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LINEAR APPROXIMATIONS TO SOME NON-LINEAR AR(1) PROCESSES

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Some methods for approximating non-linear AR(1) processes by classical linear AR(1) models are proposed. The quality of approximation is studied in special non-linear AR(1) models by means of comparisons of quality of extrapolation and interpolation in the original models and in their approximations. It is assumed that the white noise has either rectangular or exponential distribution.

1. INTRODUCTION

Consider a non-linear AR(1) process $\{X_t\}$ defined by

$$X_t = \lambda(X_{t-1}) + e_t, \qquad t \geq 1, \tag{1.1}$$

where λ is a measurable function and e_t is a strict white noise with a density h. A random variable X_0 is supposed to be given. Let $\mathsf{E}e_t^2 < \infty$. Denote $\gamma = \mathsf{E}e_t$, $\sigma^2 = \operatorname{var} e_t$. There are applications such that it is possible to obtain a long realization of the white noise without any signal. Then parameters of the white noise like γ and σ^2 can be estimated very precisely.

In some cases it can be proved that a non-linear process is stationary. Then there are two ways how to approximate it by a linear stationary process.

(i) If expectation and covariance function of the non-linear process are known then one can try to find a linear stationary process with the same characteristics. Pemberton [4] studied this approach and applied it to special threshold autoregressive models, because it was proved in [3] that the autocorrelation structure of a piecewise constant autoregressive threshold model with k regimes is the same as that of an ARMA(p,p) model with $p \leq k-1$.

(ii) If the function λ is smooth, then it is possible to expand it in a Taylor series and to use only its linear approximation. This method is discussed in [6].

In this paper we study quality of approximations of the type (ii). We demonstrate our ideas on the model

$$X_t = \omega X_{t-1}^q + e_t, \qquad t \geq 1, \tag{1.2}$$

where $\omega > 0$, $q \in (0,1)$, random variables e_t are non-negative and X_0 is also a non-negative variable. In order to compare values obtained by a linear approximation with the true ones, we need exact formulas for extrapolation and interpolation in the model (1.2). Such results can be derived only for some distributions of the white noise e_t. Here we present two such cases.

A. Rectangular distribution of the white noise. Assume that $e_t \sim \mathsf{R}(a,b)$ where $0 \le a < b < \infty$ and R denotes the rectangular distribution. The model (1.2) with this white noise will be denoted by $\mathcal{R}(q,\omega,a,b)$. Further we define $\mathcal{R}(q,\omega) = \mathcal{R}(q,\omega,0,1)$.

It is known that for $z \ge 0$ the equation $x = \omega x^q + z$ has a unique positive root x_z. Define $\alpha = x_a$, $\beta = x_b$. Then there exists a distribution of X_0 such that the process $\{X_t, t \ge 0\}$ is strictly stationary and $\alpha \le X_t \le \beta$ for all $t \ge 0$ (see [1]).

B. Exponential distribution of the white noise. Assume that $e_t \sim \mathsf{Ex}(1)$ where $\mathsf{Ex}(1)$ is the exponential distribution with parameter 1 having the density $h(x) = e^{-x}$ for $x > 0$. In this case we denote the model (1.2) by $\mathcal{E}(q,\omega)$. For simplicity, here we investigate only the model $\mathcal{E}(\frac{1}{2},\omega)$.

Our approximations are based on an AR(1) process $Z_t = \nu + \rho Z_{t-1} + \varepsilon_t$ where ε_t is a white noise with $\mathsf{E}\varepsilon_t = 0$, $\mathsf{var}\,\varepsilon_t = \sigma^2$. It is known (cf. Lemma A.1) that

$$\mu = \mathsf{E}Z_t = \frac{\nu}{1-\rho}, \qquad \mathsf{var}\,Z_t = \frac{\sigma^2}{1-\rho^2}.$$

Approximations of the parameter ν will be denoted as $\hat{\nu}$, $\tilde{\nu}$, etc. and similar denotations will be used also for approximations of other parameters of the AR(1) model.

In a simulation study 10 000 realizations of the process $\mathcal{R}\left(\frac{1}{2},1\right)$ and 10 000 realizations of the process $\mathcal{E}\left(\frac{1}{2},1\right)$ were calculated. The length of each realization was 10 000. The average $\bar{\mu}$ of arithmetic means of realizations and its standard deviation $s_{\bar{\mu}}$ were calculated as well as the average of empirical autocorrelation coefficients $\bar{\rho}$ and its standard deviation $s_{\bar{\rho}}$. The results, which will be used in the following parts of the paper for numerical illustrations, are summarized in Table 1.

Table 1. Results of simulations.

Model	$\bar{\mu}$	$s_{\bar{\mu}}$	$\bar{\rho}$	$s_{\bar{\rho}}$
$\mathcal{R}\left(\frac{1}{2},1\right)$	1.858	0.0046	0.369	0.0093
$\mathcal{E}\left(\frac{1}{2},1\right)$	2.578	0.0140	0.282	0.0095

2. METHODS OF LINEARIZATION

Consider model (1.2) with a general non-negative white noise. For $q = 0$ we have $X_t = \omega + e_t$ and for $q = 1$ we have $X_t = \omega X_{t-1} + e_t$. In both cases X_t is a linear process. It can be expected that $q = 0.5$ leads to a model which mostly differs from a linear one.

Consider the model $\mathcal{R}(q, \omega, a, b)$. The first method of linearization used in this paper is based on the approximation of the function $y = \omega x^q$ by its tangent at a point ξ, which has equation $y = \omega(1-q)\xi^q + \omega q \xi^{q-1} x$. Then the model (1.2) can be approximated by

$$Z_t = \omega(1-q)\xi^q + \omega q \xi^{q-1} Z_{t-1} + e_t = \omega(1-q)\xi^q + \gamma + \omega q \xi^{q-1} Z_{t-1} + \varepsilon_t \tag{2.1}$$

where $\varepsilon_t = e_t - \gamma$. We consider only values $\xi \in [\alpha, \beta]$. A natural choice would be $\xi = \mathsf{E}X_t$. Since the stationary distribution of X_t needed for calculation of $\mathsf{E}X_t$ is rarely known we use $\xi = (\alpha+\beta)/2$ in our paper. Lemma A.4 gives that

$$0 \le \omega q \xi^{q-1} \le \omega q \beta^{q-1} < 1$$

and thus (2.1) represents a stationary AR(1) model. If $a = 0$ and $b = 1$ then $\gamma = 0.5$, $\alpha = 1$, $\beta = 2.618$ and $\xi = (\alpha+\beta)/2 = 1.809$. In the case of $\mathcal{R}\left(\frac{1}{2}, 1\right)$, the parameters of the AR(1) process Z_t given by (2.1) are $\hat{\mu} = 1.866$, $\hat{\rho} = 0.372$ and $\hat{\nu} = 1.172$, which corresponds to model

$$Z_t = 1.172 + 0.372 Z_{t-1} + \varepsilon_t. \tag{2.2}$$

In our second method we approximate the function $y = \omega x^q$ on $[\alpha, \beta]$ by a line $y = u + vx$ derived by the least-squares method. The coefficients u, v minimizing $\int_\alpha^\beta (\omega x^q - u - vx)^2 \, dx$ are

$$\begin{aligned} u &= \frac{\omega}{(\beta-\alpha)^4} \left\{ \frac{4(\beta^3-\alpha^3)}{1+q}(\beta^{q+1} - \alpha^{q+1}) - \frac{6(\beta^2-\alpha^2)}{2+q}(\beta^{q+2} - \alpha^{q+2}) \right\}, \\ v &= \frac{\omega}{(\beta-\alpha)^4} \left\{ \frac{12(\beta-\alpha)}{2+q}(\beta^{q+2} - \alpha^{q+2}) - \frac{6(\beta^2-\alpha^2)}{1+q}(\beta^{q+1} - \alpha^{q+1}) \right\}. \end{aligned}$$

In this case we have for the model (1.2) an approximation

$$Z_t = u + vZ_{t-1} + e_t = U + vZ_{t-1} + \varepsilon_t \tag{2.3}$$

where $U = u + \gamma$ and $\varepsilon_t = e_t - \gamma$. It follows from Corollary A.3 and Lemma A.4 that $0 < v < 1$ so that (2.3) represents also a stationary AR(1) model.

Numerically, for $\mathcal{R}\left(\frac{1}{2}, 1\right)$ the second approximation gives $u = 0.650$, $v = 0.378 = \tilde{\rho}$, $\tilde{\mu} = 1.848$, $\tilde{\nu} = 1.149$ and it yields

$$Z_t = 1.149 + 0.378 Z_{t-1} + \varepsilon_t. \tag{2.4}$$

By the way, an AR(1) process $\{Z_t\}$ with parameters corresponding to values $\bar{\mu}$ and $\bar{\rho}$ given in Table 1 for the model $\mathcal{R}\left(\frac{1}{2}, 1\right)$, i.e. with expectation 1.858 and autocorrelation coefficient 0.369 is

$$Z_t = 1.172 + 0.369 Z_{t-1} + \varepsilon_t.$$

Other approximations can be based on piece-wise linear functions, which would lead to threshold autoregression.

Now, consider model $\mathcal{E}\left(\frac{1}{2}, 1\right)$. In this case the distribution of X_t has support $(1, \infty)$. The expectation of X_t is not known. Since $\sqrt{x}$ is a concave function, we have $\mathsf{E}\sqrt{X_{t-1}} \leq \sqrt{\mathsf{E}X_{t-1}}$ and thus for stationary $\{X_t\}$ we get $\mathsf{E}X_t \leq \xi$ where

$$\xi = \xi(\omega) = \sqrt{\frac{\omega + \sqrt{\omega^2 + 4}}{2}}.$$

If we approximate the function $y = \omega\sqrt{x}$ by its tangent at the point ξ, i.e. by $y = \omega\sqrt{\xi}/2 + \omega x/(2\sqrt{\xi})$, then we come to the model

$$Z_t = \frac{\omega}{2}\sqrt{\xi} + \frac{\omega}{2\sqrt{\xi}} Z_{t-1} + e_t = \left(\frac{\omega}{2}\sqrt{\xi} + 1\right) + \frac{\omega}{2\sqrt{\xi}} Z_{t-1} + \varepsilon_t$$

where $\varepsilon_t = e_t - 1$. It is easy to show that the function $r(\omega) = \omega \Big/ \left[2\sqrt{\xi(\omega)}\right]$ is increasing on $[0, \infty)$ and the equation $r(\omega) = 1$ has a unique root $\omega_0 = 2.62362$. It means that for $\omega \in [0, \omega_0]$ our process Z_t is stationary. In the special case $\omega = 1$ we have $\xi = 1.272$ and

$$Z_t = 1.564 + 0.443 Z_{t-1} + \varepsilon_t \tag{2.5}$$

with $\mathsf{E}Z_t = 2.809$.

The linear least squares approximation of the function $\omega\sqrt{x}$ on $[1, \infty)$ cannot be used and so we consider its weighted form

$$\min_{u,v} \int_1^\infty e^{-x}(\omega\sqrt{x} - u - vx)^2 \, \mathrm{d}x.$$

Since

$$\begin{aligned}
\int x e^{-x} \, \mathrm{d}x &= -xe^{-x} - e^{-x}, \\
\int x^2 e^{-x} \, \mathrm{d}x &= -(x^2 + 2x + 2)e^{-x}, \\
\int \sqrt{x} e^{-x} \, \mathrm{d}x &= -\sqrt{x}e^{-x} + \sqrt{\pi}\Phi\left(\sqrt{2x}\right), \\
\int x^{3/2} e^{-x} \, \mathrm{d}x &= -x^{3/2}e^{-x} - \frac{3}{2}\sqrt{x}e^{-x} + \frac{3}{2}\sqrt{\pi}\Phi\left(\sqrt{2x}\right),
\end{aligned}$$

we get

$$\begin{aligned}
u &= 2\omega e\sqrt{\pi}\left[1 - \Phi\left(\sqrt{2}\right)\right], \\
v &= \frac{\omega}{2}\left\{1 - e\sqrt{\pi}\left[1 - \Phi\left(\sqrt{2}\right)\right]\right\}.
\end{aligned}$$

This approximation gives

$$Z_t = u + vZ_{t-1} + e_t = (u + 1) + vZ_{t-1} + \varepsilon_t$$

with $\varepsilon_t = e_t - 1$. For $\omega = 1$ we have $u = 0.757873$, $v = 0.310532$ and

$$Z_t = 1.758 + 0.311 Z_{t-1} + \varepsilon_t \tag{2.6}$$

with $\mathsf{E}Z_t = 2.550$.

3. ESTIMATING THE PARAMETERS OF THE NON-LINEAR MODEL

Assume that α, β and γ are known parameters. If we have a realization $Z_1, \ldots, Z_n$ of the process $\mathcal{R}(q, \omega, a, b,)$ then $\mathsf{E}Z_t$ and ρ are easily estimable parameters. We denote their estimates by μ^* and ρ^*. The moment method applied to (2.1) gives

$$\frac{\omega(1-q)\xi^q + \gamma}{1-\rho^*} = \mu^*, \qquad \omega q \xi^{q-1} = \rho^*. \tag{3.1}$$

Inserting $\xi^q = \rho^* \xi / (\omega q)$ into the first equation we get for estimates $\hat{q}$ and $\hat{\omega}$ of the parameters q and ω the following formulas:

$$\hat{q} = \frac{\rho^* \xi}{\rho^* \xi + \mu^*(1-\rho^*) - \gamma}, \qquad \hat{\omega} = \frac{\rho^*}{\hat{q}\xi^{\hat{q}-1}}. \tag{3.2}$$

In the second method u and v are quite complicated functions of q and ω so that the equations

$$\frac{u+\gamma}{1-\rho^*} = \mu^*, \qquad v = \rho^* \tag{3.3}$$

must be solved numerically. The solution of these equations is denoted by $\tilde{q}$ and $\tilde{\omega}$.

Consider the model $\mathcal{R}(q, \omega)$. If we take $\mu = \mu^* = 1.858$ and $\rho = \rho^* = 0.369$ then (3.2) yields $\hat{q} = 0.498$ and $\hat{\omega} = 0.997$. From (3.3) we obtain $\tilde{q} = 0.486$ and $\tilde{\omega} = 1.013$. All the estimates are quite close to the true values of the corresponding parameters.

In the model $\mathcal{E}(\frac{1}{2}, \omega)$ we estimate only one parameter ω. The moment method applied to the first method of linearization leads to equations

$$\mu = \frac{\frac{\omega}{2}\sqrt{\xi} + 1}{1 - \frac{\omega}{2\sqrt{\xi}}}, \qquad \rho = \frac{\omega}{2\sqrt{\xi}}.$$

Their solutions will be denoted by ω_1^* and ω_2^*, respectively.

Using the second method of linearization we come to equations

$$\mu = \frac{2\omega e\sqrt{\pi}[1 - \Phi(\sqrt{2})] + 1}{1 - \frac{\omega}{2}\{1 - e\sqrt{\pi}[1 - \Phi(\sqrt{2})]\}}, \qquad \rho = \frac{\omega}{2}\{1 - e\sqrt{\pi}[1 - \Phi(\sqrt{2})]\}.$$

Their solutions will be denoted by ω_1^+ and ω_2^+, respectively. If we insert values $\bar{\mu} = 2.578$ and $\bar{\rho} = 0.282$ from our simulations, we obtain

$$\omega_1^* = 0.922, \qquad \omega_2^* = 0.608, \qquad \omega_1^+ = 1.266, \qquad \omega_2^+ = 0.908.$$

In this case the estimates considerably differ from the true value $\omega = 1$.

4. EXTRAPOLATION

Let $\hat{X}_t$ be the LS extrapolation of X_t one step ahead based on $\{X_{t-1}, X_{t-2}, \dots\}$. It is known that for model (1.1) we have

$$\hat{X}_t = \lambda(X_{t-1}) + \gamma, \qquad \mathsf{E}(X_t - \hat{X}_t)^2 = \sigma^2.$$

The LS extrapolation two or more steps ahead in model (1.1) is more complicated. Some results for $\mathcal{R}\left(\frac{1}{2}, 1\right)$ and for $\mathcal{E}\left(\frac{1}{2}, 1\right)$ can be found in the Appendix. In a simulation of the length $n = 10\,000$ of $\mathcal{R}\left(\frac{1}{2}, 1\right)$ one step ahead extrapolations $\hat{X}_t$ as well as extrapolations Z_t^* of (2.2) and Z_t^+ of (2.4) were calculated. The averages of $(X_t - \hat{X}_t)^2$, $(X_t - Z_t^*)^2$ and $(X_t - Z_t^+)^2$ are denoted by $\hat{\sigma}^2$, σ^{*2} and σ^{+2}, respectively. They are given in Table 2. Similarly, in a simulation of the length $n = 10\,000$ of $\mathcal{E}\left(\frac{1}{2}, 1\right)$ one step ahead extrapolations $\hat{X}_t$ as well as extrapolations Z_t^* of (2.5) and Z_t^+ of (2.6) were calculated together with the corresponding averages $\hat{\sigma}^2$, σ^{*2} and σ^{+2}. See Table 2.

Table 2. Results of extrapolations.

Model	σ^2	$\hat{\sigma}^2$	σ^{*2}	σ^{+2}
$\mathcal{R}\left(\frac{1}{2}, 1\right)$	0.083	0.084	0.084	0.084
$\mathcal{E}\left(\frac{1}{2}, 1\right)$	1.000	1.017	1.060	1.020

Values $\hat{\sigma}^2$ and σ^2 are close which verifies quality of simulations. Values σ^{*2} and σ^{+2} are also very close to σ^2, which shows that the LS extrapolations based on linear approximations have nearly the same quality as the LS extrapolations in the considered non-linear models.

5. INTERPOLATION

Formulas for interpolation in models $\mathcal{R}\left(\frac{1}{2}, 1\right)$ and $\mathcal{E}\left(\frac{1}{2}, 1\right)$ can be found in the Appendix. Formula for interpolation in linear AR(1) model is well known but for convenience it is remembered in Lemma A.1. Let X_t^0 be interpolation of X_t given $\{\dots, X_{t-2}, X_{t-1}, X_{t+1}, X_{t+2}, \dots\}$. We denote by Z_t^{*i} and Z_t^{+i} interpolations in processes (2.2) and (2.4), respectively, when we deal with $\mathcal{R}\left(\frac{1}{2}, 1\right)$. If we consider $\mathcal{E}\left(\frac{1}{2}, 1\right)$ then Z_t^{*i} and Z_t^{+i} are interpolations in processes (2.5) and (2.6), respectively. A simulation of $\mathcal{R}\left(\frac{1}{2}, 1\right)$ of the length $n = 10\,000$ was calculated and the averages of $(X_t - X_t^0)^2$, $(X_t - Z_t^{*i})^2$ and $(X_t - Z_t^{+i})^2$ denoted by $\hat{\sigma}_i^2$, σ_i^{*2} and σ_i^{+2}, respectively, are given in Table 3. A similar simulation was carried out also for $\mathcal{E}\left(\frac{1}{2}, 1\right)$.

Table 3. Results of interpolations.

Model	$\hat{\sigma}_i^2$	σ_i^{*2}	σ_i^{+2}
$\mathcal{R}\left(\frac{1}{2}, 1\right)$	0.069	0.075	0.075
$\mathcal{E}\left(\frac{1}{2}, 1\right)$	0.882	1.018	1.016

We can conclude that σ_i^{*2} and σ_i^{+2} are also close to $\hat{\sigma}_i^2$, but the differences are larger than in the case of extrapolation.

APPENDIX

Lemma A.1. Let $\{Z_t\}$ be the AR(1) process defined by

$$Z_t = \nu + \rho Z_{t-1} + \varepsilon_t \tag{A.1}$$

where $\rho \in (-1, 1)$ and $\{\varepsilon_t\}$ is a white noise with $\mathsf{E}\varepsilon_t = 0$, $\mathsf{E}\varepsilon_t^2 = \sigma^2 < \infty$. Then

$$\mathsf{E}Z_t = \frac{\nu}{1-\rho}, \qquad \mathsf{var}\, Z_t = \frac{\sigma^2}{1-\rho^2}, \qquad \mathsf{corr}(Z_t, Z_{t-1}) = \rho.$$

The best linear extrapolation $\hat{Z}_t$ of Z_t given $\{Z_{t-s}, s \geq 1\}$ is $\hat{Z}_t = \nu + \rho Z_{t-1}$. The residual variance of extrapolation is $\mathsf{E}(Z_t - \hat{Z}_t)^2 = \sigma^2$. The best linear interpolation Z_t^0 of Z_t given $\{Z_{t-s}, s \neq 0\}$ is

$$Z_t^0 = \frac{\rho}{1+\rho^2}(Z_{t-1} + Z_{t+1}) + \nu \frac{1-\rho}{1+\rho^2}$$

and its residual variance is $\mathsf{E}(Z - Z_t^0)^2 = \frac{\sigma^2}{1+\rho^2}$.

Proof. The assertion is well known. □

Lemma A.2. Let $-\infty < \alpha < \beta < \infty$. Let f be a continuous function on $[\alpha, \beta]$. Let $y = u + vx$ be the least squares approximation of f on $[\alpha, \beta]$. Then there exist x_1, x_2 such that $\alpha < x_1 < x_2 < \beta$ and that $u + vx_1 = f(x_1)$, $u + vx_2 = f(x_2)$.

Proof. Since u, v are such that $\int_\alpha^\beta [f(x) - U - Vx]^2 \, \mathrm{d}x$ attains its minimum at $U = u$, $V = v$, the equations

$$\int_\alpha^\beta [f(x) - u - vx] \, \mathrm{d}x = 0, \tag{A.2}$$

$$\int_\alpha^\beta x[f(x) - u - vx] \, \mathrm{d}x = 0 \tag{A.3}$$

are satisfied. It follows from (A.2) that there exists $x_1 \in (\alpha, \beta)$ such that $f(x_1) - u - vx_1 = 0$. In the remaining part of the proof we assume without loss of generality that $\alpha \geq 0$. If $f(x) - u - vx \neq 0$ for all $x \in (\alpha, \beta)$, $x \neq x_1$, the we have either

$$\begin{aligned} f(x) - u - vx > 0 \quad &\text{for} \quad \alpha < x < x_1, \\ f(x) - u - vx < 0 \quad &\text{for} \quad x_1 < x < \beta \end{aligned} \tag{A.4}$$

or

$$\begin{aligned} f(x) - u - vx < 0 \quad &\text{for} \quad \alpha < x < x_1, \\ f(x) - u - vx > 0 \quad &\text{for} \quad x_1 < x < \beta \end{aligned} \tag{A.5}$$

because f is continuous and other cases such as $f(x) - u - vx > 0$ for $x \neq x_1$ etc. would be in contradiction with (A.2). Assume that (A.4) holds. Then we can see that

$$\begin{aligned}\int_\alpha^\beta \frac{x}{x_1}[f(x) - u - vx]\,\mathrm{d}x &< \int_\alpha^{x_1}[f(x) - u - vx]\,\mathrm{d}x + \int_{x_1}^\beta [f(x) - u - vx]\,\mathrm{d}x \\ &= \int_\alpha^\beta [f(x) - u - vx]\,\mathrm{d}x = 0\end{aligned}$$

which contradicts (A.3). The case (A.5) is similar. □

Corollary A.3. Let the assumptions of Lemma A.2 be fulfilled. Moreover, let $f'(x)$ exist for all $x \in (\alpha, \beta)$. Then there exists $\xi \in (\alpha, \beta)$ such that $v = f'(\xi)$.

Lemma A.4. In the model $\mathcal{R}(q, \omega, a, b)$ we have $\omega q \beta^{q-1} < 1$.

Proof. Remember that β satisfies $\beta = \omega\beta^q + b$. Then $1 = \omega\beta^{q-1} + b/\beta$, which gives $\omega\beta^{q-1} \leq 1$. Since $q \in (0, 1)$, the assertion follows. □

Let the model (1.1) hold and let $X_0, \ldots, X_t$ be known. The naïve extrapolation of the variable X_{t+m} $(m \geq 1)$ is given by

$$X_{t+1}^* = \lambda(X_t) + \gamma, \qquad X_{t+m}^* = \lambda(X_{t+m-1}^*) + \gamma.$$

Obviously, there exists a function H_m such that $X_{t+m} = H_m(X_t)$. The functions H_m, $m \geq 1$, satisfy

$$H_1(x) = \lambda(x) + \gamma, \qquad H_{m+1}(x) = \lambda[H_m(x)] + \gamma \text{ for } m \geq 1.$$

The least squares extrapolation of X_{t+m} is

$$\hat{X}_{t+m} = K_m(X_t)$$

where

$$K_1(x) = \lambda(x) + \gamma, \qquad K_{m+1}(x) = \int_{-\infty}^{\infty} K_m(y) h[y - \lambda(x)]\,\mathrm{d}y \tag{A.6}$$

for $m \geq 1$ (see [5], p. 346, and [2]). We can see that $K_1(x) \equiv H_1(x)$ but for $m \geq 2$ we have generally that $H_m(x) \not\equiv K_m(x)$.

Assume that variables $X_0, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n$ are known. Let $X_{\setminus j} = \{x_0, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\}$ where x_j $(1 \leq j \leq n-1)$ is excluded. Then conditional densities $p(.|.)$ satisfy

$$p(x_j | X_{\setminus j}) = \frac{1}{K} p(x_j | x_{j-1}) p(x_{j+1} | x_j) \tag{A.7}$$

where

$$K = p(x_{j+1} | x_{j-1}) = \int p(x_j | x_{j-1}) p(x_{j+1} | x_j)\,\mathrm{d}x_j$$

[see [5], p. 319, formulas (5.222) and (5.223)]. Since e_t has the density h, we have $p(x_j|x_{j-1}) = h[x_j - \lambda(x_{j-1})]$. The least squares interpolation of X_j is

$$X_j^0 = \mathsf{E}(X_j|X_{\setminus j}) = \int x_j p(x_j|X_{\setminus j})\, \mathrm{d}x_j.$$

Now, we specify the general formulas to the two cases of the model (1.2) mentioned in Introduction.

A. Rectangular distribution of the white noise. In the model $\mathcal{R}(q,\omega,a,b)$ it is clear that

$$\begin{aligned} K_1(x) &= H_1(x) = \frac{a+b}{2} + \omega x^q, \\ H_2(x) &= \frac{a+b}{2} + \omega\left(\frac{a+b}{2} + \omega x^q\right)^q \end{aligned}$$

and simple calculations give

$$K_2(x) = \frac{a+b}{2} + \frac{\omega}{(b-a)(1+q)}\left[(b+\omega x^q)^{1+q} - (a+\omega x^q)^{1+q}\right].$$

If $m \geq 3$ then expressions for $K_m(x)$ are complicated even for such values of q which admit derivation of explicit formulas. In our paper we consider only the case $a = 0$ so that $e_t \sim \mathsf{R}(0,b)$. Then $h(x) = 1/b$ for $0 < x < b$ and $h(x) = 0$ otherwise. Formula (A.7) gives

$$p(x_j|X_{\setminus j}) = \frac{1}{\min\{x_{j+1}^2, b+\sqrt{x_{j-1}}\} - \max\{\sqrt{x_{j-1}}, (x_{j+1}-b)^2\}}$$

for

$$\max\{\sqrt{x_{j-1}}, (x_{j+1}-b)^2\} < x_j < \min\{x_{j+1}^2, b+\sqrt{x_{j-1}}\}$$

and 0 otherwise. From here it is clear that the least squares interpolation X_j^0 of X_j given $X_{\setminus j}$ is

$$X_j^0 = \mathsf{E}(X_j|X_{\setminus j}) = \frac{1}{2}[\min\{x_{j+1}^2, b+\sqrt{x_{j-1}}\} + \max\{\sqrt{x_{j-1}}, (x_{j+1}-b)^2\}].$$

B. Exponential distribution of the white noise. Here we consider the model $\mathcal{E}(\frac{1}{2},\omega)$. First we have

$$\begin{aligned} K_1(x) &= H_1(x) = \omega\sqrt{x} + 1, \\ H_2(x) &= 1 + \omega\sqrt{\omega\sqrt{x}+1}. \end{aligned}$$

Further computations lead to

$$K_2(x) = 1 + \omega^{3/2}\sqrt[4]{x} + \sqrt{\pi}\,\omega e^{\omega\sqrt{x}}[1 - \Phi(\sqrt{2\omega}\sqrt[4]{x})]$$

where Φ is the distribution function of $N(0,1)$. Further we have $H_1(x) = K_1(x)$.

To find explicit formulas for interpolation we need integrals

$$\begin{aligned}\int e^{-x+\omega\sqrt{x}}\,\mathrm{d}x &= -e^{-x+\omega\sqrt{x}} + \omega\sqrt{\pi}e^{\omega^2/4}\Phi\left(\sqrt{2x}-\frac{\omega}{\sqrt{2}}\right),\\ \int xe^{-x+\omega\sqrt{x}}\,\mathrm{d}x &= \frac{\omega}{2}\left(3+\frac{\omega^2}{2}\right)\sqrt{\pi}e^{\omega^2/4}\Phi\left(\sqrt{2x}-\frac{\omega}{\sqrt{2}}\right)\\ &\quad -\left(1+\frac{\omega^2}{4}+\frac{\omega}{2}\sqrt{x}+x\right)e^{-x+\omega\sqrt{x}}\end{aligned}$$

which can be obtained by direct calculation. The conditional density of X_j given $X_{\backslash j}$ is

$$q(x_j) = p(x_j|X_{\backslash j}) = Ke^{-x_j+\omega\sqrt{x_j}}$$

for

$$\omega\sqrt{x_{j-1}} < x_j < x_{j+1}^2/\omega^2$$

and zero otherwise where

$$\begin{aligned}K^{-1} &= \exp\{-\omega\sqrt{x_{j-1}}+\omega^{3/2}\sqrt[4]{x_{j-1}}\} - \exp\left\{-\frac{x_{j+1}^2}{\omega^2}+x_{j+1}\right\}\\ &\quad +\omega\sqrt{\pi}e^{\omega^2/4}\left[\Phi\left(\frac{\sqrt{2}\,x_{j+1}}{\omega}-\frac{\omega}{\sqrt{2}}\right)-\Phi\left(\sqrt{2\omega}\sqrt[4]{x_{j-1}}-\frac{\omega}{\sqrt{2}}\right)\right].\end{aligned}$$

The least squares interpolation of the random variable X_j is

$$\begin{aligned}X_j^0 = K^{-1}&\left(\frac{\omega}{2}\left(3+\frac{\omega^2}{2}\right)\sqrt{\pi}e^{\omega^2/4}\left[\Phi\left(\frac{\sqrt{2}\,x_{j+1}}{\omega}-\frac{\omega}{\sqrt{2}}\right)-\Phi\left(\sqrt{2\omega}\sqrt[4]{x_{j-1}}-\frac{\omega}{\sqrt{2}}\right)\right]\right.\\ &+\left(1+\frac{\omega^2}{4}+\frac{\omega^{3/2}}{2}\sqrt[4]{x_{j-1}}+\omega\sqrt{x_{j-1}}\right)\exp\{-\omega\sqrt{x_{j-1}}+\omega^{3/2}\sqrt[4]{x_{j-1}}\}\\ &\left.-\left(1+\frac{\omega^2}{4}+\frac{x_{j+1}}{2}+\frac{x_{j+1}^2}{\omega^2}\right)\exp\left\{-\frac{x_{j+1}^2}{\omega^2}+x_{j+1}\right\}\right).\end{aligned}$$

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