int_{R^N} \mathrm{d}^N\nu^0(x) \int_{\Omega_T^N} \mathrm{d}\mu_x^N \exp\left\{ N \int_0^T f_1 \vee f_2(S_\tau^N(\omega))\, \mathrm{d}\tau \right\}$$

$$
\begin{aligned}
= & \int_{R^N} \mathrm{d}^N \nu^0(x) \int_{\Omega_T^N} d\mu_x^N \chi_A(\omega) \exp\{N \int_0^T f_1(S_\tau^N(\omega))\, \mathrm{d}\tau\} \\
& + \int_{R^N} \mathrm{d}^N \nu^0(x) \int_{\Omega_T^N} \mathrm{d}\mu_x^N \chi_B(\omega) \exp\left\{N \int_0^T f_2(S_\tau^N(\omega))\mathrm{d}\tau\right\} \\
& + \int_{R^N} \mathrm{d}^N \nu^0(x) \int_{\Omega_T^N} d\mu_x^N [1 - (\chi_A(\omega) + \chi_B(\omega))] \\
& \times \exp\left\{N \int_0^T f_1 \vee f_2(S_\tau^N(\omega))\, \mathrm{d}\tau\right\}. \qquad (6.19)
\end{aligned}
$$

Passing to the inequality we get

$$
\begin{aligned}
& J_N(Nf_1 \vee f_2) \leq J_N(Nf_1) + J_N(Nf_2) \\
& + \int_{R^N} \mathrm{d}^N \nu^0(x) \int_{\Omega_T^N} \mathrm{d}\mu_x^N [1 - (\chi_A(\omega) + \chi_B(\omega))] \\
& \times \exp\{N \int_0^T f_1 \vee f_2(S_\tau^N(\omega))\, \mathrm{d}\tau\}, \qquad (6.20)
\end{aligned}
$$

and so we will be done if show that normalized integral in the last display is vanishing as N tends to infinity. To this end consider the last integral with $T = m\hbar$ and note that by Jensen's inequality

$$
\begin{aligned}
& \int_{R^N} \mathrm{d}^N \nu^0(x) \int_{\Omega_T^N} \mathrm{d}\mu_x^N [1 - (\chi_A(\omega) + \chi_B(\omega))] \\
& \times \exp\left\{N \int_0^{m\hbar} f_1 \vee f_2(S_\tau^N(\omega))\, \mathrm{d}\tau\right\} \\
= & \int_{R^N} \mathrm{d}^N \nu^0(x) \int_{\Omega_T^N} \mathrm{d}\mu_x^N [1 - (\chi_A(\omega) + \chi_B(\omega))] \\
& \times \exp\left\{\sum_{k=0}^{m-1} N\hbar^{-1} \int_0^{\hbar} \hbar f_1 \vee f_2(S_{\tau+k\hbar}^N(\omega))\, \mathrm{d}\tau\right\} \\
\leq & \hbar^{-1} \int_0^{\hbar} d\tau \int_{R^N} \mathrm{d}^N \nu^0(x) \int_{\Omega_T^N} \mathrm{d}\mu_x^N [1 - (\chi_A(\omega) + \chi_B(\omega))] \\
& \times \exp\left\{\sum_{k=0}^{m-1} N\hbar f_1 \vee f_2(S_{\tau+k\hbar}^N(\omega))\right\} \\
\leq & \sup_{\tau \in [0,\hbar]} \int_{R^N} \mathrm{d}^N \nu^0(x) \int_{\Omega_T^N} \mathrm{d}\mu_x^N [1 - (\chi_A(\omega) + \chi_B(\omega))] \\
& \times \exp\left\{\sum_{k=0}^{m-1} N\hbar f_1 \vee f_2(S_{\tau+k\hbar}^N(\omega))\right\}. \qquad (6.21)
\end{aligned}
$$

Consider a set of sequences $\Sigma = \{x_1, \ldots, x_m\}$ with

$$
x_k = (x_k^{(1)}, \ldots, x_k^{(N)}), \quad k = 1, \ldots, m.
$$

Let

$$\alpha = \left\{ \Sigma : f_1\left(N^{-1}\sum_{i=1}^{N} x_k^{(i)}\right) \geq f_2\left(N^{-1}\sum_{i=1}^{N} x_k^{(i)}\right),\ k=1,\ldots,m \right\},$$

$$\beta = \left\{ \Sigma : f_2\left(N^{-1}\sum_{i=1}^{N} x_k^{(i)}\right) \geq f_1\left(N^{-1}\sum_{i=1}^{N} x_k^{(i)}\right),\ k=1,\ldots,m \right\}$$

and let χ_α and χ_β be the indicators of this sets.

By definition of the measure μ^N (see (6.14)) for each $\tau \in [0,\hbar]$

$$\begin{aligned} &\int_{R^N} \mathrm{d}^N\nu^0(x) \int_{\Omega_T^N} \mathrm{d}\mu_x^N [1-(\chi_A(\omega)+\chi_B(\omega))] \\ &\times \exp\left\{ \sum_{k=0}^{m-1} N\hbar f_1 \vee f_2(S_{\tau+k\hbar}^N(\omega)) \right\} \\ = &\int_{R^N} \cdots \int_{R^N} [1-(\chi_\alpha(\Sigma)+\chi_\beta(\Sigma))] \exp\left\{ \sum_{k=1}^{m} N\hbar f_1 \vee f_2\left(N^{-1}\sum_{i=1}^{N} x_k^{(i)}\right)\right\} \\ &p(x,x_1,\hbar)p(x_1,x_2,\hbar)\ldots p(x_{m-1},x_m,\hbar)\mathrm{d}^N\nu^0(x)\mathrm{d}^N\nu^0(x_1)\ldots\mathrm{d}^N\nu^0(x_m). \qquad (6.22) \end{aligned}$$

Next note that by *ultracontractivity* the following inequality for the right hand side of (6.22) is fulfilled

$$\begin{aligned} r.h.s. \quad \leq \quad & K^m \int_{R^N} \cdots \int_{R^N} [1-(\chi_\alpha(\Sigma)+\chi_\beta(\Sigma))] \\ \exp \quad & \left\{ \sum_{k=1}^{m} N\hbar f_1 \vee f_2\left(N^{-1}\sum_{i=1}^{N} x_k^{(i)}\right)\right\} \mathrm{d}^N\nu^0(x_1)\ldots\mathrm{d}^N\nu^0(x_m). \ (6.23) \end{aligned}$$

All sequences in the integrals contains at least a pair k_1 and k_2 such that

$$f_1\left(N^{-1}\sum_{i=1}^{N} x_{k_1}^{(i)}\right) \geq f_2\left(N^{-1}\sum_{i=1}^{N} x_{k_1}^{(i)}\right)$$

and

$$f_2\left(N^{-1}\sum_{i=1}^{N} x_{k_2}^{(i)}\right) \geq f_1\left(N^{-1}\sum_{i=1}^{N} x_{k_2}^{(i)}\right).$$

Without lost of generality we may assume that the last inequality are fulfilled for $k_2 = 1$. Introduce a set $U = \left\{x_1 \in \mathsf{R}^N : f_2(N^{-1}\sum_{i=1}^N x^i) \geq f_1(N^{-1}\sum_{i=1}^N x^{(i)})\right\}$. Thus the integral in (6.23) we can rewrite as follows

$$\begin{aligned} &K^m \int_{R^N} \cdots \int_{R^N} [1-(\chi_\alpha(\Sigma)+\chi_\beta(\Sigma))] \\ &\times \exp\left\{ \sum_{k=1}^{m} N\hbar f_1 \vee f_2\left(N^{-1}\sum_{i=1}^{N} x_k^{(i)}\right)\right\} \mathrm{d}^N\nu^0(x_1)\ldots\mathrm{d}^N\nu^0(x_m) \end{aligned}$$

$$
\begin{aligned}
\leq\ & K^m \int_U \mathrm{d}^N\nu^0(x_1) \exp\left\{ N\hbar f_2 \left(N^{-1} \sum_{i=1}^{N} x_1^{(i)} \right) \right\} \\
\times & \left(\int_{R^N} \cdots \int_{R^N} \exp\left\{ \sum_{k=2}^{m} N\hbar f_1 \vee f_2 \left(N^{-1} \sum_{i=1}^{N} x_k^{(i)} \right) \right\} \mathrm{d}^N\nu^0(x_2) \ldots \mathrm{d}^N\nu^0(x_m) \right) \\
=\ & K^m \int_U d^N\nu^0(x_1) \exp\left\{ N\hbar f_2 \left(N^{-1} \sum_{i=1}^{N} x_1^{(i)} \right) \right\} \\
\times & \left(\int_{R^N} \exp\left\{ N\hbar f_1 \vee f_2 \left(N^{-1} \sum_{i=1}^{N} x^{(i)} \right) \right\} \mathrm{d}^N\nu^0(x) \right)^{m-1} . \qquad (6.24)
\end{aligned}
$$

Finally note that

$$
\begin{aligned}
& \left(\int_U \mathrm{d}^N\nu^0(x_1) \exp\left\{ N\hbar f_2 \left(N^{-1} \sum_{i=1}^{N} x_1^{(i)} \right) \right\} \right)^{1/N} \\
\leq\ & \left(\sup_{x_1 \in \mathrm{R}^N} \exp\left\{ N\hbar f_2 \left(N^{-1} \sum_{i=1}^{N} x_1^{(i)} \right) \right\} \right)^{1/N} \left(\int_U \mathrm{d}^N\nu^0(x) \right)^{1/N} \\
\leq\ & K \left(\int_U \mathrm{d}^N\nu^0(x) \right)^{1/N} . \qquad (6.25)
\end{aligned}
$$

Let $m = \int_{\mathrm{R}} y \,\mathrm{d}\nu^0(y)$ and let for the definiteness $f_1(m) \geq f_2(m)$. It means that points of the set U are such that $N^{-1}\sum_{i=1}^{N} x_1^{(i)}$ is separated from the expectation. Thus taking into account the strong ergodic property of ν^0 we get the desire result.

In this paper we restrict ourselves by proving strong sub-additive property of the functional and sup-integral representation of the limit. Full analysis will be given elsewhere.

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