Kybernetika

Celia Villanueva-Novelo; Sergej Čelikovský; Bernardino Castillo-Toledo

Structurally stable design of output regulation for a class of nonlinear systems

Kybernetika, Vol. 37 (2001), No. 5, [547]--564

Persistent URL: http://dml.cz/dmlcz/135426

KYBERNETIKA — VOLUME 37 (2001), NUMBER 5, PAGES 547-564

STRUCTURALLY STABLE DESIGN OF OUTPUT REGULATION FOR A CLASS OF NONLINEAR SYSTEMS*

CELIA VILLANUEVA–NOVELO, SERGEJ ČELIKOVSKÝ AND BERNARDINO CASTILLO–TOLEDO

The problem of output regulation of the systems affected by unknown constant parameters is considered here. The main goal is to find a unique feedback compensator (independent on the actual values of unknown parameters) that drives a given error (control criterion) asymptotically to zero for all values of parameters from a certain neighbourhood of their nominal value. Such a task is usually referred to as the structurally stable output regulation problem. Under certain assumptions, such a problem is known to be solvable using dynamical error feedback. The corresponding necessary and sufficient conditions basically include the solvability of the so-called regulator equation and the existence of an immersion of a certain system with outputs into the one having favourable observability and controllability properties. Its model is then directly used for dynamic compensator construction. Usually, such an immersion may be selected as the one to an observable linear system with outputs. In a general case, the above mentioned conditions are highly nonconstructive and difficult to check. This paper studies a certain particular class of systems, the so-called strictly triangular polynomial systems, where that immersion to a linear system can be obtained in a constructive way. Moreover, it provides computer algorithm (based on MAPLE symbolic package) to design the corresponding solution to the structurally stable output regulation problem. Examples together with computer simulations are included to clarify the suggested approach.

1. INTRODUCTION

A central problem in control theory and its applications is to design a control law to achieve asymptotic tracking with disturbance rejection in nonlinear systems. When the reference inputs and disturbances are generated by an autonomous differential equations, this problem is called nonlinear output regulation problem, or, alternatively, nonlinear servomechanism problem, see e.g. [1, 9, 10, 11]. The problem can

*Supported by the Grant Agency of the Czech Republic through the research grant No. 102/99/1368 and Proyecto CONACyT No. 31844-A and 26358-A.

be precisely formulated as follows. Consider a nonlinear plant described by

$$\begin{aligned} \dot{x} &= f(x, w, u, \mu) \\ e &= h(x, w, \mu) \end{aligned} \tag{1}$$

where $f(x, w, u, \mu)$, $h(x, w, \mu)$ are sufficiently smooth. The first equation of (1) describes the dynamics of a *plant*, whose *state* x is defined in a neighborhood U of the origin in $\mathbb{R}^n$, with *control input* $u \in \mathbb{R}^m$ and subject to a set of *exogenous* input variables $w \in \mathbb{R}^r$, which includes *disturbances* (to be rejected) and/or *references* (to be tracked) and $\mu \in \mathbb{R}^q$ is the vector of unknown parameters. The second equation defines an *error* variable $e \in \mathbb{R}^p$, which is expressed as a function of the state x, the exogenous input w and the vector of unknown parameters μ. Suppose $\mu = 0$ to be a nominal value of the parameter μ and assume $f(0, 0, 0, \mu) = 0$, $h(0, 0, \mu) = 0$, for all μ in a neighbourhood of its nominal value.

The family of the exogenous inputs $w(\cdot)$ affecting the plant (1), is the one of all functions of time which are solution of the autonomous differential equation

$$\dot{w} = s(w) \tag{2}$$

with initial condition $w(0)$ ranging in some neighborhood W of the origin of $\mathbb{R}^r$. This system is a mathematical model of the generator of all possible exogenous input functions and is called as the exogeneous one, or simply as the *exosystem*. Throughout the paper, (2) is assumed to be neutrally stable, which is a standard assumption for exogenous systems. Without going into a detailed definition (see [1, 9, 10, 11]), the local neutral stability of (2) means its local Lyapunov stability together with a property of its every point in a neighbourhood of the origin being the *nonwandering* one, [6]. In particular, every trajectory starting near the origin neither leaves it nor tends to the origin and as $t \to \infty$. Such a requirement seems to be reasonable, there is no practical interest to track unstable reference while tracking asymptotically decaying signal makes no difference to usual stabilization problem.

Beginning with the pioneering works [7, 10], the nonlinear output regulation problem has been studied intensively during the last decade. Its basic purpose is to construct a feedback compensator to drive error e in (1) asymptotically to zero while preserving acceptable internal behaviour of the overall closed-loop system. The output regulation problem has many variations, dependently on whether or not unknown parameters $\mu \in \mathbb{R}^q$ affect the systems and which kind of feedback is used to solve it. Basic results on full information feedback case, error feedback case and the so-called structurally stable regulation are collected in [9, 11], some results on full information nonsmooth feedback were obtained in [3, 4]. For the further robust aspects of the output regulation see [1, 8] and references within there.

The present paper deals with the case crucially affected by the unknown parameters influence, i. e. with the robust aspects of the output regulation. The detailed definitions are to be given in the next section. Basically, a single feedback compensator to achieve output regulation for all values of unknown parameters from a certain range is to be designed. The full solution to such a problem, called *Structurally Stable Output Regulation Problem (SSORP)*, has been provided in [9]. Nevertheless, necessary and sufficient conditions given there are quite abstract and nonconstructive. Some of these results are recalled in the next section.

Here, a more narrow class of nonlinear systems is studied. This class is the class of the so-called strictly triangular systems, where certain results concerning robust regulations have been obtained during last decade, [1, 12]. Using the ideas of [1], the constructive tests of necessary and sufficient conditions for robust output regulations are studied and the algorithm to design solutions of the SSORP is provided in this paper. The computer implementation of that algorithm constitutes the main contribution of the paper and is also tested via simulations on test examples and compared with other existing algorithms.

The paper is organized as follows. The following section provides a brief survey of output regulation problem, recalls some known facts on strictly triangular systems, polynomial systems and equivalence of more general classes of systems to them. Sections 3 and 4 describe the main contribution of the paper, including illustrative example and comparison with other existing algorithms. Final section draws conclusions and gives some outlooks for further research.

Notations. Throughout the paper we use standard notations of differential-geometric approach to nonlinear systems, see e.g. [9]. In particular, consider a real-valued function h and a vector field f, both defined on a subset U of $\mathbb{R}^n$. The Lie derivative of h along f is a new function, denoted $L_f h$, and defined as

$$L_f h = dh(x)f(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i(x)$$

at each x of U. In the case that $h = (h_1, h_2, \ldots, h_m)'$ we put

$$L_f h := (L_f h_1, L_f h_2, \ldots, L_f h_m)'.$$

If h is being differentiated k times along f, the notation $L_f^k h$ is used

$$L_f^k h := \frac{\partial \left(L_f^{k-1} h \right)}{\partial x} f(x), \ k > 1,$$

with $L_f^0 h = h(x)$. Further, the notation

$$[f(x), g(x)] := \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

stands for the so-called Lie bracket of the vector fields $f(x)$, $g(x)$, itself being a new vector field. For a multi-ply Lie bracket expressions, a compact notation

$$\mathrm{ad}_f^0 g := g, \quad \mathrm{ad}_f^i g := [f, \mathrm{ad}_f^{i-1} g], \ i \geq 1,$$

is often used. A constant-dimensional (or regular) smooth distribution is a subspace $\Delta(x)$ of tangent space T_x, smoothly depending on a point $x \in \mathbb{R}$. As a matter of fact, d-dimensional regular distribution Δ on $U \subset \mathbb{R}$ may be identified with d-tuple of smooth vector fields $f^1, f^2, \dots, f^d$, mutually linearly independent at any point of $U \subset \mathbb{R}$. In this case, notation $\Delta = \text{span}\{f^1, f^2, \dots, f^d\}$ is used. We say that the vector field f belongs to the distribution Δ, denoted $f \in \Delta$, if $f(x) \in \Delta(x)$ for all $x \in U$. The distribution Δ is said to be involutive, if for any two of vector fields $f^1, f^2 \in \Delta$ it holds that $[f^1, f^2] \in \Delta$.

2. DEFINITIONS AND PRELIMINARY RESULTS

2.1. Output regulation of nonlinear systems

Let us repeat in some detail the classical statement of the output regulation problem. Such a problem serves as a formalization for the well-known practical task of the tracking the prescribed reference signal and/or rejecting undesired disturbances. More precisely, consider nonlinear control system (1) together with the exogenous system (2). The system (1) describes the system to be controlled and the control goal is to drive its output e (usually referred to as the so-called *error*) asymptotically to zero. This system is affected also by the variable w generated by the exogeneous system (2) and may represent both desired reference to follow, the disturbance to be rejected or combination of both. Let us underline that the above setting deals basically only with references or disturbances generated by the *known* dynamical system, what restricts class of applicable exogeneous signals.

Such a setting has been studied intensively throughout the last three decades, first for linear and since the nineties also for nonlinear systems, [1, 7, 8, 9, 10, 11, 12]. There is a large variety of formulations, dependently on information available for feedback compensator that are to solve the problem. The most simple is the so-called *full information* output regulation problem where both the controlled system state x and the exosystem state w are available for the direct measurement. As the more realistic problem, the so-called *error feedback* may be considered, i. e. only the error e is available for measurement. The slight modification of the last problem enables to study also the so-called *structurally stable* output regulation problem, where unknown parameters are present in the model, namely vector μ in (1). Let us give its detailed and precise formulation.

Definition 1. Structurally Stable Output Regulation Problem (SSORP). Given a nonlinear system of the form (1) and a neutrally stable exosystem (2), find, if possible, an integer ν, two mappings $\theta(\xi)$ and $\eta(\xi, e)$ (with $\xi \in \Xi \subset \mathbb{R}^\nu, \theta : \mathbb{R}^\nu \to \mathbb{R}^m, \eta : \mathbb{R}^\nu \times \mathbb{R}^p \to \mathbb{R}^\nu$) and a neighbourhood $\mathcal{P}$ of $\mu = 0$ in $\mathbb{R}^q$ such that, for each $\mu \in \mathcal{P}$:

(S) the equilibrium $(x, \xi) = (0, 0)$ of

$$\dot{x} = f(x, 0, \theta(\xi), \mu)$$
$$\dot{\xi} = \eta(\xi, h(x, 0, \mu))$$

is asymptotically stable in the first approximation,

(R) there exists a neighborhood $V \subset U \times \Xi \times W$ of $(0,0,0)$ such that, for each initial condition $(x(0), \xi(0), w(0)) \in V$, the solution of

$$\dot{x} = f(x, w, \theta(\xi), \mu)$$
$$\dot{\xi} = \eta(\xi, h(x, w, \mu))$$
$$\dot{w} = s(w)$$

is such that

$$\lim_{t \to \infty} e(t) = 0.$$

Notice, that condition (S) of the above definition assures that a small exogeneous signal $w(t)$ generates a small steady state response of the overall closed loop system given in (R). Summarizing, "output regulation" means that the error (viewed as a certain output of the overall closed loop system) is driven asymptotically to zero while internal dynamics remains possibly nonzero, but bounded and its bound is smooth function of bound on exosignals $w(t)$. Term "structurally stable" then means that all these achievements persist small changes of unknown parameters μ, without any need of changing the feedback compensator.

To repeat the necessary and sufficient conditions for the SSORP, the concept of systems immersion, [9], is needed. Consider a pair of smooth autonomous systems with outputs

$$\dot{x} = f(x), \quad y = h(x) \in \mathbb{R}^m \tag{3}$$

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}), \quad y = \tilde{h}(\tilde{x}) \in \mathbb{R}^m \tag{4}$$

defined on two different state spaces, X and $\tilde{X}$, but having the same output space $Y = \mathbb{R}^m$. Assume $f(0) = 0$, $h(0) = 0$ and $\tilde{f}(0) = 0$, $\tilde{h}(0) = 0$ and denote those two systems as $\{X, f, h\}$ and $\{\tilde{X}, \tilde{f}, \tilde{h}\}$, respectively.

Definition 2. (Immersion). [9] System $\{X, f, h\}$ is said to be smoothly immersed into system $\{\tilde{X}, \tilde{f}, \tilde{h}\}$ if there exist a C^k mapping $\tau : X \to \tilde{X}$, with $k \geq 1$, satisfying $\tau(0) = 0$ and

$$h(x) \neq h(z) \Rightarrow \tilde{h}(\tau(x)) \neq \tilde{h}(\tau(z)), \tag{5}$$

for all $x, z \in X$, and it is such that

$$\begin{aligned} \frac{\partial \tau}{\partial x} f(x) &= \tilde{f}(\tau(x)) \\ h(x) &= \tilde{h}(\tau(x)) \end{aligned} \tag{6}$$

for all $x \in X$.

In another words, image under τ of any state trajectory of the system $\{X, f, h\}$ is the state trajectory of the system $\{\tilde{X}, \tilde{f}, \tilde{h}\}$, both these trajectories producing the same output trajectory, and, moreover, every pair of distinguishable states is mapped by τ into a pair of distinguishable states. Later on, it will be seen that immersion can replace completely the original exogeneous signal generator and at the same time to have more favourable structural properties. Of course, the best situation is when immersion into a linear system is available. The following proposition repeats the necessary and sufficient conditions for it.

Proposition 3. (Immersion into a linear system). [9] The following conditions are equivalent:

(1) $\{X, f, h\}$ is immersed into a finite dimensional and observable linear system.

(2) There exist an integer q and a set of real numbers $a_0, a_1, \ldots, a_{q-1}$ such that

$$L_f^q h(w) = a_0 h(w) + a_1 L_f h(w) + \ldots + a_{q-1} L_f^{q-1} h(w).$$

Now, we are ready to give the basic result on the structurally stable output regulation.

Theorem 4. (Necessary and sufficient condition for SSORP). [9] The Structurally Stable Output Regulation Problem is solvable if and only if there exist mappings $x = \pi(w,\mu)$ and $u = c(w,\mu)$, with $\pi(0,\mu) = 0$ and $c(0,\mu) = 0$, both defined in a neighbourhood $W^o \times \mathcal{P} \subset W \times I\!R^p$ of the origin, satisfying the conditions

$$\begin{aligned} \frac{\partial \pi(w,\mu)}{\partial w} s(w) &= f(\pi(w,\mu), w, c(w,\mu), \mu) \\ 0 &= h(\pi(w,\mu), w, \mu) \end{aligned} \tag{7}$$

for all $(w,\mu) \in W^o \times \mathcal{P}$, and such that the autonomous system with output $\{W^o \times \mathcal{P}, s^a, c\}$ i.e.

$$\begin{aligned} \begin{bmatrix} \dot{w} \\ \dot{\mu} \end{bmatrix} &= \begin{bmatrix} s(w) \\ 0 \end{bmatrix} = s^a(w) \\ u &= c(w,\mu) \end{aligned}$$

is immersed into a system

$$\begin{aligned} \dot{\xi} &= \varphi(\xi) \\ u &= \gamma(\xi) \end{aligned}$$

defined on a neighborhood Ξ^o of the origin in $I\!R^\nu$, in which $\varphi(0) = 0$ and $\gamma(0) = 0$, and the matrices

$$\Phi = \left[\frac{\partial \varphi}{\partial \xi}\right]_{\xi=0}, \quad \Gamma = \left[\frac{\partial \gamma}{\partial \xi}\right]_{\xi=0}$$

are such that the pair

$$\begin{pmatrix} A(0) & 0 \\ NC(0) & \Phi \end{pmatrix}, \begin{pmatrix} B(0) \\ 0 \end{pmatrix}$$

is stabilizable for some choice of the matrix N, and the pair

$$\begin{pmatrix} C(0) & 0 \end{pmatrix}, \quad \begin{pmatrix} A(0) & B(0)\Gamma \\ 0 & \Phi \end{pmatrix}$$

is detectable, where:

$$A(\mu) = \left[\frac{\partial f}{\partial x}\right]_{(0,0,0,\mu)} \quad B(\mu) = \left[\frac{\partial f}{\partial u}\right]_{(0,0,0,\mu)} \quad C(\mu) = \left[\frac{\partial h}{\partial x}\right]_{(0,0,\mu)}.$$

As a consequence of Proposition 3, we have the following result:

Corollary 5. [9] The SSORP is solvable by means of a linear controller if the pair $(A(0), B(0))$ is stabilizable, the pair $(C(0), A(0))$ is detectable, there exist mappings $x = \pi(w, \mu)$ and $u = c(w, \mu)$, with $\pi(0, \mu) = 0$ and $c(0, \mu) = 0$, both defined in a neighborhood $W^o \times \mathcal{P} \subset W \times I\!R^p$ of the origin, satisfying the conditions (7) and such that, for some set of q real numbers $a_0, a_1, \ldots, a_{q-1}$,

$$L_s^q c(w, \mu) = a_0 c(w, \mu) + a_1 L_s c(w, \mu) + \ldots + a_{q-1} L_s^{q-1} c(w, \mu) \tag{8}$$

for all $(w, \mu) \in W^o \times \mathcal{P}$, and, moreover, the matrix

$$\begin{pmatrix} A(0) - \lambda I & B(0) \\ C(0) & 0 \end{pmatrix}$$

is nonsingular for every complex λ which is a root of the polynomial

$$p(\lambda) = a_0 + a_1\lambda + \ldots + a_{q-1}\lambda^{q-1} - \lambda^q$$

having non-negative real part.

2.2. Strictly triangular systems and polynomial systems

Throughout this paper, we are going to study the so-called *strictly triangular form systems*:

$$\begin{aligned} \dot{x_j} &= \alpha_{j+1}(\mu)x_{j+1} + \vartheta_j(x_1, x_2, \ldots, x_j, w, \mu), \quad j = 1, \ldots, n-1, \\ \dot{x_n} &= \alpha_{n+1}(\mu)u + \vartheta_n(x_1, x_2, \ldots, x_n, w, \mu) \\ e &= g(\mu)x_1 - d(w, \mu) \end{aligned} \tag{9}$$

where and x, u, μ are defined in the same way as for the system (1) with $m = 1$, i.e. $u \in I\!R$, $\alpha_i(\mu) \neq 0$, $i = 2, 3, \ldots, n+1$ and $g(\mu) \neq 0$ for all μ in some neighbourhood of its nominal value $\mu = 0$.

The class of strictly triangular systems is frequently studied class in problems of controlling the uncertain systems, cf. [1, 12] and references within there. The following differential geometric conditions show when the internal dynamics of a more general system may be transformed into the strictly triangular one via coordinate change and feedback, cf. e.g. [12].

Theorem 6. A general affine single-input nonlinear system depending on the unknown parameter

$$\dot{x} = f(z) + g(z)u + q(z, \mu), \quad z \in I\!R^n,\ u \in I\!R,\ \mu \in I\!R^q,$$

is locally equivalent to (9) via smooth coordinate change and static state feedback if and only if for all $j = 0, \ldots, n-1$ it holds:

1. $\Delta_j(x)$ is involutive for all $x \in U$, U being a neighbourhood of the origin in $I\!R^n$;
2. $\dim \Delta_j(0) = j + 1$;
3. $\mathrm{ad}_q \Delta_j \subset \Delta_j$.

Here

$$\Delta_j(x) := \mathrm{span}\{g, \mathrm{ad}_f g, \ldots, \mathrm{ad}_f^j g\}.$$

More results and motivation on strictly triangular systems can be found in [12]. Adaptation of Theorem 6 to the case of systems with outputs is analogous as in the case of feedback linearization of certain systems with outputs, cf. [9].

The systems studied in this paper will be also assumed to have polynomial right hand side. For the problem of state equivalence to a polynomial systems, see [2] and references within there.

3. MAIN RESULTS

Throughout the rest of the paper we consider the strictly triangular systems with polynomial right hand side and linear exogeneous system. The purpose of such a significant simplification is the developing of computer-based automatic design of the regulators solving the SSORP. More precisely, consider the system (9) where ϑ_j are supposed to be polynomials in (x, w) of the degree k_j

$$\vartheta_j(x_1, x_2, \ldots, x_j, w, \mu) = \sum_{0 < i_1 + i_2 + \ldots + i_{j+r} \le k_j} a_{i_1, i_2, \ldots, i_{j+r}}(\mu) x_1^{i_1} x_2^{i_2} \ldots x_j^{i_j} w_1^{i_{j+1}} \ldots w^{i_{j+r}}$$

for all $j = 1, 2, \ldots, n$ with $a_{i_1, i_2, \ldots, i_{j+r}} : I\!R^q \to I\!R$. Analogously, we suppose that $d(w, \mu)$ is a polynomial in w with k_d degree

$$d(w, \mu) = \sum_{0 < i_1 + i_2 + \ldots + i_r \le k_d} d_{i_1, i_2, \ldots, i_r}(\mu) w_1^{i_1} w_2^{i_2} \ldots w_r^{i_r},$$
$$d_{i_1, i_2, \ldots, i_r} : I\!R^q \to I\!R.$$

Moreover, the exogenous system, which we are going to work with, is supposed to have a linear form

$$\dot{w} = Sw \tag{10}$$

where S is an $(r \times r)$ real matrix. In order to guarantee the neutral stability of (10) suppose that S is complex diagonalizable and has eigenvalues with zero real parts only. Such a system is still a possible generator of a rich variety of references and disturbances usually studied in regulation. Obviously, each row of Sw is a linear polynomial, that is, we have

$$\dot{w}_j = \sum_{j=1}^{r} s_{ij} w_j \tag{11}$$

for all $i = 1, \ldots, r$.

The solution to SSORP problem may be described for the above class in an explicit and constructive way.

Theorem 7. Consider a nonlinear system of the form (9), with $\vartheta_j(x, \mu)$ being polynomial and a neutrally stable exosystem of the form (10). Then the (SSORP) is solvable.

To prove Theorem 7 the following lemmas will be used.

Lemma 8. Consider a nonlinear system of the form (9) where $\vartheta_j(x, w, \mu)$ is polynomial in (x, w) and the exosystem (10) then $u = c(w, \mu)$ solving (7) is unique and is polynomial in w.

Proof. By assumption $\vartheta_j(x, w, \mu)$ is a polynomial for all $j = 1, \ldots, n$, and let $k_j \geq 1$ be its degree. According to (7)

$$\begin{aligned} \frac{\partial \pi(w, \mu)}{\partial w} s(w) &= f(\pi(w, \mu), w, c(w, \mu), \mu) \\ 0 &= h(\pi(w, \mu), w, \mu) \end{aligned}$$

where $x = \pi(w, \mu)$, i. e.

$$\begin{aligned} x_1 &= \pi_1(w, \mu) \\ x_2 &= \pi_2(w, \mu) \\ &\vdots \\ x_n &= \pi_n(w, \mu). \end{aligned} \tag{12}$$

Since

$$e = g(\mu)x_1 - d(w, \mu) = g(\mu)x_1 - \sum_{0 < i_1 + i_2 + \ldots + i_r \leq k_d} d_{i_1, i_2, \ldots, i_r}(\mu) w_1^{i_1} w_2^{i_2} \ldots w_r^{i_r}$$

and $h(\pi(w, \mu), w, \mu) = 0$ it holds that

$$\pi_1(w, \mu) = \frac{d(w, \mu)}{g(\mu)} = \sum_{0 < i_1 + i_2 + \ldots + i_r \leq k_d} \tilde{d}_{i_1, i_2, \ldots, i_r}(\mu) w_1^{i_1} w_2^{i_2} \ldots w_r^{i_r}.$$

Therefore, without any loss of generality

$$\pi_1(w,\mu) = \sum_{0<i_1+i_2+\ldots+i_r\leq k_d} d_{i_1,i_2,\ldots,i_r}(\mu) w_1^{i_1} w_2^{i_2} \ldots w_r^{i_r}$$

and

$$\begin{aligned}\frac{\partial \pi_1(w,\mu)}{\partial w} Sw &= f_1(\pi(w,\mu), w, c(w,\mu), \mu) \\ &= \alpha_2(\mu)\pi_2(w,\mu) + \vartheta_1(\pi_1(w,\mu), w, \mu).\end{aligned}$$

Further,

$$\pi_2(w,\mu) = \left[\frac{\partial \pi_1(w,\mu)}{\partial w} Sw - \vartheta_1(\pi_1(w,\mu), w, \mu)\right] \frac{1}{\alpha_2(\mu)} \tag{13}$$

and, moreover, at the right hand side of the equation (13) the term $\frac{\partial \pi_1(w,\mu)}{\partial w} Sw$ is a polynomial in w. Proceeding inductively, it is obvious that

$$u = c(w,\mu) = \frac{\partial \pi_n(w,\mu)}{\partial w} Sw - \vartheta_n(\pi_1(w,\mu), \ldots, \pi_n(w,\mu), w, \mu)$$

have a polynomial form. □

Lemma 9. If $c(w,\mu)$ is a polynomial and the exogenous system is linear then the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} w \\ \mu \end{bmatrix} = \begin{bmatrix} Sw \\ 0 \end{bmatrix} := S^a \tilde{w}$$

with output

$$u = c(w,\mu)$$

is such that, for some set of q real numbers $a_0, a_1, \ldots, a_{q-1}$,

$$L_s^q c(w,\mu) = a_0 c(w,\mu) + a_1 L_s c(w,\mu) + \ldots + a_{q-1} L_s^{q-1} c(w,\mu), \quad s(w) := Sw, \tag{14}$$

for all $(w,\mu) \in W^o \times \mathcal{P}$, and moreover the matrix

$$\begin{pmatrix} A(0) - \lambda I & B(0) \\ C(0) & 0 \end{pmatrix}$$

is nonsingular for every λ which is a root of the polynomial

$$p(\lambda) = a_0 + a_1\lambda + \ldots + a_{q-1}\lambda^{q-1} - \lambda^q$$

having non-negative real part.

Proof. By assumption $c(w,\mu)$ is a polynomial in w, let $k \geq 1$ be its degree. It is easy to see that $L_s c(w,\mu)$ is a polynomial of k degree too:

$$L_s c(w,\mu) = \left\langle \frac{\partial \sum_{0<i_1+i_2+\ldots+i_r\leq k_d} \alpha_{i_1,i_2,\ldots,i_r}(\mu) w_1^{i_1} w_2^{i_2} \ldots w_r^{i_r}}{\partial w}, S^a w \right\rangle.$$

Let

$$\{w_1, w_2, \ldots, w_r, w_1 w_2, \ldots, w_r^k\}$$

be the basis of the polynomials of degree k in w, and consider the following change of variables:

$$\{\xi_1, \xi_2, \ldots, \xi_{n_r}\} = \{w_1, w_2, \ldots, w_r^k\}$$

where $n_r = \binom{k+r}{k} = \binom{k+r}{r}$. Let us construct the following system:

$$\begin{bmatrix} \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_r \\ \vdots \\ \dot{\xi}_{n_r} \end{bmatrix} = \begin{bmatrix} S_1 \\ \vdots \\ S_r \\ \vdots \\ S^*_{n_r} \end{bmatrix} \xi$$

$$c(w) = [a_{0,0,\ldots,0}(\mu), \ldots, a_{0,0,\ldots,r}(\mu)]\, \xi$$

where S_i for $i = 1, \ldots, r$ are the rows of S and S^*_j for $j = (r+1), \ldots, n_r$ are some row vectors in $I\!R^r$. The characteristic equation of the above system, which characterizes the control input, is $P(\lambda) = a_0 + a_1\lambda + \ldots - a_q\lambda^q$. As a consequence, it is easy to see that

$$L_s^q c(w,\mu) = \tilde{a}_0 c(w,\mu) + \tilde{a}_1 L_s c(w,\mu) + \ldots + \tilde{a}_{q-1} L_s^{q-1} c(w,\mu)$$

where $a_i = \frac{a_i}{a_q}$, for all (w, μ).

Now,

$$A(0) = \begin{bmatrix} \frac{\partial\vartheta_1}{\partial x_1} & \alpha_2(0) & 0 & \cdots & 0 \\ \frac{\partial\vartheta_2}{\partial x_1} & \frac{\partial\vartheta_2}{\partial x_2} & \alpha_3(0) & \cdots & 0 \\ & & & \cdots & \\ \frac{\partial\vartheta_{n-1}}{\partial x_1} & \frac{\partial\vartheta_{n-1}}{\partial x_2} & \frac{\partial\vartheta_{n-1}}{\partial x_3} & \cdots & \alpha_n(0) \\ \frac{\partial\vartheta_n}{\partial x_1} & \frac{\partial\vartheta_n}{\partial x_2} & \frac{\partial\vartheta_n}{\partial x_3} & \cdots & \frac{\partial\vartheta_n}{\partial x_n} \end{bmatrix}, \quad B(0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \alpha_{n+1}(0) \end{bmatrix}$$

$$C(0) = \begin{bmatrix} g(0) & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and since $\alpha_i(0) \neq 0$, $i = 1, 2, \ldots, n+1$, the matrix

$$\begin{bmatrix} A(0) - \lambda I & B(0) \\ C(0) & 0 \end{bmatrix} .$$

$$= \begin{bmatrix} \frac{\partial\vartheta_1}{\partial x_1} - \lambda & \alpha_2(0) & 0 & \cdots & 0 & 0 & 0 \\ * & \frac{\partial\vartheta_2}{\partial x_2} - \lambda & \alpha_3(0) & \cdots & 0 & 0 & 0 \\ & & & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \frac{\partial\vartheta_{n-1}}{\partial x_{n-1}} - \lambda & \alpha_n(0) & 0 \\ * & * & * & \cdots & * & \frac{\partial\vartheta_n}{\partial x_n} - \lambda & \alpha_{n+1}(0) \\ g(0) & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

is obviously nonsingular for every λ. □

Proof of Theorem 7. Since,

$$\begin{bmatrix} A(0)-\lambda I & B(0) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial\vartheta_1}{\partial x_1}-\lambda & \alpha_2(0) & 0 & \cdots & 0 & 0 & 0 \\ * & \frac{\partial\vartheta_2}{\partial x_2}-\lambda & \alpha_3(0) & \cdots & 0 & 0 & 0 \\ & & & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \frac{\partial\vartheta_{n-1}}{\partial x_{n-1}}-\lambda & \alpha_n(0) & 0 \\ * & * & * & \cdots & * & \frac{\partial\vartheta_n}{\partial x_n}-\lambda & \alpha_{n+1}(0) \end{bmatrix}$$

has full row rank, then the pair $(A(0), B(0))$ is stabilizable. Analogously,

$$\begin{bmatrix} A(0)-\lambda I \\ C(0) \end{bmatrix} = \begin{bmatrix} \frac{\partial\vartheta_1}{\partial x_1}-\lambda & \alpha_2(0) & 0 & \cdots & 0 & 0 \\ * & \frac{\partial\vartheta_2}{\partial x_2}-\lambda & \alpha_3(0) & \cdots & 0 & 0 \\ & & & \ddots & \vdots & \vdots \\ * & * & * & \cdots & \frac{\partial\vartheta_{n-1}}{\partial x_{n-1}}-\lambda & \alpha_n(0) \\ * & * & * & \cdots & * & \frac{\partial\vartheta_n}{\partial x_n}-\lambda \\ g(0) & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

has full column rank then the pair $(A(0), C(0))$ is detectable and the proof is completed using Corollary 5 and Lemma 8, 9. □

4. COMPUTATIONAL ASPECTS

The presented proof has the constructive character, i.e. it describes the possible procedure how to construct a desired feedback controller. Therefore, a natural idea is to create an algorithm to construct the dynamic error feedback regulators solving the structurally stable output regulation problem for the strictly triangular polynomial systems.

To start with, let us first put $z_1 = c(w,\mu)$ and

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \vdots \\ \dot z_q \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ -a_0(\mu)z_1 - a_1(\mu)z_2 - \ldots - a_{q-1}(\mu)z_{q-1} \end{bmatrix} = \begin{bmatrix} L_s c(w,\mu) \\ L_s^2 c(w,\mu) \\ \vdots \\ L_s^q c(w,\mu) \end{bmatrix}$$

$$u = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} z.$$

So, the linear controller is characterized by:

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \cdots \\ \dot z_{q-1} \\ \dot z_q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{q-1} \end{bmatrix} z = \Phi z$$

$$u = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} z = \Gamma z.$$

Since, $N = \begin{bmatrix} n_1 & n_2 & \cdots & n_q \end{bmatrix}$ and

$$\begin{bmatrix} A(0) & 0 \\ NC(0) & \Phi \end{bmatrix}$$

$$= \begin{bmatrix} * & \alpha_2(0) & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & \alpha_3(0) & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & \alpha_n(0) & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \cdots & * & 0 & 0 & 0 & \cdots & 0 \\ n_1 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ n_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ n_{q-1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ n_q & 0 & 0 & \cdots & 0 & -a_0 & -a_1 & a_2 & \cdots & -a_{q-1} \end{bmatrix}$$

it is easy to see that for every $N \neq 0$

$$\begin{bmatrix} A(0) - \lambda I & 0 & B(0) \\ NC(0) & \Phi - \lambda I & 0 \end{bmatrix}$$

has full row rank, and therefore the pair

$$\begin{bmatrix} A(0) & 0 \\ NC(0) & \Phi \end{bmatrix}, \quad \begin{bmatrix} B(0) \\ 0 \end{bmatrix}$$

is stabilizable. Analogously, since,

$$\begin{bmatrix} A(0) & B(0)\Gamma \\ 0 & \Phi \end{bmatrix}$$

$$= \begin{bmatrix} * & \alpha_2(0) & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & \alpha_3(0) & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & \alpha_n(0) & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \cdots & * & \alpha_{n+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -a_0 & -a_1 & a_2 & \cdots & -a_{q-1} \end{bmatrix}$$

it is easy to see that, the pair

$$\begin{bmatrix} C(0) & 0 \end{bmatrix}, \quad \begin{bmatrix} A(0) & B(0)\Gamma \\ 0 & \Phi \end{bmatrix}$$

is detectable. That is, all the conditions of Theorem 4 holds.

So, all we need is to construct an algorithm to find real numbers $a_0, a_1, \ldots, a_{q-1}$. The following Algorithm 10 finds the minimal order of the immersion and the corresponding real numbers $a_0, a_1, \ldots, a_{q-1}$. This algorithm is a new contribution of this paper and has been realized as a MAPLE code. We sketch here its basic ideas.

Algorithm 10.

Step 1. Initializing: $Pi(1) = \frac{d(w,\mu)}{g(\mu)}$, Sol $= \{\phi\}, q = 0, C(q) = 0$.

Step 2. For $j = 1$ to $n-1$ do:
substituting $Pi(j)$ in $\vartheta_j(x_1, x_2, \ldots, x_j, w, \mu)$
$Pi(j+1) = [L_s Pi(j) - \vartheta_j(x_1, x_2, \ldots, x_j, w, \mu)] \frac{1}{\alpha_{j+1}(\mu)}$
end.

Step 3. $c^a(w,\mu) = Pi(n)$, $L(q) = c^a(w,\mu)$

Step 4. While Sol $= \{\phi\}$ do:
$L(q+1) = L_s L(q)$
$C(q+1) = a(q)L(q) + C(q)$
Sol $=$ Solve$\{L(q+1) = C(q+1)\}$
$q := q+1$
end.

We illustrate the above algorithm by the following example, already considered by the different approach in [9]. Later on, we provide a comparison of both approaches.

Example 11. Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + (1+\mu_1)x_1^2 \\ \dot{x}_2 &= x_3 + \mu_2 x_2 + x_1 \\ \dot{x}_3 &= x_4 + (1+\mu_2)x_3 \\ \dot{x}_4 &= u + x_4 \\ e &= x_1 - w_1 \end{aligned}$$

in which $\mu = (\mu_1, \mu_2)$ is a vector of unknown parameters, and suppose w_1 is generated by the linear exosystem

$$\begin{aligned} \dot{w}_1 &= -w_2 \\ \dot{w}_2 &= w_1. \end{aligned}$$

By the latter algorithm we can show that the input is

$$\begin{aligned} c(w,\mu) &= (2 + 2\mu_1 + (1+\mu_2)\mu_2(-1-\mu_1))w_1^2 \\ &\quad + ((-2\mu_2(1+\mu_1) - (1+\mu_2)(2+2\mu_1))w_2 + 3\mu_2 + 2)w_1 \\ &\quad + (-2-2\mu_1)w_2^2 + (2 - (1+\mu_2)\mu_2)w_2, \end{aligned}$$

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -5 & 0 \end{bmatrix}.$$

Using the results of linear regulation for the nominal value $\mu = 0$, and the linearized system given by:

$$A(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

we obtain a controller, simulations of which are shown in Figure 1.

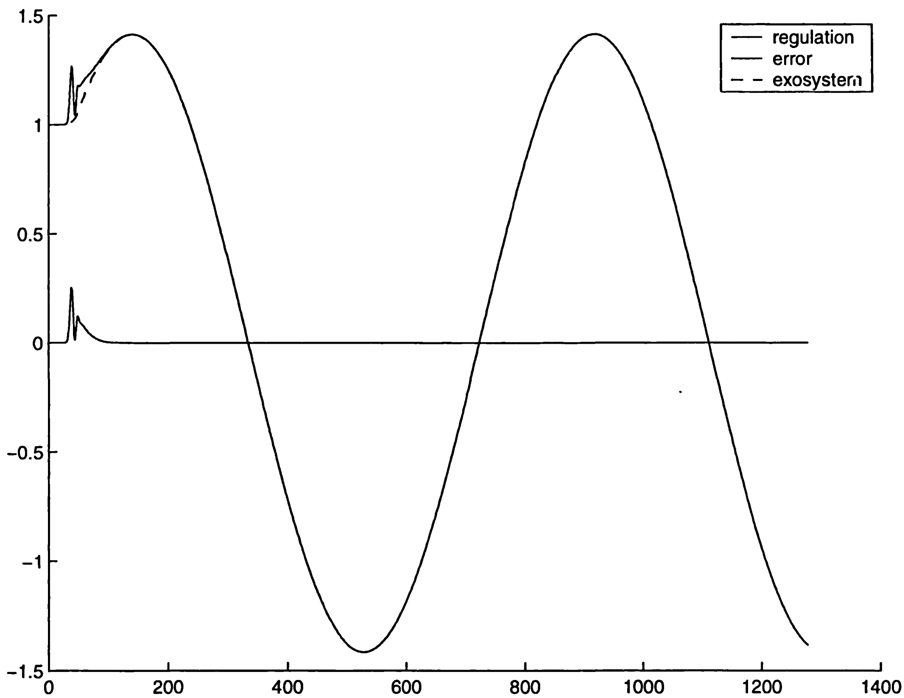

Fig. 1. Simulations of dynamic error feedback regulator for the systems in Example 11.

In [9], the following approach has been studied for the above example. Its disadvantage is that it does not provide the minimal order of the immersion and it is not clear how to generalize it to other systems than that of Example 11. That approach consists simply in finding the matrix:

$$M = \begin{bmatrix} S_1 \\ \vdots \\ S_r \\ \vdots \\ S^*_{n_r} \end{bmatrix}$$

and then determining its characteristic polynomial. Based on the characteristic polynomial, it determines the corresponding coefficients to build the immersion Φ. The following example illustrates such an approach.

Example 12. Consider again Example 11. We have:

$$M = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$$

whose characteristic polynomial is: $\lambda^5 + 5\lambda^3 + 4\lambda$, then the matrix Φ is:

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -4 & 0 & -5 & 0 \end{bmatrix}.$$

Based on the matrix Φ, the error feedback dynamic compensator solving the SSORP was built and tested, giving the analogous performance as in the case of Example 11. We may therefore conclude that our approach is more systematic, provides, in general, smaller order of dynamical compensator, with the similar performance as that of [9].

5. CONCLUSIONS AND OUTLOOKS

This paper provides for a certain class of systems constructive procedures and computer algebra based algorithms to compute and build dynamic error feedback compensators to solve the structurally stable output regulation problem. Comparing to previous similar approaches, the present paper provides for a given class a systematic procedure giving the minimal order of dynamic compensator. Simulations test shows its good performance, which is basically undistinguishable from that of previous approaches using higher order dynamical compensators.

The drawback of the presented approach is that considered class of systems is rather restrictive. To relax these requirements, the immersion to a class of systems of the form:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 = L_s c(w,\mu) \\
\xi_2 &= \xi_3 = L_s^2 c(w,\mu) \\
&\vdots \\
\dot{\xi}_q &= a_0(w)\xi_1 + \ldots + a_{q-1}(w)\xi_{q-1} = L_s^q c(w,\mu) \\
\dot{w} &= s(w) \\
u &= \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \xi
\end{aligned}$$

might be considered. In other words, one would try to find $\{a_0(w), \ldots, a_{q-1}(w)\}$ such that

$$L_s^q c(w,\mu) = a_0(w)c(w,\mu) + \ldots + a_{q-1}(w)L_s^{q-1}c(w,\mu).$$

The main difference here is that the "coefficients" $a_{1,\ldots,q-1}$ may depend on w, so that the immersion eliminates explicit appearance of unknown parameters μ, but may preserve presence of exosystem state w. One may naturally expect such a situation to be obtained under less restrictive conditions than previously. Nevertheless, such an immersion, referred tentatively as the *generalized* one, requires the more general formulation of the output regulation problem. This and other open questions are matter of the ongoing research.

(Received May 11, 2000.)

REFERENCES

[1] C. I. Byrnes, F. Delli Priscolli, and A. Isidori: Output Regulation of Uncertain Nonlinear Systems. Birkhäuser, Boston 1997.

[2] S. Čelikovský: Local stabilization and controllability of a class of nontriangular nonlinear systems. In: Proc. 36th IEEE Conference on Decision and Control, San Diego 1997, pp. 1728–1733.

[3] S. Čelikovský and J. Huang: Continuous feedback asymptotic output regulation for a class of nonlinear systems having nonstabilizable linearization. In: Proc. 37th IEEE Conference on Decision and Control, Tampa 1999, pp. 3087–3092.

[4] S. Čelikovský and J. Huang: Continuous feedback practical output regulation for a class of nonlinear systems having nonstabilizable linearization. In: Proc. 38th IEEE Conference on Decision and Control, Phoenix 2000, pp. 4796–4801.

[5] Ch. Chen: Linear System Theory and Design. Third edition. Oxford University Press, Oxford 1984.

[6] J. Guckenheimer and P. Holmes: Nonlinear Oscilations, Dynamical Systems and Bifurcations of Vector Fields. Springer–Verlag, New York 1983.

[7] J. Huang and W. J. Rugh: On a nonlinear multivariable servomechanism problem. Automatica *26* (1990), 963–972.

[8] J. Huang: Asymptotic tracking and disturbance rejection in uncertain nonlinear system. IEEE Trans. Automat. Control *40* (1995), 1118–1122.

[9] A. Isidori: Nonlinear Control Systems. Third edition. Springer–Verlag, New York 1995, pp. 385–425.

[10] A. Isidori and C. I. Byrnes: Output Regulation of nonlinear systems. IEEE Trans. Automat. Control *35* (1990), 131–140.

[11] H. Knobloch and A. Isidori et al: Topics in Control Theory. Birkhäuser, Boston 1993.

[12] R. Marino and P. Tomei: Nonlinear Control Design – Nonlinear, Robust and Adaptive. Prentice Hall, Englewood Cliffs, N.Y. 1994.

Celia Villanueva-Novelo, Universidad Autónoma de Yucatán, Facultad de Matemáticas, c. 8 s/n x 21, Colonia María Luisa, Mérida, Yucatán. México.
e-mail: vnovelo@tunku.uady.mx

RNDr. Sergej Čelikovský, CSc., Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Praha 8. Czech Republic.
e-mail: celikovs@utia.cas.cz

Dr. Bernardino Castillo-Toledo, Centro de Investigación y de Estudios Avanzados del IPN, Unidad Guadalajara, A. P. 31-438, Plaza La Luna, Guadalajara, Jalisco 44550. México.
e-mail: toledo@gdl.cinvestav.mx