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COALITIONAL FUZZY PREFERENCES[1]

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The paper deals with the concept of coalitional preferences in the group decision-making situations in which the agents and coalitions have only vague idea about the comparative acceptability of particular outcomes. The coalitional games with vague utilities (see, e. g., [7]) can serve for a good example when some types of the game solutions (e. g., the von Neumann–Morgenstern one) are to be extended to the fuzzy game case. In this paper, we consider the fuzzy analogies of coalitional preferences and coalitional domination concepts known from the deterministic optimization models. These coalitional preferences are derived from the individual preferences of the coalition members. In the fuzzy extension of the model the input individual preferences are represented by fuzzy relations and, consequently, also the coalitional preferences have to be fuzzy. The general properties of these coalitional preferences are discussed in this contribution, and they are compared with the situation in the deterministic model. Finally, the case when the fuzziness of the individual preferences follows from fuzziness of the utility functions over the outcomes of the decision-making is mentioned and discussed.

1. HEURISTIC INTRODUCTION

In many of the decision-making situations the decisions are accepted not only individually but also as a result of some coalitional mechanism. Even in this case the decision accepted by a coalition has to respect the principle of individual rationality and the demands of all individual members of the coalition. The ability of a coalition to offer such uniformly acceptable policy of its decision-making is a necessary condition for its existence or, better, for its ability to result from the negotiation process. This fact has a significant representation, e. g., in the coalitional game theory and bargaining models.

In the classical theory of games and related models of optimization the individual preferences of players (or agents) are considered to be exactly determined. Consequently, also the coalitional preferences derived from them are deterministic. Seemingly, this approach appears to be rational. Individual agent usually knows what he wants. On the other hand, if the results of the decision-making are rep-

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resented by material objects (like goods, agricultural or industrial products, etc.), working activities or business advantages, then the assumption about the strictly determined preferences need not be so selfevident. In the contrary, real agent can prefer some of the outcomes against others only with some degree of uncertainty. In such situation, his preferences appear to be vague, and we are dealing with fuzzy preference relation, i. e., with fuzzy ordering relation over outcomes of the decisions.

In the following sections, we process such individual fuzzy preferences, namely, we derive the group (coalitional) fuzzy preference relations from them, and compare them with the properties of their deterministic counterparts. In the following, second, section we briefly recall the classical concept of deterministic coalitional preferences. In the third section we, also briefly, specify the individual fuzzy preferences which are considered here. Section 4 is devoted to the coalitional fuzzy preference relation defined as a superposition of the individual preferences of the coalition members. In the next, fifth, section we present and discuss the situation in which the individual fuzzy preferences are derived from fuzzy utility functions. The paper is concluded by several brief remarks.

In the paper we respect and preserve the usual terminology of the theory of coalitional games, and use the term "domination" for particular cases of the coalitional preference relation.

2. DETERMINISTIC COALITIONAL PREFERENCES

In the theory of coalitional games (see, e. g., [4, 8, 9]) the individual preferences of players are represented by ordering relations over the set of possible outcomes of the game.

Let us denote by I the (finite and non-empty) set of *players* and by X the (non-empty) set of possible *outcomes.*

In most of the following sections we suppose that the outcomes are real objects or services like goods, comparative business advantages, working support, moral support, social position which hardly can be evaluated by a deterministic values of some utility. Anyhow, they can be in some cases connected with vague utilities. To introduce necessary formal tools for the fuzzification of both approaches, we present concepts connected with each of them.

2.1. Preferences without utilities

Let us consider a player $i \in I$. By symbol $\succsim_i$ we denote the preference scale of the player i as a complete ordering relation over the set X. It means for any two outcomes $x, y \in X$

$$x \succsim_i y \tag{1}$$

means that player i (weakly) prefers x to y and the usual properties of such ordering, namely, for x, y, $z \in X$

$$\text{(completeness)} \quad x \succsim_i y \text{ or } y \succsim_i x, \tag{2}$$

$$\text{(reflexivity)} \quad x \succsim_i x, \tag{3}$$

$$\text{(semiantisymmetry)} \quad \text{if } x \succsim_i y \text{ and } y \succsim_i x \text{ then both outcomes are considered to be equivalent,} \tag{4}$$

$$\text{(transitivity)} \quad \text{if } x \succsim_i y \text{ and } y \succsim_i z \text{ then } x \succsim_i z \tag{5}$$

are fulfilled.

This preference relation can be extended to coalitions of players. In such case it is called *domination via coalition.* If $K \subset I$, $K \neq \emptyset$, is a coalition, i. e., a set of players, and x, $y \in X$ then we say that x dominates y via coalition K and write

$$x \operatorname{dom}_K y \tag{6}$$

iff

$$\begin{aligned} &\text{for all } i \in K,\ x \succsim_i y, \\ &\text{for some } j \in K, \text{ relation } y \succsim_i x \text{ does not hold.} \end{aligned} \tag{7}$$

It is evident and generally known that such relation of domination is not generally complete (property (2) need not be fulfilled), as well as the reflexivity does not hold (property (3)). On the other hand, the semiantisymmetry (5) turns into antisymmetry

$$\text{if } x \operatorname{dom}_K y \text{ then never } y \operatorname{dom}_K x. \tag{8}$$

The transitivity (5) is always fulfilled. If one-player coalition $\{i\}$, $i \in I$, is considered then if $x \operatorname{dom}_{\{i\}} y$ when $x \succsim_i y$ but not $y \succsim_i x$.

In the coalitional games theory, it is usual to consider also *general domination.* If x, $y \in X$ are outcomes then we say that x dominates y and write

$$x \operatorname{dom} y \tag{9}$$

iff there exists a (non-empty) coalition K such that $x \operatorname{dom}_K y$.

It is easy to see that the relation of general domination is neither complete nor reflexive. Moreover, it can be neither semiantisymmetric nor antisymmetric, and it need not be even transitive. In this sense, it is difficult to consider it for an ordering. Anyhow, in the game theory it is a significant relation, e. g., in the theory of von Neumann solution (see [4, 8, 9]).

2.2. Preferences with utilities

It is possible under some conditions which are exactly formulated in the utility theory, to substitute the qualitative orderings $\succsim_i$, $i \in I$, by quantitative numerical scale whose values characterize the acceptability of particular outcomes for a given player. In our paper, we consider this situation for vague knowledge of these scales in Section 5.

Formally, the quantitative scale representing the preferences $\succsim_i$, $i \in I$, is characterized by *utility functions* u_i, $i \in I$, over the set X. Their values $u_i(x)$ give the quantitative evaluation of the utility of the outcome x for player $i \in I$. It is defined in such way that for any x, $y \in X$ and $i \in I$

$$u_i(x) \geq u_i(y) \quad \text{iff } x \succsim_i y. \tag{10}$$

Due to this condition, it is easy to transfer the properties of $\succsim_i$ into the natural properties of numerical ordering $\geq$ where condition (5) turns into

$$\text{if } u_i(x) \geq u_i(y) \text{ and } u_i(y) \geq u_i(x) \text{ then } u_i(x) = u_i(y). \tag{11}$$

The definition of domination via a coalition K easily transforms into the following relation:

$$\begin{aligned} x \operatorname{dom}_K y \quad \text{iff} \quad & u_i(x) \geq u_i(y) \quad \text{for all } i \in K, \\ & u_j(x) > u_j(y) \quad \text{for some } j \in K. \end{aligned}$$

The properties of such domination via coalition, as well as of the general domination defined by means of it are completely analogous to the previous case.

3. INDIVIDUAL FUZZY PREFERENCE

In fact, there are not so many cooperative situations (e. g., coalitional games, group decision-making procedures, some market processes, etc.) in which the preferences of their participants are as deterministic as they were described in the previous section. Namely in the case when the outcomes for particular players are not represented by money but by some material or moral advantages, the preferences need not be strictly given. There exists natural vagueness of the preferences of players which can be represented by vague ordering relation over the set of outcomes.

Even in this (and all the following) chapter we preserve the denotation I for the set of players and X for the set of outcomes.

The vague preferences of each individual player $i \in I$ are here represented by *fuzzy ordering relation* $\succsim_i^F$ over the set X. It means that $\succsim_i^F$ can be identified with a fuzzy subset of the Cartesian product $X \times X$ with membership function

$$\nu^i(\cdot,\cdot) : X \times X \to [0,1]$$

where the value $\nu^i(x,y)$ for $i \in I$, x, $y \in X$, represents the possibility that player i prefers x to y. Fuzzy orderings may be relations with a rich structure (see, e. g., [2, 3]). In this case we consider the following properties for desirable: for x, y, $z \in X$,

$i \in I$

(completeness)	value $\nu^i(x,y)$ is defined for any pair $x,\, y \in X$,	(12)
(fuzzy reflexivity)	$\nu^i(x,x) = 1$,	(13)
(fuzzy semiantisymmetry)	$\nu^i(x,y) \geq 1 - \nu^i(y,x)$, and the value $\nu^i(x,y)+\nu^i(y,x)-1$ represents the possibility that outcomes x, y are considered to be equivalent for player i,	(14)
(fuzzy tranzitivity)	$\nu^i(x,z) \geq \min(\nu^i(x,y), \nu^i(y,z))$.	(15)

The fuzzy ordering relations $\succsim_i^F$ fulfilling the previous properties will be in this paper processed as *individual fuzzy preference relation.*

Some of the just presented desired properties of fuzzy individual preferences are rather discutable. The fuzzy completeness and fuzzy reflexivity are relatively clear and they obviously reflect analogous properties of the ordering $\geq$ of real numbers. Also the fuzzy transitivity appears acceptable – it expresses the intuitive idea that there may exist also other (may be stronger) reasons for the opinion that $x \succsim_i^F z$ than deriving the relation via third outcome $y \in X$. This property may be discussed but it belongs, at least, among the most acceptable ones. Rather different situation appear in the case of fuzzy semiantisymmetry. Its deterministic counterpart (5) is rather a construction than a definition of property – it simply stresses the fact that some pairs of outcomes can be acceptable on an equal level. The dogmatic transcription of (5) into the language of fuzzy sets could be, e. g.,

$$\min\left(\nu^i(x,y),\, \nu^i(y,x)\right)$$

is the possibility that x and y are equivalent (i. e. the possibility that parallelly $x \succsim_i^F$y and $y \succsim_i^F x$). Anyhow, the heuristical interpretation of such condition seems to be rather difficult, even hardly understandable. Condition of fuzzy semiantisymmetry, as formulated in (14), has an easy interpretation: The possibility that $y \succsim_i^F x$ cannot be smaller than the possibility that $x \succsim_i^F y$ is not valid; if it is greater then the difference characterizes the possibility that both outcomes are equivalent from the point of view of player i. This interpretationally reasonable formulation can cause some problems when it is transformed into the fuzzy coalitional preferences. Nevertheless, the alternative approach mentioned above causes some other problems, as well.

4. COALITIONAL FUZZY PREFERENCES

Analogously to the deterministic case, even the individual fuzzy preferences can be extended to some coalitional relation representing the collective coalitional acceptability of particular outcomes. The coalitional fuzzy preferences are, analogously to the individual ones, represented by a binary relation over the set of outcomes X. As this relation is derived from the fuzzy relations of individual preferences, it is evident that it is to be also fuzzy. If $K \subset I$, $K \neq \emptyset$, is a coalition then we call the relation as *fuzzy domination via coalition* and for any pair x, $y \in X$ we write

$$x \operatorname{dom}_K^F y$$

and read x fuzzy dominates y via K. As the fuzzy domination is a fuzzy relation, it is represented by a fuzzy subset of the Cartesian product $X \times X$. Its membership function will be denoted by $\nu^K(\cdot,\cdot)$, where for every pair $(x,y) \in X \times X$ the value $\nu^K(x,y) \in [0,1]$ denotes the possibility with which $x \operatorname{dom}_K^F y$. The membership function is defined as follows. For every coalition $K \subset I$, $K \neq \emptyset$, and every pair of outcomes x, $y \in X$

$$\nu^K(x,y) = \min\left[\min(\nu^i(x,y) : i \in K), \max((1-\nu^i(y,x)) : i \in K)\right]. \qquad (16)$$

If we compare (16) with (6) and (7), we can see that the coalitional fuzzy domination via a coalition copies the concept of the corresponding deterministic relation formulated in the language of fuzzy sets. Namely,

$$\min\left(\nu^i(x,y) : i \in K\right)$$

determines the possibility that

$$x \succsim_i^F y \quad \text{for all } i \in K,$$

and

$$\max\left((1-\nu^i(x,y)) : i \in K\right)$$

similarly determines the possibility that the relation

$$y \succsim_j^F x \ \text{ does not hold for some } j \in K.$$

Remark 1. An immediate consequence of (16) and (14) is that for one-player coalition $\{i\}$ the possibility that $x \operatorname{dom}_{\{i\}}^F y$ fulfills

$$1-\nu^i(y,x) = \nu^{\{i\}}(x,y) = \min\left(\nu^i(x,y),\ 1-\nu^i(y,x)\right) \leq \nu^i(x,y), \quad x,\ y \in X.$$

Lemma 1. For every coalition K and outcomes x, y,

$$\nu^K(x,y) = \min\left[\min(\nu^i(x,y) : i \in K),\ 1-\min(\nu^i(y,x) : i \in K)\right],$$

and, if the property (14) is fulfilled then also

$$\min(\nu^i(x,y) : i \in K) \geq \nu^K(x,y) \geq 1-\max(\nu^i(y,x) : i \in K).$$

Proof. The former equality immediately follows from (16). The latter one is also an immediate consequence of the definitoric formula (16) and of the assumption

(14). Namely

$$\begin{aligned}
\min(\nu^i(x,y) : i \in K) &= \min\left[\min(\nu^i(x,y) : i \in K),\ \max(\nu^i(x,y) : i \in K)\right] \\
&\geq \min\left[\min(\nu^i(x,y) : i \in K),\ \max(1-\nu^i(y,x) : i \in K)\right] \\
&= \nu^K(x,y) \\
&= \min\left[\min(\nu^i(x,y) : i \in K),\ \max(1-\nu^i(y,x) : i \in K)\right] \\
&\geq \min\left[\min(1-\nu^i(y,x) : i \in K),\ \max(1-\nu^i(y,x) : i \in K)\right] \\
&= \min(1-\nu^i(y,x) : i \in K) = 1-\max(\nu^i(y,x) : i \in K). \quad \square
\end{aligned}$$

Remark 2. For any one-player coalition $\{i\}$ evidently $\nu^i(x,y) \geq \nu^{\{i\}}(x,y) \geq 1-\nu^i(y,x)$ as follows from Lemma 1.

As we would like to know how much the coalitional fuzzy preferences, i.e., the domination via coalition, can be considered for fuzzy ordering, it is useful to check in what degree it fulfills the desired properties (12), (13), (14), (15).

First, as the membership function $\nu^i(x,y)$ is defined for any pair of outcomes (x,y), the fuzzy domination via coalition is complete and (12) is fulfilled. In this sense the fuzzy domination is more satisfactory than its deterministic counterpart (6).

Lemma 2. If K is a coalition and x is an outcome then $\nu^K(x,x) = 0$, if (13) is fulfilled.

Proof. The statement follows from (13) and (16) immediately. Due to (13) $\nu^i(x,x) = 1$ for all $i \in I$ and then $1-\nu^i(x,x) = 0$. It means that the minimum (16) vanishes. $\square$

The previous lemma means that the fuzzy domination relation via a coalition is certainly not reflexive. This result is comparable with analogous property of the deterministic relation (6).

The situation becomes less clear if we analyze the fuzzy semiantisymmetry (14) of the individual preferences and its reflection in the coalitional preferences. Considering the deterministic counterparts of these fuzzy phenomena, namely (5) and (8), as well as other related properties, we can see that the "weak" individual preference $\succsim_i$ (similar to the weak inequality $\geq$) was transformed by means of (6), (7) into "strict" coalitional preferences "dom_K" (similar to strong inequality $>$). It means that the semiantisymmetry of individual preferences turns into antisymmetry of coalitional preferences. If we transmit this view on the deterministic preferences to their fuzzy analogies, we could expect the validity of a relation like

$$\nu^K(x,y) \leq 1-\nu^K(y,x). \tag{17}$$

Even more, the difference between these two values would represent the possibility that both outcomes are equivalent. This our expectation is justified by the following statement.

Lemma 3. For every $x, y \in X$ and every coalition $K \subset I$, $K \neq \emptyset$, if (14) holds then

$$\nu^K(x,y) \leq 1 - \nu^K(y,x).$$

Proof. Due to the definitoric relation (16)

$$\begin{aligned}
\nu^K(x,y) &= \min\left[\min(\nu^i(x,y) : i \in K), \max(1-\nu^i(y,x) : i \in K)\right] \\
&= \min\left[\min(1-1+\nu^i(x,y) : i \in K), \max(1-\nu^i(y,x) : i \in K)\right] \\
&= \min\left[\min(1-(1-\nu^i(x,y)) : i \in K), \max(1-\nu^i(y,x) : i \in K)\right] \\
&= \min\left[1-\max(1-\nu^i(x,y)) : i \in K), 1-\min(\nu^i(y,x) : i \in K)\right] \\
&= 1-\max\left[\max(1-\nu^i(x,y) : i \in K), \min(\nu^i(y,x) : i \in K)\right] \\
&\leq 1-\min\left[\max(1-\nu^i(x,y) : i \in K), \min(\nu^i(y,x) : i \in K)\right] \\
&= 1-\nu^K(y,x).
\end{aligned}$$

□

Finally, let us consider the property of fuzzy transitivity formalized in (15). In the deterministic case the transitivity of the individual preferences implies the same property for the coalitional preferences. If the fuzzy preferences are considered then the validity of (15) even for the membership functions $\nu^K(\cdot,\cdot)$ is not guaranteed.

Fuzziness is not friendly to the transitivity. On a more general level, this phenomenon was discussed in [6], and it appears even here. However the individual preferences are supposed to be transitive in the sense of (15), this property is not transferred to the coalitional fuzzy domination via a coalition, as illustrated by the following examples.

Example 1. Let us consider a coalition $K \subset I$, $K \neq \emptyset$, such that for some triple of outcomes $x, y, z \in X$ and for all players $i \in K$

$$\begin{aligned}
\nu^i(x,y) &= 1, & \nu^i(y,x) &= 0, \\
\nu^i(y,z) &= 1, & \nu^i(z,y) &= 1/3, \\
\nu^i(x,z) &= 1, & \nu^i(z,x) &= 1/2.
\end{aligned}$$

These individual fuzzy preferences posses all demanded properties including the transitivity. The coalitional fuzzy preferences derived from them by (16) are

$$\begin{aligned}
\nu^K(x,y) &= 1, & \nu^K(y,x) &= 0, \\
\nu^K(y,z) &= 2/3, & \nu^K(z,y) &= 0, \\
\nu^K(x,z) &= 1/2, & \nu^K(z,x) &= 0.
\end{aligned}$$

(The previous data also illustrate the transformation of the semiantisymmetry of the individual preferences into the antisymmetry of the coalitional ones.) We also see, that

$$\nu^K(x,z) = 1/2 < \min\left(\nu^K(x,y),\, \nu^K(y,z)\right) = 2/3.$$

This example illustrates also the achievability of the equality, when

$$\nu^K(z,x) = 0 = \min\left(\nu^K(z,y),\, \nu^K(y,z)\right).$$

The opposite inequality can be also achieved, as shown in the next example.

Example 2. Let $K = \{1,2\}$, and let

$$\begin{array}{llll} \nu^1(x,y) = 1, & \nu^1(y,x) = 1/2, & \nu^2(x,y) = 0, & \nu^2(y,x) = 1, \\ \nu^1(y,z) = 1, & \nu^1(z,y) = 1/2, & \nu^2(y,z) = 0, & \nu^2(z,y) = 1, \\ \nu^1(x,z) = 1, & \nu^1(z,x) = 1/2, & \nu^2(x,z) = 1/2, & \nu^2(z,x) = 1/2. \end{array}$$

Then

$$\begin{array}{ll} \nu^K(x,y) = 0, & \nu^K(y,x) = 1/2, \\ \nu^K(y,z) = 0, & \nu^K(z,y) = 1/2, \\ \nu^K(x,z) = 1/2, & \nu^K(z,x) = 1/2, \end{array}$$

and we can see that

$$\nu^K(x,z) = 1/2 > \min\left(\nu^K(x,y),\, \nu^K(y,z)\right) = 0.$$

Also in this case the equality takes place, as

$$\nu^K(z,x) = 1/2 = \min\left(\nu^K(z,y),\, \nu^K(y,x)\right).$$

Note that there is a very significant difference between the elementary properties of the individual fuzzy preferences of a player $i \in I$, represented by $\nu^i(x,y)$, and those of the one-player coalition $\{i\}$ whose coalitional fuzzy preferences are represented by fuzzy domination via $\{i\}$, represented by $\nu^{\{i\}}(x,y)$. Not only $\nu^i(x,x) = 1$ meanwhile $\nu^{\{i\}}(x,x) = 0$, but also the fuzzy semiantisymmetry of i, $\nu^i(x,y) \geq 1 - \nu^i(y,x)$, turns into antisymmetry of $\{i\}$, $\nu^{\{i\}}(x,y) \leq 1 - \nu^{\{i\}}(y,x)$. Moreover, as the coalition K in Example 1 can be also of a single player, $K = \{i\}$, then also the fuzzy transitivity of ν^i can turn into its opposite for $\nu^{\{i\}}$. This seemingly contradictory phenomenon follows quite naturally from the essential difference between "weak" ordering $\succsim_i^F$ and "strict" domination via coalition dom_K^F. It exists even in the deterministic case (except the problem with transitivity), and belongs to basic conceptual properties of both approaches to the preferences. It can be changed if we use an alternative definition of fuzzy domination via coalition, as briefly mentioned in the Conclusive Remarks. This modified domination corresponds with a "weak" ordering relation for a given coalition, and it bears corresponding methodological discrepancies.

For the one-players coalitions $\{i\}$, $i \in I$, the following relation holds:

Lemma 4. If $i \in I$, $\{i\}$ is a one-player coalition and x, y, $z \in X$ then

$$\nu^{\{i\}}(x,z) \le \max\left(\nu^{\{i\}}(x,y),\, \nu^{\{i\}}(y,z)\right).$$

Proof. Due to Remark 1, $\nu^{\{i\}}(x,y) = 1-\nu^i(y,x)$ for any x, y. Then (15) implies

$$\begin{aligned}\nu^{\{i\}}(x,z) &= 1-\nu^i(z,x) \le 1-\min\left(\nu^i(z,y),\, \nu^i(y,x)\right)\\ &= \max\left(1-\nu^i(z,y),\, 1-\nu^i(y,z)\right) = \max\left(\nu^{\{i\}}(x,y),\, \nu^{\{i\}}(y,z)\right). \quad \square\end{aligned}$$

It remains to discuss the fuzzy version of the most extensive domination concept, namely the general domination. In the deterministic case it was, for any x, $y \in X$ denoted by $x \operatorname{dom} y$ and it was valid iff there existed a nonempty coalition $K \subset I$ such that $x \operatorname{dom}_K y$. In the case of fuzzy preferences, the *general fuzzy domination* is, necessarily, a fuzzy relation over the Cartesian product $X \times X$ with membership function which we denote

$$\nu(\cdot,\cdot).$$

Analogously to the deterministic case we say that x generally fuzzy dominates y and write

$$x \operatorname{dom}^F y, \quad x,\, y \in X \tag{18}$$

with the possibility $\nu(x,y)$ given by

$$\nu(x,y) = \max(\nu^K(x,y) : K \subset I,\, K \ne \emptyset). \tag{19}$$

There are not so many provable properties of the general fuzzy domination. It is evident that it is complete as the membership function is defined for any pair of outputs x, y. Hence, property (12) is fulfilled. Moreover, as

$$\nu^K(x,x) = 0 \quad \text{for any } K \subset I,\ K \ne \emptyset,\ x \in X,$$

then also

$$\nu(x,x) = 0 \quad \text{for each } x$$

and in this sense the reflexivity (13) turns into antireflexivity. It is impossible to derive any general results regarding the antisymmetry or semiantisymmetry (see (14), (17)), and the relation of general fuzzy domination cannot be transitive (see (15)) if its deterministic pattern is not generally transitive.

The previous results display a disappointing conclusion. The fuzzification of the model of preferences underlines the phenomenon observable in the deterministic model, already. Namely, certain disconsistency between the individual, coalitional and general preferences, however they are connected by means of a consistent and natural procedure of derivation. This discrepancy is inherent for the coalitional or group decision making and the fuzzification cannot avoid it.

5. FUZZINESS OF UTILITIES

As mentioned in Subsection 2.2, in some cases the deterministic individual preferences $\succsim_i$ can be represented by a numerical scale of utilities connected with particular outcomes and associated with individual players. Then the individual preference scale can be directly derived from the utilities (or vice-versa) due to (10). If it is so then the eventual vagueness being present in the modelled approach of players need not be represented by defining fuzzy preferences $\succsim_i^F$ but, rather indirectly, by fuzzification of the utility values $u_i(x)$. They can be considered to be fuzzy quantities with membership functions

$$\mu_i^x(\cdot) : R \to [0,1].$$

If we choose this way, we can use fuzzy analogies to the procedures described in Subsection 2.2. Anyhow, even in such case we finally usually return to fuzzy ordering relations by means of the fuzzification of the inequalities between fuzzy utility values.

The ordering of fuzzy quantities is a complex and widely investigated problem. There exist several very different approaches to it and, consequently, many different definitions of the ordering relation. The most significant of them are summarized in [2] and [3]. Here, we mention one of them which is based on the idea that ordering of fuzzy quantities is to be a fuzzy relation. It means that it is characterized by a fuzzy subset of the 2-dimensional space of fuzzy quantities (see also [5, 6]).

Preserving the notation of the previous sections, we want to compare for every player i the utilities connected (from his point of view) with the outcomes x, y. It means that we want to specify in what degree (with which possibility)

$$u_i(x) \geq u_i(y)$$

in the fuzzy set theoretical sense. Here, we define this possibility as a number ${}^*\nu^i(x,y)$ defined by

$${}^*\nu^i(x,y) = \sup_{r,\, s \in R,\, r \geq s} [\min(\mu_i^x(r),\ \mu_i^y(s))] \tag{20}$$

for each $i \in I$, x, $y \in X$.

The previous considerations can be illustrated by the following example.

Example 3. Let us consider the set $X = \{x,y,z\}$ and a player $i \in I$ with preferences $\succeq_i$ such that $z \succeq_i y$ and $y \succeq_i x$. Let us suppose, further, that these preferences can be quantitatively evaluated by a deterministic utility function u_i, where $u_i(x) = 0$, $u_i(y) = 1$, $u_i(z) = 2$. Finally, let us suppose that the utility function was fuzzified to fuzzy function u_i^F the values of which are fuzzy quantities $u_i^F(x)$, $u_i^F(y)$, $u_i^F(z)$ with membership functions μ_i^x, μ_i^y, μ_i^z, respectively, where

$$\begin{array}{lllllll}
\mu_i^x(\xi) & = & \xi + 1 & \text{for } \xi \in [-1,0], & \mu_i^x(\xi) & = & 1-\xi \quad \text{for } \xi \in [0,1], \\
\mu_i^y(\xi) & = & \xi & \text{for } \xi \in [0,1], & \mu_i^y(\xi) & = & 2-\xi \quad \text{for } \xi \in [1,2], \\
\mu_i^z(\xi) & = & \xi/4 + 1/2 & \text{for } \xi \in [-2,2], & \mu_i^z(\xi) & = & 3-\xi \quad \text{for } \xi \in [2,3],
\end{array}$$

$$\mu_i^x(\xi) = 0, \quad \mu_i^y(\xi) = 0, \quad \mu_i^z(\xi) = 0 \quad \text{else.}$$

Then it is possible to derive, by means of (20), the fuzzy preference relation $\succeq_i^F$ with membership function $\nu_i(\cdot,\cdot) : X \times X \to [0,1]$. It can be easily verified that

$$\nu_i(y,x) = \nu_i(z,y) = \nu_i(z,x) = 1, \quad \nu_i(x,x) = \nu_i(y,y) = \nu_i(z,z) = 1,$$
$$\nu_i(x,y) = 1/2, \quad \nu_i(y,z) = 2/3, \quad \nu_i(x,z) = 1/3.$$

Evidently, $\nu_i(\cdot,\cdot)$ fulfils properties (12), (13) and (14) but not (15).

The above definition is not the unique possible one but it can be easily and naturally heuristically interpreted. Unfortunately, its formal properties are not as pleasant as we could wish. Namely, it does not fulfil the fuzzy transitivity property (15) (cf. [6]). Consequently, also the coalitional and general fuzzy dominations derived from it by (16) and (19) cannot possess this property. Of course, it is also possible to construct another definition of individual fuzzy preferences based on the comparison of fuzzy utilities which fulfils (12) – (15). As there exist also crisp orderings of fuzzy relations, it is possible to turn the whole problem into its deterministic version with crisp individual (and, consequently, also coalitional) preferences, as well.

6. CONCLUSIVE REMARKS

The previous sections have presented an elementary view on the fuzzification of the individual and coalitional preference concept which is, e. g., widely used in the coalitional game theory. It is, namely, the basic concept for the definition and investigation of the von Neumann–Morgenstern solution of cooperative games. If it is to be extended to coalitional games with vague idea about the expected pay-offs (i. e., with fuzzy pay-offs or fuzzy preferences), the good management of individual and coalitional fuzzy preferences is necessary.

Those readers who are specialists in the theory of triangular norms have certainly recognized the close relation between formula (14) and the Lukasiewicz t-norm. It is possible to apply some other t-norms instead of it. Generally the theory of fuzzy preferences which would be based on the theory of t-norms remains an open field for further investigation.

In the conclusive paragraph of Section 4, we have already mentioned certain inconsistency between the individual and coalitional fuzzy preferences. As it can be observed in Section 2, this discrepancy comes from the deterministic preference relations, and the fuzzification only stresses its unpleasantness. It mainly follows from the fact that meanwhile the individual preferences are introduced as a "weak" ordering with reflexivity, the coalitional preferences are "strict" and they exclude the reflexivity. This disproportionality of the deterministic model is here transmitted into its fuzzified counterpart, and its consequences are even more painful.

There exists a possibility which was avoided by the deterministic theory but which can be quite rational in the fuzzy approach thanks to more soft relation between "weak" and "strict" inequality of fuzzy objects.

In the deterministic case this modification consists in the introduction of "weak" coalitional domination via K

$$x \,\mathrm{Dom}_K\, y, \quad x,\, y \in I,\ K \subset I,\ K \neq \emptyset \tag{21}$$

being valid iff

$$x \succsim_i y \quad \text{for all } i \in K. \tag{22}$$

In the deterministic case this domination is generally not complete (2), but it is reflexive (3), the semiantisymmetry (5) holds in the same form in which it holds for the individual preferences, and the transitivity (5) is also preserved.

More important consequence of this definition of "weak" domination via coalition is that its analogy for fuzzy preferences is much simpler than (16). The "weak" coalitional fuzzy preference is even in this case defined as a fuzzy relation over the set of outputs. We call it weak coalitional fuzzy domination, write $x \,\mathrm{Dom}_K^F\, y$, and its membership function will be denoted by

$$\overline{\nu}^K(x,y), \quad K \subset I,\ K \neq \emptyset,\ x,\, y \in X,$$

where

$$\overline{\nu}^K(x,y) = \min(\nu^i(x,y) : i \in K). \tag{23}$$

Lemma 1, as well as a simple comparison of (16) and (23) show that

$$\overline{\nu}^K(x,y) \geq \nu^K(x,y) \quad \text{for any } K \subset I,\ x,\, y \in X.$$

Moreover, this weak domination displays quite pleasant formal properties. It is evidently reflexive, and transitive as follows from (23) and (13), (14), (15), and, of course, it is also complete as the values of membership function $\overline{\nu}^K(x,y)$ are defined for any K, x, y.

The situation becomes less pleasant in the case of the semiantisymmetry which we could expect from the "weak" coalitional domination. It can be shown that for $\overline{\nu}^K(\cdot,\cdot)$ defined by (23) neither $\overline{\nu}^K(x,y) \geq 1 - \overline{\nu}^K(y,x)$ nor $\overline{\nu}^K(x,y) \leq 1 - \overline{\nu}^K(y,x)$ can be guaranteed.

Example 4. Let us consider two-players coalition $K = \{1,2\}$ such that for a pair x, $y \in X$

$$\nu^1(x,y) = 1/3, \qquad \nu^1(y,x) = 1,$$
$$\nu^2(x,y) = 1, \qquad \nu^2(y,x) = 1/3.$$

Then due to (23)

$$\overline{\nu}^K(x,y) = 1/3 < 1 - \overline{\nu}^K(y,x) = 2/3.$$

If we, on the other hand, put

$$\nu^1(x,y) = 2/3, \qquad \nu^1(y,x) = 1,$$
$$\nu^2(x,y) = 1, \qquad \nu^2(y,x) = 2/3,$$

then $\overline{\nu}^K(x,y) = 2/3 > 1 - \overline{\nu}^K(y,x) = 1/3$.

It can be easily seen that also the "weak" general fuzzy domination derived from this relation does not display analogously pleasant consistency with the original individual fuzzy preferences, as even its deterministic pattern does.

The previous sections of this paper, as well as these conclusive remarks, show that the concepts of coalitional fuzzy preferences and general fuzzy preferences offer an inspirative field for further research, development of the general theory and also for its application in the fuzzy coalitional games, fuzzy group decision making and other models of the group behaviour.

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