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SEPARATION PRINCIPLE FOR NONLINEAR SYSTEMS USING A BILINEAR APPROXIMATION

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In this paper, we study the local stabilization problem of a class of planar nonlinear systems by means of an estimated state feedback law. Our approach is to use a bilinear approximation to establish a separation principle.

1. INTRODUCTION

State observation of nonlinear dynamic systems is becoming a growing topic of investigation in the specialized literature. The reconstruction of the time behaviour of state variables remains a major problem in both control theory and process diagnosis. For linear systems and for the local case, the Luenberger-like observer solve this problem. Nevertheless, the design of asymptotically stable observers remains a hard task in the nonlinear case. For bilinear systems, one can design an observer provided that the inputs are small. It is well known that for nonlinear systems, there exists a local exponential observer if and only if the linear approximation of the system at the origin is detectable. If moreover it is stabilizable by a state feedback, the problem of feedback stabilization with state detection is solvable with a linear observer and a linear control law. In this paper we consider the planar nonlinear systems of the form

$$\begin{cases} \dot{x} = f(x) + ug(x) \\ y = h(x) \end{cases} \tag{1}$$

where $x \in U$ is a neighborhood of the origin in $\mathbb{R}^2$, u is a scalar input and f, g are smooth vector fields and h is a real analytic function on $\mathbb{R}^2$, such that $f(0) = g(0) = 0$ and $h(0) = 0$. Many authors [2, 7] investigate the stabilizability problem when $g(0) \neq 0$. However few results are known in the case where $g(0) = 0$, [1, 4, 5, 11]. The principal difficulties arise from the fact that the linearized system is independent of the control and the vector field g is not locally rectifiable. Our approach is to consider the bilinear approximation system of (1):

$$\begin{cases} \dot{x} = Ax + uBx \\ \tilde{y} = Cx \end{cases} \tag{2}$$

where

$$A = \frac{\partial f}{\partial x}(0), \quad B = \frac{\partial g}{\partial x}(0) \quad \text{and} \quad C = \frac{\partial h}{\partial x}(0)$$

to study the stabilizability of the system (1) by means of a state estimated feedback law. In [6], the author studied the problem of finding a global state space transformation to transform a given single-input homogeneous bilinear system to a controllable linear system. A local state space transformation and a complete analysis of globally state linearizable bilinear systems in the plane are given. The authors, in [3, 8, 10], solved the problem of stabilizing in observer design for some classes of nonlinear systems. Suppose that we have a stabilizable and observable bilinear system with states x. We use a state feedback law $u = u(x)$ to asymptotically stabilize the system (1). If the states are not available, we must construct a bilinear observer for (2) which is expected to produce the estimation $\hat{x}(t)$ of the state $x(t)$. It turns out that for planar systems, one can consider bilinear systems with bad inputs (inputs for which the system is not observable). There is at most only one input which is constant that makes the system unobservable. Then we apply the feedback $u = u(\hat{x})$ which not nearly to the bad one to show that the system is asymptotically stabilizable.

2. STABILIZATION USING STATE DETECTION

Consider the single-input single-output nonlinear systems of the form (1). Since f, g, h are of C^1, one can write

$$\begin{aligned} f(x) &= Ax + f_1(x) \\ g(x) &= Bx + g_1(x) \end{aligned}$$

and

$$h(x) = Cx + h_1(x)$$

where f_1, g_1 and h_1 satisfy

$$\|f_1(x)\| \leq M_1\|x\|, \ \|g_1(x)\| \leq M_2\|x\| \ \text{ and } \ \|h_1(x)\| \leq M_3\|x\|, \ \forall\, x \in U' \subset U \tag{3}$$

with M_1, M_2 and M_3 some positive constants. We shall call (2), the approximating system for the system (1).

If the states of the bilinear system are available, we can formulate the stabilization problem of the system (1) as follows: Consider the system (1) defined on a neighborhood of the origin of $\mathbb{R}^2$, where we suppose that $f(0) = g(0) = 0$.

A function φ is said to be positively homogeneous of degree $m \geq 0$, if for any vector x and any real positive λ, we have

$$\varphi(\lambda x) = \lambda^m \varphi(x).$$

If the bilinear approximation system (2) is stabilizable by means of a positively homogeneous feedback of degree zero and of class C^1 on $\mathbb{R}^2 \setminus \{0\}$, then the system (1) is locally stabilizable.

Indeed, let $u(x)$ be a positively homogeneous stabilizing feedback of degree zero for system (2). Set

$$F(x) = Ax + u(x)\,Bx$$

and

$$G(x) = f_1(x) + u(x)\,g_1(x).$$

Since u is of class C^1 on $U' \setminus \{0\}$ then F and G are locally Lipschitz. Moreover,

$$|u(x)| \leq M_0$$

for every x and using (3),

$$||G(x)|| \leq (M_1 + M_0 M_2)||x||$$

for all $x \in U'$. It follows from Massera's theorem [14] that the origin of the differential equation

$$\dot{x} = F(x) + G(x)$$

is asymptotically stable equilibrium point.

Notice that, in [4] the authors gave a complete classification of planar homogeneous bilinear systems, where for stabilizable bilinear systems, a smooth on $\mathbb{R}^2 \setminus \{0\}$ homogeneous of degree zero feedback u is given.

Stabilization of a class of planar bilinear system:

In a suitable basis of $\mathbb{R}^2$, the matrices A and B can be written as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}.$$

First of all, we wish to write the matrices A and B as simply as possible. Consider the linear change of coordinates whose transformation matrix is given by

$$T = \begin{pmatrix} e_1 & e_2 \\ -e_2 & e_1 \end{pmatrix} \quad \text{where } e_1 = (a-d) - \sqrt{(b+c)^2 + (a-d)^2} \text{ and } e_2 = b + c.$$

Under this transformation, matrix B remains unchanged whereas matrix A becomes

$$A = \begin{pmatrix} \tilde{a} & (b-c)/2 \\ (c-b)/2 & \tilde{d} \end{pmatrix} \quad \text{where} \quad \begin{array}{l} \tilde{a} = (ae_1^2 + (c+b)\,e_1 e_2 + de_2^2)/(e_1^2 + e_2^2) \\ \tilde{d} = (ae_2^2 - (c+b)\,e_1 e_2 + de_1^2)/(e_1^2 + e_2^2). \end{array}$$

Suppose that the following assumption holds:

(i) $\mathrm{Tr}(A) \geq 0 \quad \mathrm{Tr}(B) = 0$ and $-\tilde{a}\tilde{d} = (b+c)^2 - 4ad > 0$.

Proposition 1. Under the above assumption, the following feedback law:

$$u(x_1,x_2)=\frac{t_1x_1^2+(\tilde{d}-\tilde{a})x_1x_2+t_2x_2^2}{\mu(x_1^2+x_2^2)}+\frac{c-b}{2\mu}\quad \text{where } t_1>0 \quad t_2>0$$

and

$$\tilde{d}\sqrt{\frac{t_1}{t_2}}+\tilde{a}>0$$

stabilizes the system (2).

Proof. Consider the following system:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}Y_1(x_1,x_2)\\ Y_2(x_1,x_2)\end{pmatrix}=(x_1^2+x_2^2)\left[A\begin{pmatrix}x_1\\ x_2\end{pmatrix}+u(x_1,x_2)\,B\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right].$$

Y is a homogeneous vectors fields of degree three. Since $(x_1^2+x_2^2)$ is definite positive, then it is not hard to see that, there is equivalence between stability of (2) and the above system. Set

$$F(x)=x_1Y_2(x_1,x_2)-x_2Y_1(x_1,x_2)$$

a simple computation gives

$$F(x_1,x_2)=-(t_1x_1^2+t_2x_2^2)\,(x_1^2+x_2^2).$$

Furthermore, F has not a linear factor then the phase portrait of Y is determined by the flow near (0,0). It is a global centre if $I=0$, and it is a global stable (resp. unstable) focus if and only if $I<0$ (resp. $I>0$), (for the proof see [13], where I is given by

$$I=\int_{-\infty}^{+\infty}\frac{Y_1(1,x)}{F(1,x)}\mathrm{d}x=-\frac{\pi}{t_2}\frac{\tilde{\mathrm{d}}t+\tilde{a}}{t(t+1)}\quad \text{where} \quad t=\sqrt{\frac{t_1}{t_2}}.$$

Observer design:

Consider now the system (1). If the linear pair is observable, the bilinear system is also observable for small controls. In this case, one can design an observer for (2) of the form

$$\dot{\hat{x}}=A\hat{x}+uB\hat{x}-L(C\hat{x}-y) \tag{4}$$

where L is the gain matrix such that $\operatorname{Rel}\lambda(A-LC)<0$, then there exists $P=P^T>0$ such that

$$P(A-LC)+(A-LC)^TP=-I$$

which is possible by observability of the pair (A,C). Let $e=\hat{x}-x$, a Lyapunov function can be chosen as:

$$V(e)=e^TPe.$$

The derivative of V along the trajectories of the error equation

$$\dot{e}=Ae+uBe-LCe$$

is given by

$$\dot{V}(e) = -e^T e + 2ue^T PBe.$$

Thus,

$$\dot{V}(e) \leq -||e||^2 + 2|u|||PB||||e||^2$$
$$\dot{V}(e) \leq (-1 + 2|u|||PB||)||e||^2.$$

It follows that for u sufficiently small $u < u_0 = \frac{1}{2PB}$, when $e \notin \operatorname{Ker} PB$, (otherwise if $e \in \operatorname{Ker} PB$ one gets $\dot{V}(e) \leq -||e||^2$). It follows that, the origin of the error equation is globally exponentially stable. Hence the system (4) is an exponential observer for (2) with the following estimate:

$$||e(t)|| \leq \lambda_1 ||e(0)|| e^{-\lambda_2 t}, \quad \lambda_1, \lambda_2 > 0. \tag{5}$$

Stabilization in the presence of the bad input:

In the two dimensional case, there is at most only one input u_b which is constant that makes the bilinear system unobservable. It is given by the linear equation

$$\det \begin{pmatrix} C \\ C(A+uB) \end{pmatrix} = 0.$$

From [12], for bounded and analytic (on $\mathbb{R}^2 \setminus \{0\}$) stabilizing feedback law $u(x)$, there exists $\delta > 0$ such that

$$u(x) > u_b - \delta \text{ and } u(x) < u_b + \delta, \ \forall x \in \mathbb{R}^2,$$

where u_b is the bad input. Assume that there exists a bounded and analytic (on $\mathbb{R}^2 \setminus \{0\}$) stabilizing law $u(x)$ for (2) such that, for any bad input u_b, there exists $\epsilon > 0$ such that

$$u(x) \notin (u_b - \epsilon, u_b + \epsilon), \ \forall x \in \mathbb{R}^2$$

and $u(x)$ is homogeneous of degree zero. Then, the system (2) is globally asymptotically stabilizable. Indeed, since the feedback law $u(x_1, x_2)$ is bounded and analytic (on $\mathbb{R}^2 \setminus \{0\}$) then there exists $\delta > 0$, such that

$$-\delta < u(x_1, x_2) < u_b - \epsilon, \ \forall (x_1, x_2) \in \mathbb{R}^2$$

or

$$u_b + \epsilon < u(x_1, x_2) < \delta, \ \forall (x_1, x_2) \in \mathbb{R}^2$$

where $x = (x_1, x_2)$. Suppose that,

$$u_b + \epsilon < u(x_1, x_2) < \delta, \ \forall (x_1, x_2) \in \mathbb{R}^2,$$

(the same proof in the second case), then, under a change in the input space of the form

$$u \to u + \delta$$

the system (2) becomes

$$\begin{cases} \dot{x} = \tilde{A}x + uBx \\ \tilde{y} = Cx \end{cases} \qquad \text{where} \quad \tilde{A} = A + \delta B.$$

Denoting $\tilde{u}_b$ the only bad input of the above system, and $\tilde{u}(x_1, x_2)$ the stabilizing feedback law, then

$$\tilde{u}_b = u_b - \delta \quad \text{and} \quad \tilde{u}(x_1, x_2) = u(x_1, x_2) - \delta.$$

However, $\tilde{u}(x_1, x_2)$ is a homogeneous feedback of degree zero which satisfies

$$|\tilde{u}(x_1, x_2)| < |\tilde{u}_b| - \epsilon < |\tilde{u}_b|.$$

A separation principle:

Now, in order to investigate the stabilizability problem in observer design, one can consider the observer (4) for the bilinear system. For the stabilization purpose, we shall suppose that,

$(\mathcal{H})$: There exists a homogeneous feedback law of degree zero $u(x)$, $(u(\lambda x) = u(x)$ for $\lambda \neq 0)$, and of C^1 (on $U \setminus \{0\}$), stabilizing the bilinear system (2).

It can be remarked that, by the assumption $(\mathcal{H})$, the closed-loop system

$$\dot{x} = Ax + u(x)Bx$$

is a continuous homogeneous vector field of degree one. Therefore, according to [9], there exists a homogeneous Lyapunov function V for the above differential equation. Since, its partial derivatives are also homogeneous, it follows that, there exists a positive constant α such that

$$\|\nabla V(x)\| \leq \alpha(1 + V(x)).$$

Theorem 1. Suppose that the pair (A, C) is observable. Then, under the assumption $\mathcal{H}$, the following system:

$$\begin{cases} \dot{x} = Ax + u(x - e)Bx \\ \dot{e} = (A + u(x - e)\,B - LC)\,e \end{cases} \tag{6}$$

is globally asymptotically stable.

Proof. By (5), there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that, $\|e(t)\| \leq \lambda_1 \|e(0)\| e^{-\lambda_2 t}$. Taking into account, this estimation which implies the global exponential stability of the error equation and the fact that the system $\dot{x} = \tilde{F}(x) = Ax + u(x)Bx$ is globally asymptotically stable, then the system (6) is locally asymptotically stable [16]. In order to show the global asymptotic stability, by using the argument of Seibert–Suarez [15], it suffices to prove the boundedness of any trajectories $(e(t), \hat{x}(t))$, $t \geq 0$, of the system (6). Since $e(t)$ given in (5) is bounded, then it suffices to show the boundedness of the component $\hat{x}(t)$. From $\mathcal{H}$ and using the fact that $\tilde{F}$ is a homogeneous vector fields, there exists a C^1-homogeneous Lyapunov function V such that

$$V(x) > 0,\ \forall\, x \neq 0, \quad V(0) = 0 \quad \text{and} \quad \dot{V}(x) = \nabla V_x(\tilde{F}(x)) < 0, \quad \forall\, x \neq 0$$

and $\alpha > 0$ such that

$$\|\nabla V_x\| \leq \alpha(1 + V(x)), \quad \forall\, x \in I\!R^n.$$

These properties can be found in [9]. Therefore, the derivative of V along the trajectories of time varying differential equation

$$\dot{\hat{x}} = \tilde{F}(\hat{x}) - LCe(t)$$

satisfies

$$\dot{V}(\hat{x}) = \nabla V_{\hat{x}}(\tilde{F}(\hat{x}) - \nabla V_{\hat{x}}(LCe(t)).$$

Since $\nabla V_{\hat{x}}(\tilde{F}(\hat{x}) < 0$, it follows that

$$\dot{V}(\hat{x}) \leq \|\nabla V_{\hat{x}}\| \cdot \|LC\| \cdot \|e(t)\|.$$

Then, one obtains

$$\dot{V}(\hat{x}) \leq \mu e^{-\lambda_2 t}(1 + V(\hat{x})), \quad \mu > 0.$$

Therefore, $\mathrm{Log}(1 + V(\hat{x}))$ is bounded by a positive constant. Hence, $\hat{x}(t)$ is bounded. It follows that, (6) is globally asymptotically stable. □

Now, let us consider the equation

$$\dot{\hat{x}} = A\hat{x} + uB\hat{x} - L(C\hat{x} - y) \tag{7}$$

where we take $y = h(x)$ as the output of the original system (1). Letting $\varepsilon = x - \hat{x}$, where x is the state of (1) and $\hat{x}$ satisfies the above equation (6). The derivative of the error ε is given by

$$\begin{aligned}
\dot{\varepsilon} &= f(x) + ug(x) - A\hat{x} - uB\hat{x} + L(C\hat{x} - y) \\
&= Ax + f_1(x) + uBx + ug_1(x) - A\hat{x} - uB\hat{x} + L(C\hat{x} - Cx - h_1(x)) \\
&= (A + uB)\varepsilon + f_1(x) + ug_1(x) - LC\varepsilon - Lh_1(x) \\
&= (A + uB - S^{-1t}CC)\varepsilon + f_1(x) + ug_1(x) - Lh_1(x).
\end{aligned}$$

Then, the latter expression in conjunction with the system (1) in closed-loop with the estimated feedback law

$$u = u(x - \varepsilon) \tag{8}$$

yields

$$\begin{pmatrix} \dot{x} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} (A + u(x - \varepsilon)B)x \\ (A + u(x - \varepsilon)B - LC)\varepsilon \end{pmatrix} + \begin{pmatrix} f_1(x) + u(x - \varepsilon)\, g_1(x) \\ f_1(x) + u(x - \varepsilon)\, g_1(x) - Lh_1(x) \end{pmatrix}.$$

Set

$$\phi(x,\varepsilon) = \begin{pmatrix} (A+u(x-\varepsilon)B)x \\ (A+u(x-\varepsilon)B-LC)\varepsilon \end{pmatrix}$$

and

$$\psi(x,\varepsilon) = \begin{pmatrix} f_1(x)+u(x-\varepsilon)\,g_1(x) \\ f_1(x)+u(x-\varepsilon)\,g_1(x)-Lh_1(x) \end{pmatrix}.$$

Since u is C^1 on $U' \setminus \{0\}$, it can be seen that ϕ and ψ are locally Lipschitz.

Moreover, there exists M_0 such that

$$|u(z)| \le M_0 \quad \text{for every} \quad z.$$

Furthermore, it can be seen that ϕ is homogeneous of degree one, and by using (3), one can verify that ψ satisfies

$$\|\psi(x,\varepsilon)\| \le M\|(x,\varepsilon)\|, \ \ \forall\,(x,\varepsilon) \in U' \times U'$$

where M is a positive constant which depends on M_0, M_1, M_2, M_3, η and $\|C\|$. It follows from a theorem of Massera [14], that the solution $(x,\varepsilon) = (0,0)$ of the differential equation

$${}^t(\dot{x},\dot{\varepsilon}) = \phi(x,\varepsilon) + \psi(x,\varepsilon)$$

is asymptotically stable.

Hence, using this fact, one can state the following theorem.

Theorem 2. If the approximating system (2) is observable for any input and stabilizable by means of a homogeneous feedback $u(x)$ of degree zero and of a class C^1 on $U \setminus \{0\}$, then it is stabilizable by means of a state estimate feedback given by the bilinear observer (4), and that the feedback law $u = u(x-\varepsilon)$ given in (8), makes the origin of the original system (1) locally asymptotically stable.

Proposition 2. If the (i) condition is met then, the system (2) is stabilizable thanks to a homogeneous feedback of degree zero which is analytic on $\mathbb{R}^2 \setminus \{0\}$.

Proof. There is at most only one input u_b which is constant that makes the system unobservable. Furthermore for t_1 positive constant large enough and $\tilde{d}\sqrt{\frac{t_1}{t_2}} + \tilde{a} > 0$, then the following proposition is useful

$$|u(x_1,x_2)| > |u_b| + 1. \qquad \square$$

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