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REPRESENTATION OF LOGIC FORMULAS BY NORMAL FORMS[1]

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In this paper, we deal with the disjunctive and conjunctive normal forms in the frame of predicate BL-logic and prove theirs conditional equivalence to appropriate formulas. Our aim is to show approximation ability of special normal forms defined by means of reflexive binary predicate.

1. INTRODUCTION

In this paper, we deal with fuzzy logic formulas, which formalize linguistically expressed collections of fuzzy "IF-THEN" rules, namely disjunctive (DNF) and conjunctive (CNF) normal forms (see [8]). Both normal forms are regarded to be suitable for equivalent transformation of formulas of specific fuzzy logic theory. This transformation is called in [8] "logical approximation".

It is worth noticing that actually three different ways led us to the construction of normal forms in fuzzy logic. The first is the way of generalization of classical construction. On the second way, we have generalized logical formulas, which are used in the formalization of fuzzy "IF-THEN" rules. Finally, there are constructions of classical algebraic formulas used for interpolation or approximation of continuous functions and they have common structure which can be represented using logical means.

Let us remind that in classical logic, each formula can be transformed into its simplified disjunctive and conjunctive normal form such that both normal forms are equivalent to the initial one. In fuzzy logic, the situation is different . Here, formulations of normal forms are no more equivalent. Further, we will look at concrete conditions under which the considered normal forms will become equivalent. This led us to the term "conditional equivalence".

We will consider Basic Logic (BL for short) introduced in [2]. BL-logic can be viewed as a basic logic for all logics based on continuous t-norms.

In BL-logic, the notion of disjunctive and conjunctive normal form is still not well established. There are many authors dealing with problems connected with

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them, e.g. I. Perfilieva, D. Mundici, V. Kreinovich etc. We can find an implicit definition of a disjunctive normal form in [1] arising from the constructive proof of McNaughton theorem. In [5], the explicit definition is given for Łukasiewicz logic. In [8], the conditional equivalence between any extensional formula and its normal forms has been proved formally. The extensionality has been defined w.r.t. similarity (the predicate characterized by axioms of reflexivity, symmetry and transitivity). In the present paper, we will concentrate ourselves on extensionality w.r.t. reflexive predicate.

In the sequel, we will use the definition of normal forms for predicate BL-logic in the sense of [8]. Moreover, we will look for such requirements from which result the information about the extensionality property of an arbitrary formula. The next subsection will be devoted to the study of the conditions under which DNF is equivalent to CNF, and both are equivalent to the initial formula.

The paper is divided into the following sections. At the beginning the fuzzy predicate logic is introduced and the general definitions of disjunctive and conjunctive normal forms in predicate fuzzy logic are given. The next Section 2 is devoted to the basic notions and properties. In the following section, the extensionality property and its determining is studied. And finally, in Section 4 the conditional equivalence between an extensional formula and its normal forms is proved. Also, the relationship of normal forms to initial formula is shown there.

2. BL-LOGIC AND BL-NORMAL FORMS

We will consider BL-logic introduced by P. Hájek in [2]. This book is regarded as a fundamental one (see [1]). Therefore, we will follow the notation used in this book.

We will deal with some fixed language J of predicate BL-logic. Recall that it consists of a non-empty set of predicates, set of object constants, object variables, set of connectives $\{\neg, \&, \rightarrow, \vee, \wedge, \equiv\}$ and quantifiers, and it does not contain functional symbols.

The predicate BL-calculus (BL$\forall$) contains a set of logical axioms on connectives (see 2.2.4 in [2]), quantifiers (see 5.1.7 in [2]) and the usual deduction rules (modus ponens and generalization rule).

An **L**-structure $\mathcal{M} = \langle M, (r_P)_{P\in J}, (m_c)_{c\in J}\rangle$ for the language J consists of a non-empty domain M, **L**-fuzzy relations $r_P : M^n \rightarrow \mathbf{L}$ assigned to each n-ary predicate symbol P, and designated elements $m_c \in M$ assigned to each object constant $\boldsymbol{c}$, where **L** is a linearly ordered BL-algebra. As a special case we will take $\mathbf{L} = [0, 1]$.

Now we are able to introduce the disjunctive and conjunctive normal forms in the frame of BL-logic. The basic definitions of normal forms for Łukasiewicz logic are well established by I. Perfilieva in [5]. For additional properties, see [7]. The extension of this notation to predicate BL-logic and the next definition is taken from [8].

Definition 1. Let $P_1, \ldots, P_k$ be unary predicate symbols and $E_{i_1\ldots i_n}$, $1 \le i_j \le k$, $1 \le j \le n$, $n \ge 1$, be either truth constants or closed instances of some formula. The following formulas of fuzzy predicate logic are called the disjunctive normal form

$$\mathrm{DNF}(x_1, \dots, x_n) = \bigvee_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1) \mathbin{\&} \cdots \mathbin{\&} P_{i_n}(x_n) \mathbin{\&} E_{i_1\dots i_n}) \tag{1}$$

and the conjunctive normal form

$$\mathrm{CNF}(x_1, \dots, x_n) = \bigwedge_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1) \mathbin{\&} \cdots \mathbin{\&} P_{i_n}(x_n) \to E_{i_1\dots i_n}). \tag{2}$$

We can see that this definition is too general for proving any properties, therefore further, we will specify all P_i and E_i in the previous definition.

3. EXTENSIONALITY AND ADDITIONAL PROPERTIES

We will extend the language J by the binary predicate symbol R and the predicate BL-theory T by the additional reflexivity axiom for R, i. e.

$$T \vdash (\forall x)R(x, x).$$

Our theory T may also include additional axioms for the considered predicate symbol R, for example

$$T \vdash (\forall x, y)(R(x, y) \to R(y, x)), \quad \text{(symmetry)}$$
$$T \vdash (\forall x, y, z)((R(x, y) \mathbin{\&} R(y, z)) \to R(x, z)). \quad \text{(transitivity)}$$

Note that R satisfying all three axioms is called similarity and is usually denoted by $\approx$. If R is reflexive and transitive then R is called quasiorder ($\preceq$).

It is easy to verify that the theory over BL$\forall$ with the reflexivity axiom for R proves the same axiom for R replaced by R^K, where R^K stands for $R(x, y) \mathbin{\&} \cdots \mathbin{\&} R(x, y)$ (K times).

Definition 2. A predicate P of arity n is called extensional w.r.t. a binary predicate R if

$$T \vdash R(x_1, y_1) \mathbin{\&} \cdots \mathbin{\&} R(x_n, y_n) \to (P(x_1, \dots, x_n) \to P(y_1, \dots, y_n)). \tag{3}$$

Lemma 1. Let P be a predicate of arity n extensional w.r.t. a binary predicate R. If R is symmetric then

$$T \vdash R(x_1, y_1) \mathbin{\&} \cdots \mathbin{\&} R(x_n, y_n) \to (P(x_1, \dots, x_n) \equiv P(y_1, \dots, y_n)). \tag{4}$$

Analogous formulation of Lemma 5.6.8 introduced in [2] can be proved under the reflexivity assumption on R.

Lemma 2. Let T be a theory containing the reflexivity axiom for R and the extensionality axioms for $P_1, \dots, P_n$ w.r.t. R. Let φ be a formula built from the predicates $P_1, \dots, P_n$ and let $K = \text{degree}(\varphi)$. Let $x_1, \dots, x_n$ be variables including all free variables of φ and let y_i be substituable for x_i in φ $(i = 1, \dots, n)$. Then,

$$T \ \vdash \ R^K(x_1, y_1) \mathbin{\&} \cdots \mathbin{\&} R^K(x_n, y_n) \ \rightarrow \ (\varphi(x_1, \dots, x_n) \ \rightarrow \ \varphi(y_1, \dots, y_n)).$$

Such formulas can also be called extensional w.r.t. R^K. It is worth of noticing that this lemma holds also for ordinary binary predicate R if the generalization rule in not used.

4. DETERMINATION OF EXTENSIONALITY

We will start with specification of normal forms. Further, two types of formulas will be presented in order to show that they become related with the initial formula in the special case and we will see that this relations are contingent on extensionality property of the initial formula.

We will deal with a theory T over BL$\forall$ extended by a reflexive binary predicate R. The language $J(T)$ of the theory T is supposed to contain a finite number of object constants $C = \{c_i \mid i = 1 \dots k\}$.

Suppose an arbitrary formula $\varphi(x_1, \dots, x_n)$. Let us specify $P_{i_j}(x)$ and $E_{i_1 \dots i_n}$ by $R^K(x, c_{i_j})$ in CNF, $R^K(c_{i_j}, x)$ in DNF and $\varphi(c_{i_1} \dots c_{i_n})$, respectively. This specifications will change the expressions for DNF and CNF of the forms (1), (2) into

$$\text{DNF}_\varphi(x_1, \dots, x_n) = \bigvee_{i_1, \dots, i_n = 1}^{k} (R(c_{i_1}, x_1) \mathbin{\&} \cdots \mathbin{\&} R(c_{i_n}, x_n) \mathbin{\&} \varphi(c_{i_1} \dots c_{i_n})), \tag{5}$$

$$\text{CNF}_\varphi(x_1, \dots, x_n) = \bigwedge_{i_1, \dots, i_n = 1}^{k} (R(x_1, c_{i_1}) \mathbin{\&} \cdots \mathbin{\&} R(x_n\, c_{i_n}) \rightarrow \varphi(c_{i_1} \dots c_{i_n})). \tag{6}$$

The next form of formulas are important for determining the extensionality property of initial formula.

Definition 3. Consider an arbitrary formula φ with n variables. The CNF$_\varphi$-closing($\overline{\text{CNF}_\varphi}$) and DNF$_\varphi$-closing($\overline{\text{DNF}_\varphi}$) of formula φ w.r.t. R are defined by formulas

$$\overline{\text{CNF}_\varphi}(x_1, \dots, x_n) = (\forall y_1, \dots, y_n)\, (R(x_1, y_1) \mathbin{\&} \cdots \mathbin{\&} R(x_n\, y_n) \rightarrow \varphi(y_1 \dots y_n)) \tag{7}$$

$$\overline{\text{DNF}_\varphi}(x_1, \dots, x_n) = (\exists y_1, \dots, y_n)\, (R(y_1, x_1) \mathbin{\&} \cdots \mathbin{\&} R(y_n\, x_n) \mathbin{\&} \varphi(y_1 \dots y_n)) \tag{8}$$

Note that both formulas need not be closed w.r.t. all free variables. In the following lemma we will establish the relation between closing formulas and DNF$_\varphi$, CNF$_\varphi$ for the same initial formula φ.

Lemma 3. Let T be a theory over $BL\forall$ containing a binary predicate R and $\varphi(x_1, \ldots, x_n)$ be an arbitrary formula. Moreover, let the language $J(T)$ be extended by $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k$ as object constants.Then

$$T \vdash \overline{\mathrm{CNF}_\varphi}(x_1, \ldots, x_n) \to \mathrm{CNF}_\varphi(x_1, \ldots, x_n) \tag{9}$$

$$T \vdash \mathrm{DNF}_\varphi(x_1, \ldots, x_n) \to \overline{\mathrm{DNF}_\varphi}(x_1, \ldots, x_n). \tag{10}$$

Proof. For the simplicity, we will consider $n = 1$. From the substitution axiom follows that

$$T \vdash (\forall y)\,(R(x,y) \to \varphi(y)) \to (R(x,c_i) \to \varphi(c_i)),$$
$$T \vdash (R(c_i,x)\ \&\ \varphi(c_i)) \to (\exists y)\,(R(y,x)\ \&\ \varphi(y)),$$

for all i, which gives us

$$T \vdash (\forall y)\,(R(x,y) \to \varphi(y)) \to \bigwedge_{i=1}^{k} (R(x,c_i) \to \varphi(c_i)),$$
$$T \vdash \bigvee_{i=1}^{k} (R(c_i,x)\ \&\ \varphi(c_i)) \to (\exists y)\,(R(y,x)\ \&\ \varphi(y)),$$

and hence

$$T \vdash \overline{\mathrm{CNF}_\varphi}(x) \to \mathrm{CNF}_\varphi(x),$$
$$T \vdash \mathrm{DNF}_\varphi(x) \to \overline{\mathrm{DNF}_\varphi}(x).$$

The rest of the proof follows by the mathematical induction. □

Next, we will take into account only extensional formulas and study their relationship with closing formulas. We will see that if a formula is extensional then it determine the relation to its closing formulas.

Lemma 4. Let T be a theory over $BL\forall$ containing a reflexive binary predicate R and $\varphi(x_1, \ldots, x_n)$ be an arbitrary formula extensional w.r.t. R. Then

$$T \vdash \varphi(x_1, \ldots, x_n) \to \overline{\mathrm{CNF}_\varphi}(x_1, \ldots, x_n), \tag{11}$$

$$T \vdash \overline{\mathrm{DNF}_\varphi}(x_1, \ldots, x_n) \to \varphi(x_1, \ldots, x_n). \tag{12}$$

Proof. Let $n = 1$. To prove the lemma we have to show that $\varphi \to \overline{\mathrm{CNF}_\varphi}$ and $\overline{\mathrm{DNF}_\varphi} \to \varphi$. We start with two variants of extensionality axiom:

$$T \vdash R(x,y) \to (\varphi(x) \to \varphi(y)),$$
$$T \vdash (R(y,x)\ \&\ \varphi(y)) \to \varphi(x).$$

From the first formula we obtain

$$\begin{aligned} &T \vdash \varphi(x) \rightarrow (R(x,y) \rightarrow \varphi(y)), \\ &T \vdash \varphi(x) \rightarrow (\forall y)\,(R(x,y) \rightarrow \varphi(y)), \\ &T \vdash \varphi(x) \rightarrow \overline{\mathrm{CNF}_\varphi}(x). \end{aligned}$$

Proof of the second implication is following

$$\begin{aligned} &T \vdash (\forall y)\,((R(y,x) \,\&\, \varphi(y)) \rightarrow \varphi(x)) \\ &T \vdash (\exists y)\,(R(y,x) \,\&\, \varphi(y)) \rightarrow \varphi(x) \\ &T \vdash \overline{\mathrm{DNF}_\varphi}(x) \rightarrow \varphi(x). \end{aligned}$$ □

Inverse problem of extensionality determining is solved in the following theorem. We start from the assumption knowing nothing about the considered formula and we want to appoint the extensionality property.

Theorem 1. Let T be a theory over $BL\forall$ containing a reflexive binary predicate R and let $\varphi(x_1, \dots, x_n)$ be an arbitrary formula. Then $\varphi(x_1, \dots, x_n)$ is extensional w.r.t. R if and only if

$$T \vdash \overline{\mathrm{DNF}_\varphi}(x_1, \dots, x_n) \rightarrow \varphi(x_1, \dots, x_n) \quad \text{or} \tag{13}$$

$$T \vdash \varphi(x_1, \dots, x_n) \rightarrow \overline{\mathrm{CNF}_\varphi}(x_1, \dots, x_n). \tag{14}$$

Proof. Let $n = 1$. We need to show that formula $\varphi(x) \rightarrow \overline{\mathrm{CNF}_\varphi}$ implies extensionality of φ w.r.t. R.

$$\begin{aligned} &T \vdash \varphi(x) \rightarrow \overline{\mathrm{CNF}_\varphi}(x), \\ &T \vdash \varphi(x) \rightarrow (\forall y)\,(R(x,y) \rightarrow \varphi(y)), \\ &T \vdash \varphi(x) \rightarrow (R(x,y) \rightarrow \varphi(y)), \end{aligned}$$

which is just the extensionality property. It is necessary to prove the analogous claim for the $\overline{\mathrm{DNF}_\varphi}$.

$$\begin{aligned} &T \vdash \overline{\mathrm{DNF}_\varphi}(x) \rightarrow \varphi(x), \\ &T \vdash (\exists y)\,(R(y,x) \,\&\, \varphi(y)) \rightarrow \varphi(x), \\ &T \vdash (R(y,x) \,\&\, \varphi(y)) \rightarrow \varphi(x), \end{aligned}$$

and the rest of the proof follows by the mathematical induction. □

Two facts should be noticed when we are observing previous theorem. First, from the formula (13) follows (14) and vice versa. Second, extensionality gives the confidence to sufficient representation by normal forms.

5. CONDITIONAL EQUIVALENCE OF NORMAL FORMS

In this section, we will search a sufficient condition for the conditional equivalence between normal forms. In the sequel, we will take into account only the special normal forms defined by (5) and (6). We shall prove various results on the relations of these two formulas.

Theorem 2. Let T be a theory over BL∀ containing a reflexive binary predicate R and the language $J(T)$ be extended by $\boldsymbol{c}_1, \dots, \boldsymbol{c}_k$ as object constants. Let $\varphi(x_1, \dots, x_n)$ be extensional formula(w.r.t. R^K, $K \geq 1$). Let $P_{i_j}(x)$ and $E_{i_1 \dots i_n}$ stand for $R^K(x, c_{i_j})$ in CNF, $R^K(c_{i_j}, x)$ in DNF and $\varphi(c_{i_1} \dots c_{i_n})$ respectively. Then

$$T \vdash \mathrm{DNF}_\varphi(x_1, \dots, x_n) \to \varphi(x_1, \dots, x_n), \tag{15}$$

$$T \vdash \varphi(x_1, \dots, x_n) \to \mathrm{CNF}_\varphi(x_1, \dots, x_n). \tag{16}$$

Proof. Consider the case of one free variable ($n = 1$) and $K = 1$. From the extensionality axiom for φ (3) it follows that

$$T \vdash R(x, c_i) \to (\varphi(x) \to \varphi(c_i)),$$
$$T \vdash \varphi(x) \to (R(x, c_i) \to \varphi(c_i))$$
$$T \vdash R(c_i, x) \to (\varphi(c_i) \to \varphi(x)),$$
$$T \vdash (R(c_i, x) \,\&\, \varphi(c_i)) \to \varphi(x).$$

Hence, using the properties of & and → (see 2.2.11 in [2]) we obtain

$$T \vdash \bigvee_{i=1}^{k} (R(c_i, x) \,\&\, \varphi(c_i)) \to \varphi(x),$$

$$T \vdash \varphi(x) \to \bigwedge_{i=1}^{k} (R(x, c_i) \to \varphi(c_i)),$$

which gives us

$$T \vdash \mathrm{DNF}_\varphi(x) \to \varphi(x), \tag{17}$$

$$T \vdash \varphi(x) \to \mathrm{CNF}_\varphi(x). \tag{18}$$

□

In particular, Theorem 2 states that (without any additional conditions) $\mathrm{DNF}_\varphi \to \varphi$ and $\varphi \to \mathrm{CNF}_\varphi$. Further, also the next result holds.

Corollary 1. Under the assumptions of Theorem 2 it can be proved that

$$T \vdash (\forall x_1, \dots, x_n)\, (\mathrm{DNF}_\varphi(x_1, \dots, x_n) \to \mathrm{CNF}_\varphi(x_1, \dots, x_n)). \tag{19}$$

In our case, it is impossible to prove extensionality of DNF_φ or CNF_φ w.r.t. R. The following corollary states that if two different variables are related then DNF_φ for the first variable implies CNF_φ for the other one.

Corollary 2. Let the theory T fulfill the assumptions of Theorem 2. Then

$$T \vdash (R^K(x_1, y_1) \,\&\, \dots \,\&\, R^K(x_n, y_n)) \to (\mathrm{DNF}_\varphi(x_1, \dots, x_n) \to \mathrm{CNF}_\varphi(y_1, \dots, y_n)).$$

Proof. For the simplicity, let us consider $n = 1$ and $K = 1$.

$T \vdash R(x, y) \to (\varphi(x) \to \varphi(y))$ (extensionality)
$T \vdash (R(x, y) \,\&\, \varphi(x)) \to \varphi(y)$ (using axioms of BL for $\to$)
$T \vdash (R(x, y) \,\&\, \varphi(x)) \to \mathrm{CNF}_\varphi(y)$ (by transitivity of $\to$ and (18))
$T \vdash R(x, y) \to (\varphi(x) \to \mathrm{CNF}_\varphi(y))$
$T \vdash \varphi(x) \to (R(x, y) \to \mathrm{CNF}_\varphi(y))$ (by changing of assumptions)
$T \vdash \mathrm{DNF}_\varphi(x) \to (R(x, y) \to \mathrm{CNF}_\varphi(y))$ (by transitivity of $\to$ and (17))

and finally by changing of the assumptions we obtain

$$T \vdash (R(x, y)) \to (\mathrm{DNF}_\varphi(x) \to \mathrm{CNF}_\varphi(y)).$$ □

However, the one-way implication between normal forms and the initial formula is not satisfactory. The conditional equivalence is proved in the following theorem.

Theorem 3. Let T be a theory over BL∀ containing a reflexive and symmetric binary predicate R and the language $J(T)$ be extended by $c_1, \dots, c_k$ as object constants. Let $\varphi(x_1, \dots, x_n)$ be extensional formula(w.r.t. R^K, $K \geq 1$). Let $P_{i_j}(x)$ be $R^K(x, c_{i_j})$ in CNF, $R^K(c_{i_j}, x)$ in DNF and $E_{i_1 \dots i_n}$ stands for $\varphi(c_{i_1}, \dots, c_{i_n})$. Then

$$T \cup \{(\forall x_1, \dots, x_n) (\bigvee_{i_1, \dots, i_n = 1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n))\} \vdash (\forall x_1, \dots, x_n)\, (\mathrm{DNF}_\varphi(x_1, \dots, x_n) \equiv \varphi(x_1, \dots, x_n)) \quad (20)$$

and

$$T \cup \{(\forall x_1, \dots, x_n) (\bigvee_{i_1, \dots, i_n = 1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n))\} \vdash (\forall x_1, \dots, x_n)\, (\mathrm{CNF}_\varphi(x_1, \dots, x_n) \equiv \varphi(x_1, \dots, x_n)). \quad (21)$$

Proof. For simplicity let us consider ($n = 1$) and $K = 1$. Using $(\varphi \to \varphi)$ and $(((\varphi_1 \to \psi_1) \,\&\, (\varphi_2 \to \psi_2)) \to ((\varphi_1 \,\&\, \varphi_2) \to (\psi_1 \,\&\, \psi_2)))$ we obtain

$$T \vdash (R(c_i, x) \to \varphi(c_i)) \to (R^2(c_i, x) \to (R(c_i, x) \,\&\, \varphi(c_i))) \quad (22)$$

$$T \vdash R^2(c_i, x) \to ((R(c_i, x) \to \varphi(c_i)) \to (R(c_i, x) \,\&\, \varphi(c_i))) \quad (23)$$

by $((\varphi\to(\psi\to\chi))\to(\psi\to(\varphi\to\chi)))$. The next formulas are the extensionality axiom and its modification.

$$T \vdash R(c_i, x)\to(\varphi(c_i) \equiv \varphi(x)),$$
$$T \vdash \varphi(x)\to(R(c_i, x)\to\varphi(c_i)).$$

From the last formula, (23) and using properties of $\to$ and $\vee$ we conclude

$$T \vdash R^2(c_i, x)\to(\varphi(x)\to \bigvee_{i=1}^{k} (R(c_i, x) \mathbin{\&} \varphi(c_i)).$$

From the modification of the extensionality axiom and $(((\varphi_1\to\psi)\wedge(\varphi_2\to\psi))\to((\varphi_1\vee\varphi_2)\to\psi))$ we get

$$T \vdash \bigvee_{i=1}^{k} (R(c_i, x) \mathbin{\&} \varphi(c_i))\to\varphi(x)).$$

From the previous formulas it follows that

$$T \vdash R^2(c_i, x)\to(\varphi(x) \equiv \bigvee_{i=1}^{k} (R(c_i, x) \mathbin{\&} \varphi(c_i))$$

and finally

$$T \vdash \bigvee_{i=1}^{k} R^2(c_i, x)\to(\varphi(x) \equiv \bigvee_{i=1}^{k} (R(c_i, x) \mathbin{\&} \varphi(c_i)),$$

which allows us to state that

$$T \vdash \bigvee_{i=1}^{k} R^2(c_i, x)\to(\varphi(x) \equiv \mathrm{DNF}_{\varphi}(x)),$$
$$T \vdash (\forall\, x) \bigvee_{i=1}^{k} R^2(c_i, x)\to(\forall\, x)\,(\varphi(x) \equiv \mathrm{DNF}_{\varphi}(x)),$$

or equivalently

$$T \cup \{(\forall\, x) \bigvee_{i=1}^{k} R(c_i, x)\} \vdash (\forall\, x)\,(\varphi(x) \equiv \mathrm{DNF}_{\varphi}(x)).$$

The proof of (21) is analogous to the previous one.

$$T \vdash (R(x, c_i)\to\varphi(c_i))\to(R^2(x, c_i)\to(R(x, c_i) \mathbin{\&} \varphi(c_i))),$$
$$T \vdash R^2(x, c_i)\to((R(x, c_i)\to\varphi(c_i))\to\varphi(x)).$$

By property of $\to$ and $\vee$ we obtain

$$T \vdash R^2(x, c_i) \to \left(\bigwedge_{i=1}^{k} (R(x, c_i) \to \varphi(c_i)) \to \varphi(x) \right)$$

$$T \vdash \varphi(x) \to (R(x, c_i) \to \varphi(c_i)) \quad \text{(extensionality axiom)}$$

$$T \vdash \varphi(x) \to \bigwedge_{i=1}^{k} (R(x, c_i) \to \varphi(c_i))$$

$$T \vdash R^2(x, c_i) \to \left(\bigwedge_{i=1}^{k} (R(x, c_i) \to \varphi(c_i)) \equiv \varphi(x) \right)$$

$$T \vdash \bigvee_{i=1}^{k} R^2(x, c_i) \to [\varphi(x) \equiv \mathrm{CNF}_\varphi(x)]$$

$$T \vdash (\forall x) \bigvee_{i=1}^{k} R^2(x, c_i) \to (\forall x)[\varphi(x) \equiv \mathrm{CNF}_\varphi(x)]$$

and hence

$$T \cup \{(\forall x) \bigvee_{i=1}^{k} R(x, c_i)\} \vdash (\forall x)[\varphi(x) \equiv \mathrm{CNF}_\varphi(x)]. \qquad \square$$

Corollary 3. Let T be a theory fulfilling the assumptions of Theorem 3 and let R be a reflexive binary predicate. Moreover, let us define two new n-ary predicates

$$C_1(x_1, \dots, x_n) := \bigvee_{i_1, \dots, i_n = 1}^{k} (R^{2K}(c_{i_1}, x_1) \,\&\, \dots \,\&\, R^{2K}(c_{i_n}, x_n)), \tag{24}$$

$$C_2(x_1, \dots, x_n) := \bigvee_{i_1, \dots, i_n = 1}^{k} (R^{2K}(x_1, c_{i_1}) \,\&\, \dots \,\&\, R^{2K}(x_n, c_{i_n})). \tag{25}$$

Then

$$T \vdash C_1(x_1, \dots, x_n) \to [\varphi(x_1, \dots, x_n) \equiv \mathrm{DNF}_\varphi(x_1, \dots, x_n)],$$
$$T \vdash C_2(x_1, \dots, x_n) \to [\varphi(x_1, \dots, x_n) \equiv \mathrm{CNF}_\varphi(x_1, \dots, x_n)].$$

Based on earlier described conditions (24) and (25), Theorem 3 states that the normal forms presented above become equivalent. But what does the conditions actually mean? If all x_i are equal to the corresponding c_i then both normal forms coincide with the original formula. Or, from a different point of view, conditions say that equivalence of the normal forms and initial formula depends foremost on the domain covering quality of the respective fuzzy relation. For example, we can regulate the truth degree with the number of used constants in the definition of normal forms.

Corollary 4. Let T be a theory fulfilling the assumptions of Theorem 3 and let R be a reflexive binary predicate. Then

$$T \cup \{(\forall x_1, \dots, x_n)\,(\bigvee_{i_1,\dots,i_n=1}^{k} (P_{i_1}(x_1)\,\&\,\cdots\,\&\,P_{i_n}(x_n))\} \vdash$$
$$(\forall x_1, \dots, x_n)\,(\mathrm{DNF}_\varphi(x_1, \dots, x_n) \equiv \mathrm{CNF}_\varphi(x_1, \dots, x_n)). \quad (26)$$

6. CONCLUSIONS

In this work, we have proposed two special formulas by means of which the extensionality of an arbitrary formula can be determined. Further, we have worked with a specific set of formulas only, namely extensional. Extensionality of a concrete formula is usually defined w.r.t. a similarity predicate, which gives us a continuity in a certain sense (see [4]). In our case, the extensionality is considered w.r.t. reflexive binary predicate.

Finally, we have found the special conditions (24) and (25) under which the normal forms (5) and (6) become equivalent. Our aim is to obtain the highest truth degree of formula postulating that DNF_φ or CNF_φ is equivalent to its former formula φ. All requirements are shown in section concerning the conditional equivalence of normal forms. On the semantical level, the conditional equivalence means the approximation. The quality of the approximation can be extracted from a truth value of the respective conditions.

It is evident that truth degree depends foremost on the choice of the predicate R in the structure of normal forms. The main contribution of this work is the generalization of the binary predicates R fitting the logical approximation, as given in [8], relaxing the transitivity requirement for R. The question of maximality of truth degree is left unsolved in this paper and it is appropriate topic for further study.

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