Zuzana Prášková

Wild bootstrap in RCA(1) model

Kybernetika, Vol. 39 (2003), No. 1, [1]--12

Persistent URL: `http://dml.cz/dmlcz/135504`

KYBERNETIKA — VOLUME 39 (2003), NUMBER 1, PAGES 1 – 12

WILD BOOTSTRAP IN RCA(1) MODEL[1]

ZUZANA PRÁŠKOVÁ

In the paper, a heteroskedastic autoregressive process of the first order is considered where the autoregressive parameter is random and errors are allowed to be non-identically distributed. Wild bootstrap procedure to approximate the distribution of the least-squares estimator of the mean of the random parameter is proposed as an alternative to the approximation based on asymptotic normality, and consistency of this procedure is established.

Keywords: random coefficient autoregression, heteroskedasticity, wild bootstrap
AMS Subject Classification: 62M10, 62G09, 62E20

1. INTRODUCTION

A random coefficient autoregressive process of order p (RCA(p)) is defined by

$$X_t = \sum_{i=1}^{p} b_{it} X_{t-i} + Y_t, \qquad t = 0, \pm 1, \dots ,$$

where Y_t are independent random variables with zero mean and a constant variance σ^2, $\mathbf{b}_t = (b_{1t}, \dots , b_{pt})^T$ are independent random vectors with $E\,\mathbf{b}_t = \beta = (\beta_1, \dots , \beta_p)^T$, $\operatorname{Var} \mathbf{b}_t = \mathbf{C}_{p\times p}$, independent of Y_t. Alternatively, one can write

$$X_t = \sum_{i=1}^{p} (\beta_i + B_{it}) X_{t-i} + Y_t, \qquad t = 0, \pm 1, \dots ,$$

$\mathbf{B}_t = (B_{1t}, \dots , B_{pt})^T$ are independent zero mean random vectors, $\mathbf{B}_t = \mathbf{b}_t - \beta$.

Stability and stationarity conditions were studied by Anděl [1]; estimators of parameters and their asymptotic properties for these type of processes as well as for multivariate versions of them have been systematically studied and presented in Nicholls and Quinn [8].

There exist various generalizations of the basic model, releasing assumptions both on the random parameters and the error process, studying new types of estimators

[1]Partially supported by the Grant Agency of the Czech Republic under Grant No. 201/00/0769 and by the Ministry of Education, Youth and Sports of the Czech Republic under Grant MSM 113200008.

and tests of randomness of parameters, that all are based on the assumption of stationarity and ergodicity of RCA processes.

In this paper we consider the heteroskedastic RCA(1) process

$$X_t = (\beta + B_t)X_{t-1} + Y_t, \qquad t = 0, 1, \ldots, \tag{1}$$

where X_0 is a random variable with zero mean and variance $0 < \sigma_0^2 < \infty$, $Y_t, t = 1, 2, \ldots$ are independent random variables with zero mean and finite variances σ_t^2, that are independent of X_0, and $B_t, t = 1, 2, \ldots$ are independent random variables with zero mean and finite variance σ_B^2, independent both of X_0 and of all Y_t.

The process (1) is not stationary in general. This type of the RCA process was studied by Jürgens [5], who obtained asymptotic normality of the least-squares estimator (LSE) of parameter β under quite strong moment conditions. Recently, Janečková [3, 4], established asymptotic normality of the LSE of β under minimum moment conditions and also in cases where either errors or random parameters are martingale differences. She obtained conditions for asymptotic normality of weighted LSE of β both with known and unknown values of nuisance parameters and in [4] studied properties of maximum likelihood estimator of β.

Let us recall some known results.

Model (1) can be written in the form

$$X_t = \beta X_{t-1} + u_t, t = 0, 1, \ldots, \tag{2}$$

where

$$u_t = B_t X_{t-1} + Y_t, t = 0, 1, \ldots. \tag{3}$$

It can be easily seen that $\{u_t\}$ is a martingale difference sequence with respect to the filtration $\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots\}$, where $\mathcal{F}_0 = \sigma(X_0)$ is the σ-field generated by X_0 and similarly $\mathcal{F}_t = \sigma(X_0, Y_1, B_1, \ldots, Y_t, B_t)$, $t = 1, 2, \ldots$, and

$$E\, u_t | \mathcal{F}_{t-1} = 0, \;\; E\, u_t^2 | \mathcal{F}_{t-1} = X_{t-1}^2 \sigma_B^2 + \sigma_t^2.$$

Model (2) can be viewed as an AR(1) model with a constant coefficient β and errors u_t. Obviously, the LSE of parameter β in model (2) and thus in (1) is

$$\widehat{\beta} = \frac{\sum_{t=1}^n X_{t-1} X_t}{\sum_{t=1}^n X_{t-1}^2}. \tag{4}$$

Now, let us introduce the following assumptions.

A1: $E\,|X_0|^{4+\delta}| < \infty$, $E\,|Y_t|^{4+\delta} \le K \; < \infty$ for some $\delta > 0$ and a constant $K > 0$.

A2: $\sup_t E|b_t|^{4+\delta} < 1$ for some $\delta > 0$, where $b_t := \beta + B_t$.

A3: $E\, B_t^4$ is constant for all t.

A4: $\frac{1}{n} \sum_{t=1}^n \sigma_t^2 \to \sigma^2 > 0 \quad$ as $n \to \infty$.

A5: $\frac{1}{n} \sum_{t=1}^n \sigma_t^2 E X_{t-1}^2 \to \overline{\sigma}^2 > 0 \quad$ as $n \to \infty$.

A6: $\frac{1}{n} \sum_{t=1}^n E\, Y_t^4 \to \gamma > 0 \quad$ as $n \to \infty$.

Theorem 1. Under Assumptions A1 – A6, as $n \to \infty$

$$\widehat{\beta} \to \beta \quad \text{a.s.} \tag{5}$$

and

$$\sqrt{n}(\widehat{\beta} - \beta) \to \mathcal{N}\left(0, \frac{(1-\beta^2-\sigma_B^2)^2}{\sigma^4}\Delta^2\right) \tag{6}$$

where

$$\Delta^2 = \sigma_B^2 \frac{6(\beta^2+\sigma_B^2)\overline{\sigma}^2+\gamma}{1-\gamma_b} + \overline{\sigma}^2 \tag{7}$$

and $\gamma_b = E\, b_t^4$.

Proof. See Janečková [4] Theorems 3.1 and 3.4. □

Remark 1. Strong consistency result (5) can be proved under a set of weaker assumptions. In fact, it suffices to consider A1 and A2 with moments of order $2+\delta$, and A4, only (see Janečková [3], Theorem 3.1.)

Remark 2. From Assumption A2 the inequality $\beta^2 + \sigma_B^2 < 1$ easily follows, which is the usual condition for stability and stationarity of homoskedastic RCA(1) model (see Nicholls and Quinn [8]).

It is clear that the asymptotic variance of $\widehat{\beta}$ implied by (6) and (7) is rather complicated and inference about $\widehat{\beta}$ based on asymptotic normality could be difficult.

In this paper we deal with an alternative approach how to approximate the distribution of $\widehat{\beta}$, namely with the wild bootstrap, which reflects the heteroskedasticity of the error process. The reason for considering this procedure is that the residual based bootstrap, that is other bootstrap procedure commonly used in autoregressive models, is not consistent under heteroskedasticity even in cases when autoregressive parameters are constant (Prášková [9]).

In the next sections we shall define the wild bootstrap procedure and establish its consistency. We shall introduce some preliminary results before it.

2. PRELIMINARY RESULTS

Lemma 1. Under Assumptions A1 and A2 there exists a positive constant c such that

$$E\,|X_t|^{4+\delta} \le c < \infty \text{ for all } t \text{ and given } \delta.$$

Proof. We can see from (1) that X_t can be written, if we denote $Y_0 := X_0$, as

$$X_t = \sum_{j=0}^{t} C_{t,j-1} Y_{t-j}$$

where

$$C_{t,j} = \prod_{i=0}^{j}(\beta + B_{t-i}), \quad C_{t,-1} := 1.$$

Then the result follows by using Minkowski inequality and independence assumptions on $\{Y_t\}$ and $\{B_t\}$. □

Lemma 2. Under Assumptions A1, A2, for any bounded deterministic function g of t, the following relations hold as $n \to \infty$:

$$\frac{1}{n}\sum_{t=1}^{n} g(t)X_t \to 0 \text{ a.s.} \tag{8}$$

and

$$\frac{1}{n}\sum_{t=1}^{n} g(t)(X_t^2 - E\,X_t^2) \to 0 \text{ a.s.} \tag{9}$$

Proof. It follows from the fact that both $\{g(t)X_t\}, \{g(t)(X_t^2 - E\,X_t^2)\}$ are mixingales with respect to $\mathcal{F}$ (see Janečková [4], Lemmas 3.4 and 3.5) and Davidson [2], Theorem 20.16.) □

Lemma 3. Under Assumptions A1 – A6, as $n \to \infty$

$$\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2 \to \frac{\sigma^2}{1-\beta^2-\sigma_B^2} \quad \text{a.s.} \tag{10}$$

$$\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2 Y_t^2 \to \overline{\sigma}^2 \quad \text{a.s.} \tag{11}$$

$$\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^4 \to \frac{6\overline{\sigma}^2(\beta^2+\sigma_B^2)+\gamma}{1-\gamma_b} \quad \text{a.s.} \tag{12}$$

Proof. Consider (1) in the form $X_t = b_t X_{t-1} + Y_t$ with $b_t = \beta + B_t, E\,b_t^2 = \beta^2 + \sigma_B^2$. From here we get after simple algebra

$$\begin{aligned}(1-\beta^2-\sigma_B^2)\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2 \qquad (13)\\ = \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2(b_t^2 - E\,b_t^2) + 2\frac{1}{n}\sum_{t=1}^{n} X_{t-1}b_tY_t + \frac{1}{n}\sum_{t=1}^{n} Y_t^2 + \frac{1}{n}(X_0^2 - X_n^2).\end{aligned}$$

The first two terms on the right-hand side of (13) converge to 0 almost surely according to the strong law of large numbers for martingale differences $\{X_{t-1}^2(b_t^2 -$

$E\,b_t^2)\}, \{X_{t-1}b_tY_t\}$, respectively (see e.g. Davidson [2], Chapter 20). The strong law of large numbers for $\{Y_t^2\}$ and A4 yields

$$\frac{1}{n}\sum_{t=1}^{n} Y_t^2 \to \sigma^2 \quad \text{a.s.} \tag{14}$$

The last term on the right-hand side of (13) converges to 0 almost surely according to Borel–Cantelli lemma which completes the proof of (10).

Proof of (12) can be obtained in a similar way. Simultaneously, we will prove relation (11). Again, from $X_t = b_tX_{t-1} + Y_t$ we get

$$\begin{aligned}(1-\gamma_b)\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^4 &= \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^4(b_t^4 - E\,b_t^4) + 4\frac{1}{n}\sum_{t-1}^{n} X_{t-1}^3 b_t^3 Y_t \\ &+ 6\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2 Y_t^2(b_t^2 - E\,b_t^2) + 6(\beta^2+\sigma_B^2)\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2 Y_t^2 \\ &+ 4\frac{1}{n}\sum_{t=1}^{n} X_{t-1}b_t(Y_t^3 - E\,Y_t^3) + 4\frac{1}{n}\sum_{t=1}^{n} X_{t-1}b_t E\,Y_t^3 \\ &+ \frac{1}{n}\sum_{t=1}^{n} Y_t^4 + \frac{1}{n}(X_0^4 - X_n^4).\end{aligned} \tag{15}$$

The strong law of large numbers for $\{Y_t^4\}$ and A6 gives

$$\frac{1}{n}\sum_{t=1}^{n} Y_t^4 \to \gamma \text{ a.s.} \tag{16}$$

Further we have

$$\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2Y_t^2 = \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2(Y_t^2-\sigma_t^2) + \frac{1}{n}\sum_{t=1}^{n}(X_{t-1}^2 - E\,X_{t-1}^2)\sigma_t^2 + \frac{1}{n}\sum_{t=1}^{n}\sigma_t^2 E\,X_{t-1}^2.$$

The first term on the right-hand side of the last equation tends to 0 according to the strong law of large numbers for martingale differences; the second term tends to 0 according to (9) and the last one to $\overline{\sigma}^2$ according to A5. As a result we have proved (11).

Next,

$$\frac{1}{n}\sum_{t=1}^{n} X_{t-1}b_tE\,Y_t^3 = \frac{1}{n}\sum_{t=1}^{n} X_{t-1}B_tE\,Y_t^3 + \beta\frac{1}{n}\sum_{t=1}^{n} X_{t-1}E\,Y_t^3.$$

From the strong law of large numbers for martingale differences $\{X_{t-1}B_tE\,Y_t^3\}$ and from mixingale property (8) with $g(t) = E\,Y_{t+1}^3$ we can conclude that $\frac{1}{n}\sum_{t=1}^{n} X_{t-1}b_tE\,Y_t^3$ converges to 0 almost surely.

Finally, we can apply the strong law of large numbers for martingale differences to all the remaining terms on the right-hand side of (15) but the last one, that will vanish according to Borel–Cantelli lemma. □

Corollary 1. Under Assumptions A1 – A6, as $n \to \infty$

$$\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2 u_t^2 \to \Delta^2 \text{ a.s.} \tag{17}$$

where Δ^2 is given by (7).

Proof. It can be easily shown that

$$\begin{aligned} & \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2 u_t^2 \\ = \; & \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^4 (B_t^2 - \sigma_B^2) + \sigma_B^2 \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^4 + 2\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^3 B_t Y_t + \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2 Y_t^2. \end{aligned}$$

The result follows from (11), (12) and the strong law of large numbers for martingale differences $\{X_{t-1}^4(B_t^2 - \sigma_B^2)\}$ and $\{X_{t-1}^3 B_t Y_t\}$. □

3. WILD BOOTSTRAP PROCEDURE

Considering RCA(1) model given by (2) we can define wild bootstrap procedure similarly as in constant coefficient AR(1) process (see e.g. Kreiss [6] or Prášková [9]):

Estimate residuals

$$\widehat{u}_t = X_t - \widehat{\beta} X_{t-1} \tag{18}$$

with $\widehat{\beta}$ given by (4). Construct bootstrap residuals

$$u_t^w = \widehat{u}_t K_t, t = 1, \ldots, n \tag{19}$$

where K_t are iid with zero means, unit variances and finite moments of order $2 + \delta$, $\delta > 0$, independent of $\{B_t, Y_t, 1 \le t \le n\}$ and of X_0. Compute bootstrap observations

$$X_t^w = \widehat{\beta} X_{t-1} + u_t^w, t = 1, \ldots, n \tag{20}$$

and then the LSE $\widehat{\beta}^w$ of β in regression of X_t^w on X_{t-1}, i.e.

$$\widehat{\beta}^w = \frac{\sum_{t=1}^n X_{t-1} X_t^w}{\sum_{t=1}^n X_{t-1}^2}.$$

The bootstrap approximation of the distribution of $\widehat{\beta}$ is given by the following theorem.

Theorem 2. Under assumptions A1 – A6, as $n \to \infty$,

$$\sup_x |P(\sqrt{n}(\widehat{\beta} - \beta) \leq x) - P^w(\sqrt{n}(\widehat{\beta}^w - \widehat{\beta}) \leq x)| \to 0 \text{ a.s.} \tag{21}$$

where P^w means the conditional probability given $X_0, \dots, X_n$.

Proof. We shall prove that the limiting conditional distribution of $\sqrt{n}(\widehat{\beta}^w - \widehat{\beta})$ given $X_0, \dots, X_n$ is the same as that of $\sqrt{n}(\widehat{\beta} - \beta)$ obtained in Theorem 1.

Notice that given $X_0, X_1, \dots, X_n$,

$$\sqrt{n}(\widehat{\beta}^w - \widehat{\beta}) = \frac{\frac{1}{\sqrt{n}} \sum_{t=1}^n X_{t-1} u_t^w}{\frac{1}{n} \sum_{t=1}^n X_{t-1}^2} = \frac{\frac{1}{\sqrt{n}} \sum_{t=1}^n X_{t-1} \widehat{u}_t K_t}{\frac{1}{n} \sum_{t=1}^n X_{t-1}^2} \tag{22}$$

is a linear combination of independent random variables u_t^w, respectively of $\widehat{u}_t K_t$, for which $E^w u_t^w = E(\widehat{u}_t K_t | X_0, \dots, X_n) = \widehat{u}_t E K_t = 0$, $Var^w(u_t^w) = \widehat{u}_t^2 \operatorname{Var} K_t = \widehat{u}_t^2$ and $E^w |u_t^w|^{2+\delta} = c|\widehat{u}_t|^{2+\delta}$ where $c = E|K_1|^{2+\delta}$. Due to (10) it suffices to prove asymptotic normality of $\sum_{t=1}^n X_{t-1} u_t^w / \sqrt{n}$.

Denote

$$S_n^2 = \sum_{t=1}^n Var^w \left(\frac{X_{t-1} u_t^w}{\sqrt{n}} \right) = \frac{1}{n} \sum_{t=1}^n X_{t-1}^2 \widehat{u}_t^2.$$

From (18) we have

$$S_n^2 = \frac{1}{n} \sum_{t=1}^n X_{t-1}^2 u_t^2 - 2(\widehat{\beta} - \beta) \frac{1}{n} \sum_{t=1}^n X_{t-1}^3 u_t + (\widehat{\beta} - \beta)^2 \frac{1}{n} \sum_{t=1}^n X_{t-1}^4$$

and combining (17), strong consistency of $\widehat{\beta}$, (12) and the strong law for martingale differences $\{X_{t-1}^3 u_t\}$ we get that $S_n^2 \to \Delta^2$ a.s.

Now, we verify the Feller–Lindeberg condition.

$$\frac{1}{S_n^2} \sum_{t=1}^n E^w \left[\left(\frac{X_{t-1} u_t^w}{\sqrt{n}} \right)^2 I \left\{ \left| \frac{X_{t-1} u_t^w}{\sqrt{n}} \right| > \epsilon S_n \right\} \right]$$
$$\leq \frac{1}{\epsilon^\delta S_n^{2+\delta}} \cdot \frac{1}{n^{1+\frac{\delta}{2}}} \sum_{t=1}^n E^w |X_{t-1} u_t^w|^{2+\delta} = c \frac{1}{\epsilon^\delta S_n^{2+\delta}} \cdot \frac{1}{n^{1+\frac{\delta}{2}}} \sum_{t=1}^n |X_{t-1} \widehat{u}_t|^{2+\delta}, \tag{23}$$

further,

$$\frac{1}{n^{1+\frac{\delta}{2}}} \sum_{t=1}^n |X_{t-1} \widehat{u}_t|^{2+\delta} \leq 4 \frac{1}{n^{1+\frac{\delta}{2}}} \sum_{t=1}^n |X_{t-1} u_t|^{2+\delta} + 4 |\widehat{\beta} - \beta|^{2+\delta} \frac{1}{n^{1+\frac{\delta}{2}}} \sum_{t=1}^n |X_{t-1}|^{4+2\delta} \tag{24}$$

and

$$\frac{1}{n^{1+\frac{\delta}{2}}} \sum_{t=1}^n |X_{t-1} u_t|^{2+\delta} \leq 4 \frac{1}{n^{1+\frac{\delta}{2}}} \sum_{t=1}^n |X_{t-1}|^{4+2\delta} |B_t|^{2+\delta} + 4 \frac{1}{n^{1+\frac{\delta}{2}}} \sum_{t=1}^n |X_{t-1} Y_t|^{2+\delta}. \tag{25}$$

From Assumptions A1, A2 and from Lemma 1 in Liu [7] it follows that

$$\frac{1}{n^{1+\frac{\delta}{4}}}\sum_{t=1}^{n}|Y_t|^{4+\delta}\to 0 \ \text{ a.s.}$$

which implies that $\max_{1\le t\le n}|Y_t| = o(n^{\frac{1}{4}})$. The same considerations yield $\max_{1\le t\le n}|B_t| = o(n^{\frac{1}{4}})$ and thus $\max_{1\le t\le n}|X_t| = o(n^{\frac{1}{4}})$. Now we have

$$\frac{1}{n^{1+\frac{\delta}{2}}}\sum_{t=1}^{n}|X_{t-1}|^{4+2\delta}\le\frac{\max_{1\le t\le n}|X_t|^{2\delta}}{n^{1+\frac{\delta}{2}}}\sum_{t=1}^{n}|X_{t-1}|^{4}=\frac{o(n^{\frac{\delta}{2}})}{n^{\frac{\delta}{2}}}\cdot\frac{1}{n}\sum_{t=1}^{n}X_{t-1}^{4}$$

which together with (12) implies that

$$\frac{1}{n^{1+\frac{\delta}{2}}}\sum_{t=1}^{n}|X_{t-1}|^{4+2\delta}\to 0 \ \text{ a.s.} \tag{26}$$

Next, we can write

$$\begin{aligned}&\frac{1}{n^{1+\frac{\delta}{2}}}\sum_{t=1}^{n}|X_{t-1}Y_t|^{2+\delta} &(27)\\ =\ &\frac{1}{n^{1+\frac{\delta}{2}}}\sum_{t=1}^{n}|X_{t-1}|^{2+\delta}(|Y_t|^{2+\delta}-E\,|Y_t|^{2+\delta})+\frac{1}{n^{1+\frac{\delta}{2}}}\sum_{t=1}^{n}|X_{t-1}|^{2+\delta}E\,|Y_t|^{2+\delta}.\end{aligned}$$

The first term on the right-hand side of (27) converges to 0 a.s. according to the strong law of large numbers for martingale differences. For the second term we have, due to $E\,|Y_t|^{2+\delta}\le C$ where C is a constant,

$$\frac{1}{n^{1+\frac{\delta}{2}}}\sum_{t=1}^{n}|X_{t-1}|^{2+\delta}E\,|Y_t|^{2+\delta}\le C\frac{\max_{1\le t\le n}|X_t|^{\delta}}{n^{\frac{\delta}{2}}}\frac{1}{n}\sum_{t=1}^{n}X_{t-1}^{2} \tag{28}$$

and from here together with (10) we can easily conclude that

$$\frac{1}{n^{1+\frac{\delta}{2}}}\sum_{t=1}^{n}|X_{t-1}Y_t|^{2+\delta}\to 0 \ \text{ a.s.} \tag{29}$$

It remains to prove that

$$\frac{1}{n^{1+\frac{\delta}{2}}}\sum_{t=1}^{n}|X_{t-1}|^{4+2\delta}|B_t|^{2+\delta}\to 0 \text{ a.s.} \tag{30}$$

We have

$$\frac{1}{n^{1+\frac{\delta}{2}}}\sum_{t=1}^{n}|X_{t-1}|^{4+2\delta}|B_t|^{2+\delta}\le\frac{\max_{1\le t\le n}|X_t|^{2\delta}}{n^{\frac{\delta}{2}}}\frac{1}{n}\sum_{t=1}^{n}X_{t-1}^{4}|B_t|^{2+\delta}.$$

and

$$\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^{4}|B_t|^{2+\delta} = \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^{4}(|B_t|^{2+\delta} - E\,|B_t|^{2+\delta}) + \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^{4}E\,|B_t|^{2+\delta}.$$

Then we can proceed in the same way as before, utilizing the strong law of large numbers for martingale differences, boundedness of $E|B_t|^{2+\delta}$ and (12).

As a consequence of all these considerations we see that the left-hand side of (23) converges to 0 a.s. We have proved that the conditional distribution of $\frac{1}{S_n}\sum_{t=1}^{n}\frac{X_{t-1}u_t^w}{\sqrt{n}}$ is asymptotically $\mathcal{N}(0,1)$. The rest of the proof follows from the fact that $S_n^2 \to \Delta^2$ a.s. and from (10). □

4. SIMULATIONS

We generated nonstationary process $X_t = b_t X_{t-1} + Y_t$ with independent errors $Y_t \sim \mathcal{N}(0,\sigma_t^2)$ for $\sigma_t^2 = 1 + (0.5)(-1)^t$ and with iid random parameters b_t having either normal distribution $\mathcal{N}(\beta,\sigma_B^2)$ or uniform distribution on the interval $[0,1](\mathcal{R}[0,1])$. The parameters β and σ_B^2 were chosen in such a way to satisfy Assumptions A1 – A6.

Further we generated wild bootstrap observations according to (20) with iid $K_t \sim \mathcal{N}(0,1)$. Some results are presented in Tables 1 – 4 and Figures 1, 2. In Tables 1 – 4, 95 % confidence intervals for β are introduced, computed either on the basis of the asymptotic normality result (6) either on the basis of the wild bootstrap approximation for which 5000 simulations of $\sqrt{n}(\widehat{\beta}^w - \widehat{\beta})$ were used. In Tables 1 – 3 the results are shown for $b_t \sim \mathcal{N}(\beta,\sigma_B^2)$ with various values of β and σ_B^2 (case $\sigma_B^2 = 0$ corresponds to the nonrandom autoregression $AR(1)$) and for various sample sizes n. In Table 4, similar results are presented for $b_t \sim \mathcal{R}[0,1]$, i. e. with $\beta = 0.5, \sigma_B^2 = 1/12$. It can be seen that the wild bootstrap works well and for milder sample sizes gives better results than the normal approximation.

In Figure 1, the distribution of 5000 wild bootstrap values of $\sqrt{n}(\widehat{\beta}^w - \widehat{\beta})$ (dark bars) are compared with the true distribution of $\sqrt{n}(\widehat{\beta} - \beta)$ computed by Monte Carlo using 5000 sampling values (white bars) and with the corresponding asymptotic distribution (normal curve). The number of observation was chosen to be $n = 200$ and for b_t we chose $\mathcal{N}(0;0.25), \mathcal{N}(0.1;0.25), \mathcal{N}(0.5;0.25), \mathcal{R}(0,1)$.

In Figure 2 we compared the true, wild bootstrap and asymptotic distribution of $\sqrt{n}(\widehat{\beta} - \beta)$ with the same value of $\beta^2 + \sigma_B^2$ for $n = 200$ (the top panels) and $n = 400$ (the bottom panels), in the left panels for $\beta = 0, \sigma_B^2 = 0.5$, in the right panels for $\beta = \sqrt{0}.5$ and $\sigma_B^2 = 0$ (nonrandom autoregression).

Results demonstrate the appropriateness of the application of the wild bootstrap that does not require the knowledge of asymptotic variance of $\widehat{\beta}$, as well as the influence of the variance of the random parameter upon the asymptotic distribution of $\widehat{\beta}$.

(Received November 11, 2002.)

Table 1. 95 % confidence intervals for β; $b_t \sim \mathcal{N}(\beta, \sigma_B^2)$; $n = 100$.

β	0.5	$\sqrt{2}/2$	0
σ_B^2	0	0	0.5
norm	(0.3223, 0.6353)	(0.5390, 0.8042)	(-0.3093, 0.5745)
boot	(0.3433, 0.6181)	(0.5513, 0.7951)	(-0.1047, 0.3678)
β	0	0.1	0.5
σ_B^2	0.25	0.25	0.25
norm	(-0.4046, 0.0882)	(-0.1143, 0.3786)	(0.1852, 0.7396)
boot	(-0.3802, 0.0636)	(-0.0681, 0.3399)	(0.2633, 0.6516)

Table 2. 95 % confidence intervals for β; $b_t \sim \mathcal{N}(\beta, \sigma_B^2)$; $n = 200$.

β	0.5	$\sqrt{2}/2$	0
σ_B^2	0	0	0.5
norm	(0.4671, 0.6885)	(0.6731, 0.8639)	(-0.4080, 0.2169)
boot	(0.4732, 0.6814)	(0.6878, 0.8558)	(-0.2795, 0.0788)
β	0	0.1	0.5
σ_B^2	0.25	0.25	0.25
norm	(0.0084, 0.3570)	(-0.0084, 0.3380)	(0.2628, 0.6548)
boot	(0.0144, 0.3437)	(-0.0102, 0.3472)	(0.2445, 0.6786)

Table 3. 95 % confidence intervals for β; $b_t \sim \mathcal{N}(\beta, \sigma_B^2)$; $n = 400$.

β	0.5	$\sqrt{2}/2$	0
σ_B^2	0	0	0.5
norm	(0.4386, 0.5951)	(0.6132, 0.7459)	(-0.1153, 0.3266)
boot	(0.4371, 0.5949)	(0.6094, 0.7512)	(-0.0414, 0.2512)
β	0	0.1	0.5
σ_B^2	0.25	0.25	0.25
norm	(-0.1790, 0.0674)	(0.0177, 0.2630)	(0.3366, 0.6138)
boot	(-0.1627, 0.0506)	(0.0076, 0.2748)	(0.3122, 0.6423)

Table 4. 95 % confidence intervals for β; $b_t \sim \mathcal{R}[0, 1]$.

n	100	200	400
norm	(0.3919, 0.6498)	(0.4496, 0.6641)	(0.4319, 0.5836)
boot	(0.3975, 0.6203)	(0.4542, 0.6639)	(0.4141, 0.6051)

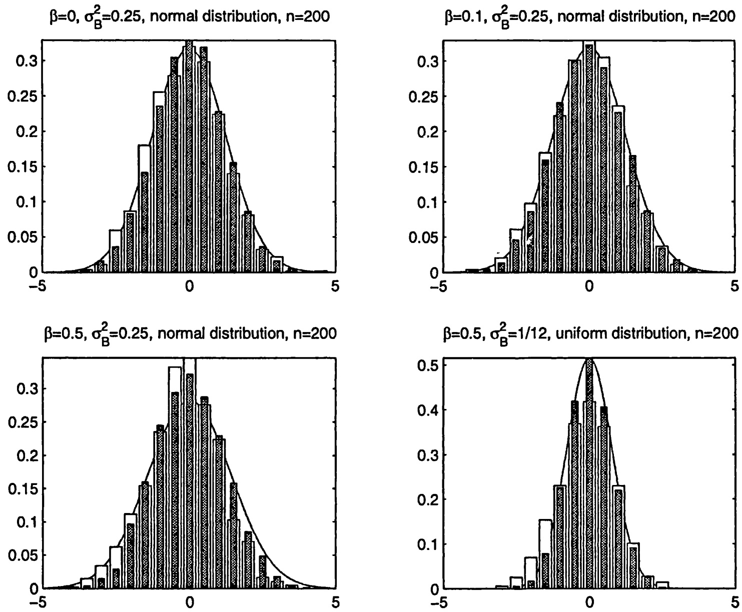

Fig. 1. True, asymptotic and wild bootstrap distribution of $\sqrt{n}(\widehat{\beta}-\beta)$.

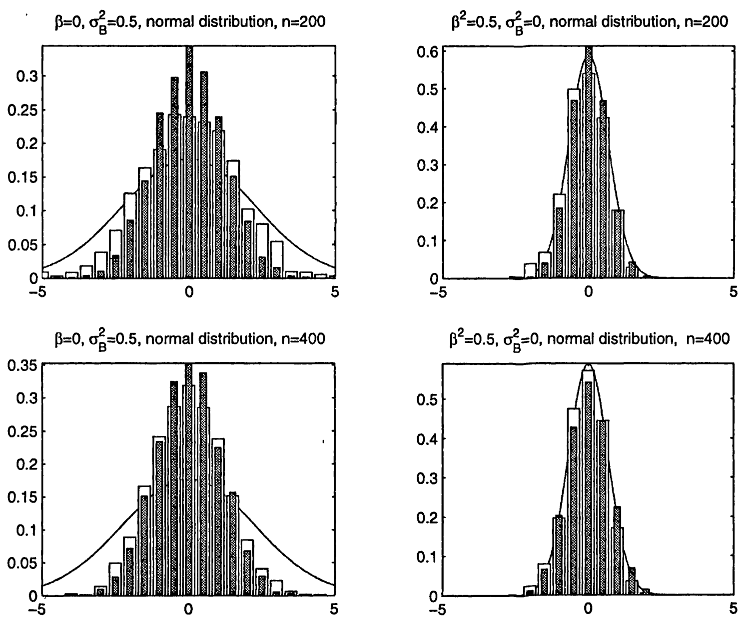

Fig. 2. True, asymptotic and wild bootstrap distribution of $\sqrt{n}(\widehat{\beta}-\beta)$; $\beta^2+\sigma_B^2=\frac{1}{2}$.

REFERENCES

[1] J. Anděl: Autoregressive series with random parameters. Math. Operationsforch. Statist. *7* (1976), 735–741.
[2] J. Davidson: Stochastic Limit Theory. Oxford University Press, New York 1994.
[3] H. Janečková: Time Series with Changing Parameters. Ph.D. Thesis, Charles University, Prague 2002.
[4] H. Janečková: Some generalizations in a heteroscedastic RCA(1) model. Acta Univ. Carolin. – Math. Phys. *43* (2002), 21–47.
[5] U. Jürgens: The estimation of a random coefficient AR(1) process under moment conditions. Statist. Hefte *16* (1985), 237–249.
[6] J. P. Kreiss: Asymptotical properties of residual bootstrap for autoregression. Preprint, TU Braunschweig 1997.
[7] R. Y. Liu: Bootstrap procedures under some non-i.i.d. models. Ann. Statist. *16* (1988), 1696–1708.
[8] D. F. Nicolls and B. G. Quinn: Random Coefficient Autoregressive Models: An Introduction (Lecture Notes in Statistics 11). Springer–Verlag, New York 1982.
[9] Z. Prášková: Bootstrapping in nonstationary autoregression. Kybernetika *38* (2002), 389–404.

Doc. RNDr. Zuzana Prášková, CSc., Department of Probability and Statistics, Faculty of Mathematics and Physics – Charles University, Sokolovská 83, 186 75 Praha 8. Czech Republic.
e-mail: praskova@karlin.mff.cuni.cz