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VARIABLE MEASUREMENT STEP IN 2–SLIDING CONTROL

ARIE LEVANT

Sliding mode is used in order to retain a dynamic system accurately at a given constraint and features theoretically-infinite-frequency switching. Standard sliding modes are known to feature finite time convergence, precise keeping of the constraint and robustness with respect to internal and external disturbances. Having generalized the notion of sliding mode, higher order sliding modes preserve or generalize its main properties, improve its precision with discrete measurements and remove the chattering effect. However, in their standard form, most of higher order sliding controllers are sensitive to measurement errors. A special measurement step feedback is introduced in the present paper, which solves that problem without loss of precision. The approach is demonstrated on a so-called twisting algorithm. Its asymptotic properties are studied in the presence of vanishing measurement errors. A model illustration and simulation results are presented.

1. INTRODUCTION

Sliding mode control is well known as one of the most effective ways to overcome uncertainty problems. The resulting so-called variable structure systems (VSS) feature high precision performance, their design is rather simple and clear [16, 17]. Yet, sliding mode implementation is restricted by an intrinsic drawback. Providing for keeping an uncertain dynamic system accurately within a given constraint, sliding modes exist due to theoretically infinite frequency of control switching. In practice this leads to the so-called chattering effect which is exhibited by potentially dangerous high-frequency vibrations of the controlled plant.

To avoid chattering some approaches were proposed. The main idea is to change the dynamics in a small vicinity of the discontinuity surface in order to avoid real discontinuity and at the same time to preserve the main properties of the whole system [15]. The idea, exploited here, is to hide the discontinuity in higher derivatives of the control. In the simplest case it may be realized by implanting a fast stable actuator between the relay and the plant [8]. In the resulting mode the corresponding state and velocity vibration magnitudes both tend to zero when switching imperfections vanish, and at the same time the plant behavior is described by the sliding mode equations. Corresponding modes are called higher order sliding modes [1, 4, 5, 10, 13]. However, such mode is unstable if the implanted dynamics is chosen improperly. In the above case convergence to that special mode is not faster than exponential [8] but it may feature a finite time as well if proper controllers are used [1, 2, 5, 8, 10, 13].

A higher order sliding mode (HOSM) is actually a movement on a discontinuity set of a dynamic system, the sliding order characterizing the dynamics smoothness degree in the vicinity of the set. If the task is to provide for keeping equality of a smooth function σ to zero, the sliding order is a number of continuous total derivatives of σ (including the zero one) in the vicinity of the sliding mode. Hence, the rth order sliding mode is determined by the equalities

$$\sigma = \dot{\sigma} = \ddot{\sigma} = \ldots = \sigma^{(r-1)} = 0, \tag{1}$$

forming an r-dimensional condition on the state of the dynamic system. The words "rth order sliding" are often abridged to "r-sliding". It is also known that with discrete measurements r-sliding mode realization may provide for up to the rth order of sliding precision with respect to the measurement interval [6, 10, 12].

The standard sliding mode has the first order, for $\dot{\sigma}$ is discontinuous. Trivial cases of asymptotically stable HOSM are easily found in many classic VSSs. For example, there is an asymptotically stable 2-sliding mode with respect to the constraint $x = 0$ at the origin $x = \dot{x} = 0$ (at one point only) of a 2-dimensional VSS keeping the constraint $x + \dot{x} = 0$ in a standard 1-sliding mode. It was mentioned above that asymptotically stable or unstable HOSMs inevitably appear in VSSs with fast actuators [8], revealing themselves by spontaneous disappearance of the chattering effect in the stability case. Thus, examples of asymptotically stable or unstable sliding modes of any order are well known [4, 5, 8, 10, 14]. Examples of r-sliding modes attracting in finite time are known for $r = 1$ (which is trivial), for $r = 2$ [1, 2, 8, 10, 13] and for $r = 3$ [8]. Arbitrary order sliding controllers with finite-time convergence were also presented [12].

Generally speaking, any r-sliding controller needs σ, $\dot{\sigma}$, $\ddot{\sigma}$, ..., $\sigma^{(r-1)}$ to be available. The only known exception is a 2-sliding controller [11, 10] which needs only measurements of σ. As a matter of fact, values of some expression like $\mathrm{sign}(\sigma^{(r-1)} - h(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-2)}))$ are needed and not $\sigma^{(r-1)}$ itself. Therefore, in realization the expression $\mathrm{sign}(\Delta\sigma^{(r-2)} - \Delta t \cdot h(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-2)}))$ is substituted for the previous one, only first differences of $\sigma^{(r-2)}$ being practically used. Nevertheless, those controllers are sensitive to measurement errors of $\sigma^{(r-2)}$. Indeed, let the maximum possible error in the measurements of $\sigma^{(r-2)}$ be $\delta > 0$. It may be shown that, with Δt fixed and δ sufficiently small, measurement errors do not interfere with the algorithm performance. But the sliding accuracy deteriorates when Δt decreases or δ increases. It happens, for $|\sigma^{(r-1)}|$ is bounded, and the measurement error influence starts to dominate in the above expression. Hence, measurement time step Δt is to be adjusted in accordance with δ evaluation which may appear to be complicated.

Due to smaller information requirements, 2-sliding algorithms look promising for applications. Indeed, a few recent publications [1, 2, 3] are devoted to their implementation. As it was marked above, most of those controllers use first differences of σ. They also provide for the second order sliding precision with respect to the measurement time step. Unfortunately, according to the above reasoning any uncontrolled measurement-step reduction inevitably leads to system failure as a result of small measurement inaccuracies. Thus, in solving a real control problem one has to check that the measurement step be larger than some critical value. It is shown in the present paper that the measurement step may be taken proportional to the square root of the maximal error of σ measurements. However, that approach needs some information on the measurement error which is not always available.

The problem is solved in the present paper by introducing a measurement step feedback $t_{i+1} - t_i = \tau_{i+1} = \Theta(\sigma(t_i))$. The idea is that τ should be small with small $|\sigma|$ and increase for large $|\sigma|$. Certainly, there are upper and lower limits of τ: $\tau_M \geq \tau \geq \tau_m > 0$.

Only one controller – twisting algorithm [8, 10, 13] – is considered in the paper, nevertheless the results may be extended to other higher order sliding controllers. The step feedback $\tau = \lambda|\sigma|^\rho$ is shown to make the algorithm robust with respect to measurement errors for certain positive values of ρ and λ. The utmost precision $\sigma = O(\tau_m^2)$, $\dot{\sigma} = O(\tau_m)$ is proved to be attained in finite time when $\delta = 0$ and $\rho \geq 0.5$ (such controllers are called second order real sliding algorithms [10, 13]). Thus, there is no need for δ evaluation and appropriate τ adjustment. The corresponding dependence on ρ is calculated of sliding precision asymptotics with respect to δ.

Some of these results have long been known to the author qualitatively as a recipe and were mentioned as a remark in [10]. Nevertheless, they were not rigorously formulated and proved, and the asymptotic dependences on ρ were not known. In particular, it was not known that the best choice is $\rho = 0.5$ providing for $\sigma = O(\delta)$ with $\tau \to 0$. A model illustration and simulation results are presented.

2. GENERALIZED CONSTRAINT FULFILLMENT PROBLEM

Our intention is to replace the standard relay algorithm $u = -\operatorname{sign}\sigma$ by a continuous output of some dynamic subsystem. To simplify and detail the constraint fulfillment problem, consider the dynamic system given by the equation

$$\dot{x} = f(t, x, u) \tag{2}$$

where $x \in X$ is a state variable, X is a smooth finite dimensional manifold, t is time, $u \in I\!R$ is control, f is a C^1-function. Let $\sigma(t, x) \in I\!R$ be a C^2-function. The only available current information consists of the current values of t, $u(t)$ and $\sigma(t)$ ($\sigma(t) := \sigma(t, x(t))$). There is also a number of known constants defined below. The goal is to force the constraint function σ to vanish in finite time by means of control continuously dependent on time.

Let K_m, K_M, σ_0, C_0 be positive constants, $K_m < K_M$, and assume the following:

1. $|u| < \kappa$, $\kappa = \text{const} > 1$. Any solution of (2) is well defined for all t, provided $u(t)$ is continuous and $|u(t)| < \kappa$ for each t.
2. There exists $u_1 \in (0, 1)$ such that for any continuous function $u(t)$ with $|u(t)| > u_1$, there is t_1, such that $\sigma(t)\, u(t) > 0$ for each $t > t_1$. Hence, the control $u(t) = -\operatorname{sign}\sigma(t_0)$, where t_0 is the initial value of time, provides for hitting the manifold $\sigma = 0$ in finite time.

   Denote $\Lambda_u(\cdot) = \frac{\partial}{\partial t}(\cdot) + \frac{\partial}{\partial x}(\cdot)\, f(t, x, u)$, $\dot{\sigma}(t, x, u) = L_u\sigma(t, x)$.

3. There is positive u_0, $u_0 < 1$, such that if $|\sigma(t,x)| < \sigma_0$, then

$$0 < K_m \leq \frac{\partial}{\partial u}\dot{\sigma}(t,x,u) \leq K_M$$

for all u, $|u| < \kappa$, and the inequality $|u| > u_0$ entails $\dot{\sigma}u > 0$.

4. Within the region $|\sigma| < \sigma_0$ the inequality $|L_u L_u \sigma(t,x)| < C_0$ holds for all t, x, and u. It means that, calculated with fixed values of control u, the second time derivative of constraint function σ is uniformly bounded.

It follows from the theorem on implicit function that there is a function $u_{\text{eq}}(t,x)$ (equivalent control [16]) satisfying the equation $\dot{\sigma} = 0$. Once $\sigma = 0$ is achieved, the control $u = u_{\text{eq}}(t,x)$ would provide for exact constraint fulfillment. Conditions 3 and 4 mean that $|\sigma| < \sigma_0$ implies $|u_{\text{eq}}| < u_0 < 1$, and that the velocity of u_{eq} changing is bounded. This provides for a possibility to approximate u_{eq} by a Lipschitzian control. Note also that linear dependence on control u is not required.

The proposed controllers depend on few constant parameters. These parameters are to be tuned in order to control the whole class of processes and constraint functions defined by the concrete values of σ_0, K_M, K_m, C_0. By increasing the constants K_M, C_0 and reducing K_m, σ_0 at the same time, we enlarge the controlled class too. Such algorithms are obviously insensitive to any model perturbations and external disturbances which do not stir the dynamic system from the given class.

The variable structure system theory deals usually with systems of the form $\dot{x} = a(t,x) + b(t,x)v$, where $x \in \mathbb{R}^n$, v is control. Under conventional assumptions the task of keeping the constraint $\varphi(t,x) = 0$ is reduced to the task stated above. A new control u and a constraint function σ are to be defined in that case by the transformation

$$v = k\Phi(x)\,u, \quad \sigma = \varphi(t,x)/\Phi(x), \quad \Phi(x) = \sqrt{xDx^t + h}, \quad k,\, h > 0, \tag{3}$$

where $k, h > 0$ are constants, D is a non-negative definite matrix.

In the simple case when $\dot{x} = A(t)\,x + b(t)\,u$, $\varphi = c(t)\,x + \xi(t)$ all conditions are reduced to the boundedness of c, $\dot{c}$, $\ddot{c}$, ξ, $\ddot{\xi}$, A, $\dot{A}$, b, $\dot{b}$ and to the inequality $cb >$ const > 0 (i.e., the relative degree equals 1). The corresponding constants determine the controlled class.

3. TWISTING ALGORITHM

Return to the generalized sliding problem stated above. The algorithm $u = -\operatorname{sign}\sigma$ is a standard 1-sliding algorithm. If values of σ are measured at discrete times t_0, t_1, t_2, $\ldots$, $t_i - t_{i-1} = \tau =$ const > 0, we get a real sliding algorithm $u(t) = -\operatorname{sign}\sigma(t_i)$, $t \in [t_i, t_{i+1})$. After some transient process first order real sliding is achieved, $\sup|\sigma(t)| = O(t)$.

Remind that 2-sliding mode is characterized by the equality $\sigma = \dot{\sigma} = 0$ and smoothness of σ, $\dot{\sigma}$. The simplest way to achieve such a mode is to keep a new constraint $\sigma + \dot{\sigma} = 0$ in a 1-sliding mode provided by discontinuity of $\ddot{\sigma}$. In that case, however, the mode would be attained only in infinite time. One of the controllers, featuring a finite-time transient process, is the so-called "twisting algorithm" [5, 6, 10, 13]

$$\dot{u} = \begin{cases} -u, & |u| > 1, \\ -\alpha_M \operatorname{sign} \sigma, & \sigma\dot{\sigma} > 0, |u| \leq 1, \\ -\alpha_m \operatorname{sign} \sigma, & \sigma\dot{\sigma} \leq 0, |u| \leq 1, \end{cases} \tag{4}$$

where $\alpha_M > \alpha_m > 0$, $\alpha_m > 4K_M/\sigma_0$, $\alpha_m > C_0/K_m$, $K_m\alpha_M - C_0 > K_M\alpha_m + C_0$ (these conditions will always be satisfied from now on). Any admissible value of u may be taken here as an initial value. Trajectories of algorithm (4) twist around the second order sliding manifold and converge to it in finite time (see Appendices).

In the steady state the process is described by the zero dynamics [9] $\dot{x} = f(t, x, u_{\mathrm{eq}}(t, x))$. That means that 2-sliding mode may be used instead of the standard one $u = -\operatorname{sign} \sigma$ without any change in the ideal behavior of the system.

The exact value of the derivative is not available in practice. Instead of $\dot{\sigma}$ a first difference $\Delta\sigma_i$ may be used.

Let $t \in [t_i, t_{i+1})$, $t_{i+1} - t_i = \tau$, $\Delta\sigma_i = \sigma(t_i) - \sigma(t_{i-1})$ and

$$\dot{u} = \begin{cases} -u(t_i), & |u(t_i)| > 1, \\ -\alpha_M \operatorname{sign} \sigma(t_i), & \sigma(t_i)\Delta\sigma_i > 0, |u(t_i)| \leq 1, \\ -\alpha_m \operatorname{sign} \sigma(t_i), & \sigma(t_i)\Delta\sigma_i \leq 0, |u(t_i)| \leq 1. \end{cases} \tag{5}$$

Theorem 1. [6, 10] Let τ be sufficiently small, then after a finite-time transient process algorithm (5) guarantees sliding accuracy $|\sigma| < a_1\tau^2$, $|\dot{\sigma}| < a_2\tau$ for some $a_1, a_2 > 0$.

In comparison, the standard 1-sliding algorithm guarantees only the inequalities of the form $|\sigma| < a_1\tau$, $|\dot{\sigma}| < a_2$.

Let $\delta > 0$ be the maximum of the possible error in the measurements of σ. It may be shown that, with τ fixed and δ sufficiently small, measurement errors do not interfere with the algorithm performance. But the sliding accuracy deteriorates when τ decreases and takes on values $\tau < \frac{1}{2}\delta/K_M$. It happens because $|\dot{\sigma}| < K_M|u - u_{\mathrm{eq}}| < 2K_M$, and the measurement error is certain to exceed the increment of σ. The problem is aggravated in case δ cannot be estimated. A typical dependence of the sliding error on δ is shown qualitatively in Figure 1.

To overcome the problem, introduce the following measurement step feedback:

$$\tau = t_{i+1} - t_i = \begin{cases} \tau_M, & \lambda\,|\sigma(t_i)|^{\,\rho} > \tau_M, \\ \lambda\,|\sigma(t_i)|^{\,\rho}, & \tau_m \leq \lambda\,|\sigma(t_i)|^{\,\rho} \leq \tau_M, \\ \tau_m, & \lambda\,|\sigma(t_i)|^{\,\rho} < \tau_m, \end{cases} \tag{6}$$

where $0 < \tau_m < \tau_M$, $\lambda > 0$, $\rho > 0$.

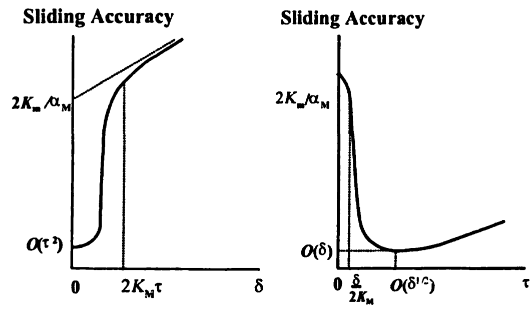

Fig. 1. Sliding accuracy of controller (5) with measurement errors.

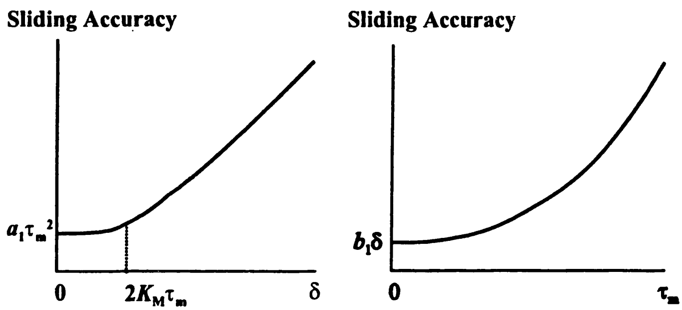

Fig. 2. Accuracy of controller (5) (6) with $\rho = 0.5$.

Theorem 2. Let constant parameters τ_M, λ be sufficiently small, $\lambda > 1/2$. Then, after a finite-time transient process, algorithm (5), (6) guarantees, for $\delta < \delta_0 < \sigma_0$, and τ_m sufficiently small, the following:

$$|\sigma| \leq \max\left(a_1 \tau_m^2, b_1 \delta^{2/(2\rho+1)}\right), \quad |\dot{\sigma}| \leq \max\left(a_2 \tau_m, b_2 \delta^{1/(2\rho+1)}\right), \tag{7}$$

where a_1, a_2, b_1, b_2 are some positive constants dependent on ρ, λ.

Theorem 2 means that algorithm (5), (6) is a second order real sliding algorithm [10] which is robust with respect to measurements errors. The new typical dependence of the sliding error on δ is shown qualitatively in Figure 2. Note that this algorithm does not need any evaluation of the measurement errors.

Having substituted $\tau_m = 0$ into (7), receive some ideal dependence on δ, which is shown qualitatively for different ρ in Figure 3. With $\rho < 1/2$ algorithm (5), (6) does not guarantee $\sup|\sigma| \to 0$ with $\tau_m \to 0$ even when $\delta \to 0$. Whereas the best choice of ρ is obviously $\rho = 0.5$, the proper choice of λ is certainly a subject for some optimization problem. Naturally, the algorithm may be simplified when δ is given a priori. In that case τ may be chosen as a function of δ.

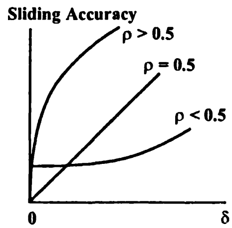

Fig. 3. Accuracy of controller (5), (6) for different $\rho > 0$.

Theorem 3. Let $\tau = \lambda_0 \delta^{1/2}$, $\lambda_0 > 0$, and α_M/α_m be sufficiently large, then algorithm (5) guarantees after a finite-time transient process that $|\sigma| < a_1\delta$, $|\dot{\sigma}| < a_2\delta^{1/2}$ for some positive constants a_1, a_2.

Twisting algorithm in systems with relative degree 2. There are two ways to provide for $\sigma = 0$ by means of the twisting algorithm when the system has relative degree 2 with respect to σ. The latter means that $\sigma'_u = \dot{\sigma}'_u = 0$, and $\ddot{\sigma}'_u > 0$ for definiteness. One way is to keep some auxiliary constraint like $\sigma + \dot{\sigma} = 0$ in the second order sliding, providing, thus, for keeping $\sigma = 0$ in an asymptotically stable 3-sliding mode. The other is to form a discontinuous control signal by means of a modified twisting algorithm

$$u = \begin{cases} -\alpha_M \operatorname{sign} \sigma(t_i), & \sigma(t_i)\Delta\sigma_i > 0; \\ -\alpha_m \operatorname{sign} \sigma(t_i), & \sigma(t_i)\Delta\sigma_i \leq 0; \end{cases}$$

where $\alpha_M > \alpha_m > 0$. The corresponding ideal sliding algorithm using values of $\dot{\sigma}$ is formed in an obvious way, also some formal statement of the problem similar to the one in Section 2 may be easily developed.

Weakening the smoothness conditions. The smoothness conditions on the functions f and σ may be significantly weakened [6]: only Lipschitzian property is required for f and partial derivatives of σ. It may be shown that in case a system is linearly dependent on control, $\Phi = (a_i|x_i| + h)$, $a_i > 0$, $h > 0$ may be used in (3) instead of a smooth Φ described in Section 2.

4. ILLUSTRATIVE EXAMPLE

Consider a simple example of robot manipulator control (Figure 4). Let a light hard rod be suspended by its end O and assume that it rotates around this end without any friction. All motions are restricted to some vertical plane. A load of known mass m is moving along the rod. Its distance from O equals $R(t)$ and is not measured. An engine is connected to the rod and transmits a torque v to it. Torque v is considered as control. The task is to track function x_c given in real time by the angular coordinate x of the rod.

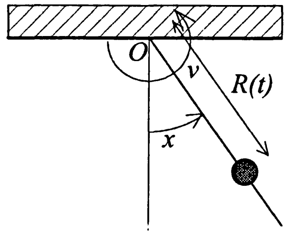

Fig. 4. Illustrative example.

It is easy to see that the system is described by the equation

$$\ddot{x} = -2\frac{\dot{R}}{R}\dot{x} - g\frac{1}{R}\sin x + \frac{1}{mR^2}v$$

where $g = 9.81$ is the gravitational constant. Suppose that $0 < R_m \leq R \leq R_M$, $\dot{R}$, $\ddot{R}$, $\dot{x}_c$ and $\ddot{x}_c$ are bounded, $x - x_c$ is available. In the following $\dot{x}$ - $\dot{x}_c$ is supposed to be measured, otherwise some special robust differentiators may be implemented [3, 11]. The corresponding constants determine a class of objects to be controlled by the algorithm under design.

All parameters of the algorithm may be evaluated in accordance with the above-mentioned constants restricting unknown functions $R(t)$ and $x_c(t)$ and their derivatives. Experience shows that the parameter values are usually excessively large in this case. The easiest way to find the parameters is to tune the parameters during simulation. Of course, the controlled class may occur to be some-what smaller in that case, but it will still allow significant disturbances of the considered realizations of R and x_c. It was taken for simulation that $m = 1$, and

$$R = 1 + 0.25\sin 4t + 0.5\cos t, \qquad x_c = 0.08\sin t + 0.12\cos 0.3t.$$

A new control u is introduced

$$v = 30(1 + |x| + |\dot{x}|)\,u, \qquad \sigma = [(\dot{x} - \dot{x}_c) + 2(x - x_c)]/(1 + |x| + |\dot{x}|),$$

$$\dot{u}(t) = \begin{cases} -\,u(t_i), |u(t_i)| > 1, \\ -\,7\mathrm{sign}\,\sigma(t_i), \sigma(t_i)\Delta\sigma_i > 0, |u(t_i)| \leq 1, \\ -\mathrm{sign}\,\sigma(t_i), \sigma(t_i)\Delta\sigma_i \leq 0, |u(t_i)| \leq 1. \end{cases}$$

$$t_{i+1} - t_i = \begin{cases} 0.02,\ 0.015\,|\sigma(t_i)|^{0.5} > 0.02, \\ 0.015\,|\sigma(t_i)|^{0.5},\ \tau_m \leq 0.015\,|\sigma(t_i)|^{0.5} \leq 0.02, \\ \tau_m,\ 0.015\,|\sigma(t_i)|^{0.5} < \tau_m, \end{cases}$$

where τ_m is $2{\cdot}10^{-4}$ and less. The initial conditions $x(0) = 0$, $x_c(0) = 3$, $u(0) = 0$ were taken.

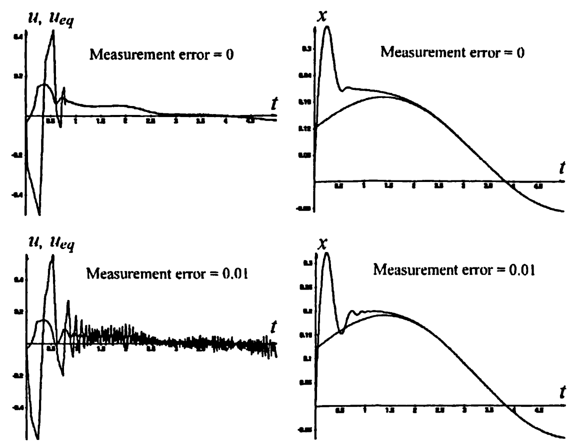

Fig. 5. Algorithm performance with $\delta = 0$ and $\delta = 0.01$.

The tracking precision $|x - x_c| \leq 5.7 \cdot 10^{-5}$ and the sliding accuracy $|\sigma| \leq 5.7 \cdot 10^{-5}$ were achieved with $\tau_m = 2 \cdot 10^{-4}$. τ_m having been changing from $2 \cdot 10^{-4}$ to $2 \cdot 10^{-5}$ and $2 \cdot 10^{-6}$, the sliding accuracy changed from $7.08 \cdot 10^{-6}$ to $7.52 \cdot 10^{-8}$ and $7.51 \cdot 10^{-10}$ respectively.

It follows from the simulation data that in the steady state $\sup |\sigma|$ is proportional to δ with a coefficient close to $2 - 2.5$. For example, for $\delta = 0.05$: $\sup |\sigma| = 0.12$, for $\delta = 0.01$: $\sup |\sigma| = 0.024$, for $\delta = 0.001$: $\sup |\sigma| = 0.0025$, for $\delta = 0.0005$: $\sup |\sigma| = 0.00088$. Functions $u(t)$, $u_{\text{eq}}(t)$ and $x(t)$, $x_c(t)$ for the measurement errors $\delta = 0$, $\delta = 0.01$ are shown in Figure 5. It has to be mentioned that with $\tau = \text{const} = 2 \cdot 10^{-4}$ a system failure happens already with $\delta = 0.003$. The smaller constant τ, the smaller critical δ in that case (Figure 1).

The algorithm considered is a second order real sliding algorithm with respect to the constraint function σ. It also provides for the second order precision of tracking in the steady mode, but it does not satisfy the definition of a second order real sliding algorithm with respect to the constraint function $\sigma' = x - x_c$, for its convergence time tends to infinity when algorithm parameter $\tau_m \to 0$ (its convergence is exponential-like).

5. CONCLUSIONS AND REMARKS

The measurement step feedback principle was proposed and was shown to make the twisting algorithm insensitive with respect to measurement errors. That method may be applied to any real-time sliding controllers using first differences [1, 2, 5, 6, 10, 12], providing the necessary basis for their practical applications.

Like its predecessor, the achieved modified twisting algorithm provides under uncertainty conditions, the second order of sliding accuracy with respect to the measurement step in the absence of measurement errors. Asymptotic estimations of the sliding accuracy having been obtained, the best choice of a key parameter is found.

The proposed algorithm may be successfully used in solving various tracking problems and problems of VSS theory. It features the main advantages of the standard sliding mode control and at the same time precludes discontinuity of control. It also provides for higher accuracy of constraint fulfillment with small measurement errors. However, with significant measurement errors that new sliding algorithm may prove to be less precise than the standard 1-sliding mode.

6. APPENDICES

6.1. Auxiliary notions

Introduce some notions and reasoning useful for further consideration. Let Γ be a segment of a piece-wise smooth curve lying in the plane σ, $\dot\sigma$, its ends being the only intersections with the axis $\sigma = 0$. We call it a majorant curve for differential inclusion $\frac{\mathrm{d}}{\mathrm{d}t}(\sigma, \dot\sigma) \in F(\sigma, \dot\sigma)$, if no phase trajectory of the inclusion may leave the compact part of the plane bounded by Γ and the axis without intersecting the axis.

Assume now that the half-plane $\sigma \geq 0$ is partitioned into open sets O_i by a finite number of smooth curves γ_j, including the ray $\dot\sigma = 0$, $\sigma \geq 0$ and the line $\sigma = 0$. Let the constants $R_{Mi} > 0$, $R_{mi} < R_{Mi}$ be juxtaposed with every O_i, $K > 0$. Consider a differential inclusion

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sigma, \dot\sigma) \in F(\sigma, \dot\sigma) = \left(K\dot\sigma, \left[\min_{j:(\sigma,\dot\sigma)\in O_j} R_{mj}, \max_{j:(\sigma,\dot\sigma)\in O_j} R_{Mj}\right]\right). \tag{8}$$

In all cases of further consideration phase trajectories of the vector field

$$M_F(\sigma, \dot\sigma) = (K\dot\sigma, -M_R), \qquad M_R = \begin{cases} R_{mi}, \dot\sigma > 0, \\ R_{Mi}, \dot\sigma \leq 0, \end{cases} \quad (\sigma, \dot\sigma) \in O_i \tag{9}$$

constitute majorants of the differential inclusion (8) in the half-plane $\sigma \geq 0$. That reasoning may obviously be transferred to the case $\sigma \leq 0$.

Let G_η be an operator constituted by a combination of the linear coordinate transformation $g_\eta : (\sigma, \dot\sigma) \longmapsto (\eta^2 s, \eta\dot\sigma)$, $\eta > 0$ and the time transformation $t \longmapsto \eta t$. Operator G_η performs transformation of any inclusion of the form $\ddot\sigma \in Q(\sigma, \dot\sigma)$ into the inclusion $\ddot\sigma \in Q(\eta^{-2}\sigma, \eta^{-1}\dot\sigma)$.

6.2. Trajectories of the twisting algorithm

It is easy to demonstrate that, with τ sufficiently small, every trajectory of system (2), (5) reaches the constraint $\sigma = 0$ in finite time with $|u| \leq 1 + \alpha_M\tau$. After that the system state will stay in the region $|\sigma| \leq \sigma_0$, $|u| \leq 1 + \alpha_M\tau$ forever. For the ideal twisting algorithm (4) the region $|\sigma| \leq \sigma_0$, $|u| \leq 1$ will be attractive.

Consider the performance of the ideal twisting algorithm (4) in the region $|\sigma| \leq \sigma_0$, $|u| \leq 1$. Calculate, according to Section 3,

$$\ddot\sigma \in [-C_0, C_0] + [K_m, K_M]\dot u. \tag{10}$$

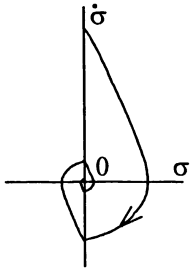

Fig. 6. Twisting algorithm trajectories.

The operations on the sets are understood here in a natural way. Assume now that at the initial time $\sigma = 0$, $|u| \leq 1$. It is easy to see that with $\sigma \neq 0$, $|\sigma| \leq \sigma_0$, the equality $u = 1$ may be reached only with $\dot{\sigma}\,\mathrm{sign}\,\sigma \leq -K_m(1 - u_0)$. Therefore

$$\ddot{\sigma} \in \begin{cases} -[K_m\alpha_M - C_0, K_M\alpha_M + C_0]\,\mathrm{sign}\,\sigma, & \sigma\dot{\sigma} \geq 0, \\ -[K_m\alpha_m - C_0, K_M\alpha_m + C_0]\,\mathrm{sign}\,\sigma, & -K_m(1 - u_0) \leq \dot{\sigma}\,\mathrm{sign}\,\sigma < 0, \\ -[K_m - C_0, K_M\alpha_m + C_0]\,\mathrm{sign}\,\sigma, & \dot{\sigma}\,\mathrm{sign}\,\sigma < -K_m(1 - u_0). \end{cases} \tag{11}$$

According to the reasonings of 6.1., the majorant of inclusion (11) may be determined as a continuous curve consisting of the curves $|\sigma| + 0.5\dot{\sigma}^2/(K_m\alpha_M - C_0\,\mathrm{sign}\,\sigma) = \mathrm{const}$. Continuing the majorant from one half-plane to another a twisting curve, shown in Figure 6, is achieved. Any real trajectory of the system will inevitably twist “inside“ such a majorant curve. Designate by $\dot{\sigma}_0$, $\dot{\sigma}_1$, $\dot{\sigma}_2$, ...the points of the majorant intersections with the axis $\sigma = 0$. It is easy to see that

$$\left|\frac{\dot{\sigma}_{i+1}}{\dot{\sigma}_i}\right| = \sqrt{\frac{K_M\alpha_m + C_0}{K_m\alpha_M - C_0}} < 1.$$

The convergence time is estimated by the sum $\sum|\dot{\sigma}_i|/(K_m\alpha_m - C_0)$, which is bounded. Details may be found in [6, 13].

Now consider algorithm (5) of real sliding. The movement is now described by inclusion (10), where

$$\dot{u} = \begin{cases} -\alpha_M\,\mathrm{sign}\,\sigma(t_i), & \sigma(t_i)\,\Delta\sigma_i > 0, \\ -\alpha_m\,\mathrm{sign}\,\sigma(t_i), & \sigma(t_i)\,\Delta\sigma_i \leq 0,\ \dot{\sigma}\,\mathrm{sign}\,\sigma > -K_m(1 - u_0 - \alpha_M\tau_M), \\ -[1-\alpha_M\tau_M, \alpha_m]\,\mathrm{sign}\,\sigma(t_i), & \sigma(t_i)\,\Delta\sigma_i \leq 0,\ \dot{\sigma}\,\mathrm{sign}\,\sigma \leq -K_m(1-u_0-\alpha_M\tau_M). \end{cases}$$

Here $\tau_M > 0$ is some sufficiently small upper bound of τ. For sufficiently small τ all the trajectories after a finite time stay in the region $|\sigma| < \sigma_0$, $|\dot{\sigma}| < K_m(1 - u_0 - \alpha_M\tau_M)$. This may be shown by the majorant technique. After that $|u| < 1$ and

$$\dot{u} = \begin{cases} -\alpha_M\,\mathrm{sign}\,\sigma(t_i), & \sigma(t_i)\Delta\sigma_i > 0, \\ -\alpha_m\,\mathrm{sign}\,\sigma(t_i), & \sigma(t_i)\Delta\sigma_i \leq 0. \end{cases} \tag{12}$$

The motion is now described by the differential inclusion (10), (12). Fix a concrete small value $\tau = \tau_0$. There is a bounded set Ω_{τ_0} such that all trajectories of inclusion (10), (12) within finite time penetrate into Ω_{τ_0} to stay there. The corresponding sliding accuracy is given by $\sup\{\sigma|(\sigma,\dot\sigma) \in \Omega_{\tau_0}\}$ and $\sup\{\dot\sigma|(\sigma,\dot\sigma) \in \Omega_{\tau_0}\}$.

6.3. Plan of the proofs

All the Theorems are proved in the same way. Consider the proof of Theorem 3, which is the most difficult. Examine the trajectories of the controlled differential inclusion (10), (5), (6), $\rho \geq 0.5$. It is easy to show that with $\delta + K_M(\kappa+1)\tau_M + 0.5(K_M\alpha_M + C_0)\tau_M^2 < \sigma_0$ no trajectory leaves the region $|\sigma| < \sigma_0$ once the manifold $\sigma = 0$ has been reached.

Let us say that there is a *switching error* at the time $t \in [t_i, t_{i+1})$, where t_i, t_{i+1} are the measurement times, if $\operatorname{sign}(\underline{\sigma}(t_i) - \underline{\sigma}(t_{i-1})) \neq \operatorname{sign}\dot\sigma(t)$, or $\operatorname{sign}\underline{\sigma}(t_i) \neq \operatorname{sign}\sigma(t)$. Here $\underline{\sigma}(t_i)$, $\underline{\sigma}(t_{i-1})$ are the measurement results, $t_{i+1} - t_i = \tau(\underline{\sigma}(t_i))$. Denote by E_σ and $E_{\dot\sigma}$ some sets lying in the plane σ, $\dot\sigma$, and including all the points of possible error in $\operatorname{sign}\sigma$ and $\operatorname{sign}\dot\sigma$ correspondingly, $O_r = \{(\sigma,\dot\sigma)|\sigma^2 + \dot\sigma^2 < r\}$.

Here are the main stages of the proof of Theorem 2.

Lemma 4. For any $k > 1$ with λ, τ_m sufficiently small there exists $r = r(\delta, \tau_m)$, such that $r \to 0$ when $\delta \to 0$, $\tau_m \to 0$, and the sets E_σ, $E_{\dot\sigma}$ may be taken in the form

$$E_\sigma = \{(\sigma,\dot\sigma)|\dot\sigma^2/|\sigma| > k\} \cup O_r, E_{\dot\sigma} = \{(\sigma,\dot\sigma)|\dot\sigma^2/|\sigma| \notin [k^{-1}, k]\} \cup O_r. \tag{13}$$

Lemma 5. Let E_σ, $E_{\dot\sigma}$ be given by (13). Then with k sufficiently large any trajectory of (13), (5), (6) accesses O_r in finite time, which does not depend on r and after that it stays inside the ball O_{r_1}, where $r_1 = r_1(r) > r$, $r_1 \to 0$ when $r \to 0$.

Choose some $\eta = \eta(\delta, \tau_m)$ so that $G_\eta(O_r)$ be bounded and the diameter of $G_\eta(O_r)$ not tend to 0 when $\delta \to 0$, $\tau_m \to 0$. Apply operator G_η and consider the movement on the image plane. Any set of the form $\dot\sigma^2/|\sigma| \in \Omega$, $\Omega \subset I\!R$, is invariant with respect to the operator G_η for any η. According to Lemma 5 there is a ball O_R attracting the trajectories in finite time. Then after the inverse transformation $G_{\eta^{-1}}$ achieve that the sliding accuracy is given by the inequalities $|\sigma| < \eta^{-2}R$, $|\dot\sigma| < \eta^{-1}R$.

6.4. Proof of Lemma 5

Instead of the differential inclusion (10), (5), (6) consider a differential equation

$$\ddot\sigma = \begin{cases} (K_M\alpha_M + C_0)\operatorname{sign}\dot\sigma, & & (\sigma,\dot\sigma) \in E_{\dot\sigma}, \\ -(K_m\alpha_m - C_0)\operatorname{sign}\sigma, & \sigma\dot\sigma > 0, & (\sigma,\dot\sigma) \in E_{\dot\sigma} \setminus E_\sigma, \\ -(K_M\alpha_M + C_0)\operatorname{sign}\sigma, & \sigma\dot\sigma \leq 0, & (\sigma,\dot\sigma) \in E_{\dot\sigma} \setminus E_\sigma, \\ -(K_m\alpha_M - C_0)\operatorname{sign}\sigma, & \sigma\dot\sigma > 0, & (\sigma,\dot\sigma) \notin E_{\dot\sigma} \cup E_\sigma, \\ -(K_M\alpha_m + C_0)\operatorname{sign}\sigma, & \sigma\dot\sigma \leq 0, & (\sigma,\dot\sigma) \notin E_{\dot\sigma} \cup E_\sigma, \end{cases} \tag{14}$$

It is easy to see that the phase trajectories of (14) constitute majorant curves for inclusion (10), (5), (6) in the half-planes $\sigma > 0$, $\dot{\sigma} < 0$.

The trajectory of (14) has successive intersections $\dot{\sigma}_1, \dot{\sigma}_2, \dots$ with the axis $\sigma = 0$. The region $\dot{\sigma}^2/|\sigma| \in [k^{-1}, k]$ is invariant with respect to G_η. This means that the linear operator G_η transforms trajectories of (14) into trajectories of (14) for points of image and preimage being outside of O_r. Hence, the value of $|\dot{\sigma}_{i+1}/\dot{\sigma}_i|$ is constant outside the ball O_r. Obviously, for k sufficiently large, $|\dot{\sigma}_{i+1}/\dot{\sigma}_i| < 1$, and that proves the Lemma. □

6.5. Proof of Lemma 4

Lemma 6. Lemma 4 is true with respect to E_σ.

Any error in $\operatorname{sign}\sigma$ may occur in the area $|\sigma| \leq \delta$ and at the points which may be accessed from this area with not more than one measurement in the area (the point is accessed without measurement if the last measurement was performed before the trajectory entered the area).

Lemma 6 follows from a number of simple propositions. Consider a differential inclusion

$$\ddot{\sigma} \in [A_m, A_M], \tag{15}$$

and assume that the region of admissible points is bounded: $|\sigma| \leq \sigma_M$, $|\dot{\sigma}| \leq \dot{\sigma}_M$. Let $(\sigma(t), \dot{\sigma}(t))$ be a trajectory of (15), $(t_i, \sigma_i, \dot{\sigma}_i)$, $i = 1, 2, 3$ be the times and the coordinates of the switching points, $\tau_i = t_{i+1} - t_i > 0$.

Proposition 7. Let $\tau_1 = \lambda(|\sigma_1| + \delta)^\rho$, $A_m > 0$, $\dot{\sigma}_1 \geq 0$, $\sigma_1 < 0$, then

1. with $\rho \geq 1$, λ sufficiently small: $\sup\{\sigma_2 | \sigma_1 < 0\} \leq \delta$;
2. with $0.5 \leq \rho < 1$, λ sufficiently small, $\sigma_1 < 0$: $\sigma_2 > \delta$, $\dot{\sigma}_2 \geq (1-\rho)^{1-\rho}(2\rho)^\rho \lambda^{-1}(\sigma_2 - \delta)^{1-\rho}$.

Proposition 8. Let $\tau_1 = \tau(\underline{\sigma}_1)$, $|\underline{\sigma}_1 - \sigma_1| \leq \delta$, τ be given by (11), $\lambda\delta^\rho < \tau_M$, $A_m > 0$, $\dot{\sigma}_1 > 0$, $\sigma_1 \in [-\delta, \delta]$. Then $\sigma_2 \leq \delta + \dot{\sigma}_2 \tau(2\delta) + 0.5 A_M (\tau(2\delta))^2$.

Proposition 9. Let $\tau_1 = \tau_2 = \tau_m$, $\dot{\sigma}_1 \geq 0$. Then

$$|\sigma_3 - \sigma_1| \leq 2\dot{\sigma}_1 \tau_m + 2A_M \tau_m^2, \; |\dot{\sigma}_3 - \dot{\sigma}_1| \leq 2A_M \tau_m.$$

The Propositions are proved by successive use of the trivial inequalities

$$\dot{\sigma}_1 \tau_1 + 0.5 A_m \tau_1^2 \leq \sigma_2 - \sigma_1 \leq \dot{\sigma}_1 \tau_1 + 0.5 A_M \tau_1^2, \tag{16}$$

$$A_m \tau_1 \leq \dot{\sigma}_2 - \dot{\sigma}_1 \leq A_M \tau_1. \tag{17}$$

In the second statement of Proposition 7 A_M is excluded by the simple reasoning that with λ sufficiently small, $\rho \geq 0.5$ and σ bounded $|\sigma| + \delta \geq A_M \lambda^2 (|\sigma| + \delta)^{2\rho}$. □

Lemma 10. Lemma 4 is true with respect to $E_{\dot\sigma}$.

It is easy to see that any error in the sign of $\dot\sigma_1$ may occur only within the area $|\dot\sigma_1| \leq 2\delta/\min\{\tau(\zeta)|\zeta \in [\sigma-\delta, \sigma+\delta]\}$ and at the points which may be accessed from that area with not more than two measurements. The following Propositions are needed here.

Proposition 11. Let $|\sigma_1| > \delta$, $\tau_i = \lambda(|\sigma_i| + \delta)^\rho$, $i = 1, 2$, then for any $k > 0$ the increments of σ and $\dot\sigma$ satisfy

$$|\sigma_3 - \sigma_1| \leq \lambda\xi_1(k)\,|\dot\sigma_1|^{2\rho+1}, \quad |\dot\sigma_3 - \dot\sigma_1| \leq \lambda\xi_2(k)\,|\dot\sigma_1|^{2\rho} \quad \text{with } \frac{\dot\sigma_1^2}{|\sigma_1|} \geq k,$$

or

$$|\sigma_3 - \sigma_1| \leq \lambda\zeta_1(k)\,|\sigma_1|^{(2\rho+1)/2}, \quad |\dot\sigma_3 - \dot\sigma_1| \leq \lambda\zeta_2(k)\,|\sigma_1|^{\rho} \quad \text{with } \frac{\dot\sigma_1^2}{|\sigma_1|} \leq k^{-1}.$$

Here ξ_1, ξ_2, ζ_1, ζ_2 are some decreasing positive functions of k.

This Proposition is a result of routine successive calculation with usage of (16), (17). The next Proposition is a direct consequence of the previous one.

Proposition 12. With sufficiently small λ under the conditions of Proposition 11 the following inequalities are satisfied:

1. with $\frac{\dot\sigma_1^2}{|\sigma_1|} \geq k > 0$ $\quad \frac{\dot\sigma_3^2}{|\sigma_3|} \geq \min\left(k\frac{(1-\lambda D_1)^2}{2}, \frac{(1-\lambda D_1)^2}{2\lambda D_2}\right)$,
2. with $\frac{\dot\sigma_1^2}{|\sigma_1|} \leq k^{-1}$ $\quad \frac{\dot\sigma_3^2}{|\sigma_3|} \leq \max\left(k^{-1}\frac{4}{1-\lambda E_1}, \frac{4\lambda^2}{1-\lambda E_1}\right)$.

Here $D_1 = \xi_1\dot\sigma_M^{2\rho-1}$, $D_2 = \xi_2\dot\sigma_M^{2\rho-1}$, $E_1 = \zeta_1\sigma_M^{2\rho-1}$.

The same propositions are true with increments $\sigma_2 - \sigma_1$, $\dot\sigma_2 - \dot\sigma_1$. It follows from Propositions 9, 12 that for any region $\Omega = \{(\sigma, \dot\sigma)|\dot\sigma^2/|\sigma| \in [k^{-1}, k]\} \cup O_r$ the region of the points accessible from Ω with not more than 2 measurements is included into another set $\Omega_1 = \{(\sigma, \dot\sigma)|\dot\sigma 2/|\sigma|[k_1^{-1}, k_1]\} \cup O_{r_1}$ where $k_1 \to \infty$, and $r_1 \to 0$ while $k \to \infty$, $r \to 0$, $\lambda \to 0$, $\tau_m \to 0$, $\delta \to 0$. Lemma 10 follows now from Propositions 7 to 9 and from the remark at the beginning of Lemma 10 proof. □

6.6. Completion of Theorem 2 proof

Substantiate the last stage of the plan 6.3.. According to Lemma 4, the error region consists of the set $\dot\sigma^2/|\sigma| \notin [k^{-1}, k]$ which is invariant with respect to operator G_η, and of a bounded set included into O_r. In agreement with 6.4, the latter set may be represented as a union of specific sets. Calculate the images of these sets after the transformation G_η. Operator G_η transforms any set given by an inequality $P(\sigma, \dot\sigma) < 0$ into the set $P(\eta^{-2}\sigma, \eta^{-1}\dot\sigma) < 0$.

The sets are the intersections of the set $\dot\sigma^2/|\sigma| \in [k^{-1}, k]$ with the following sets.

1. $|\sigma| \le \delta$,
2. $|\dot\sigma| < 2\delta/\lambda/(|\sigma| - \delta)^\rho \& |\sigma| > \delta$,
3. the set corresponding to Proposition 7:
   $|\ \dot\sigma\ | \ge (1-\rho)^{1-\rho}(2\rho)\rho\lambda^{-1}(|\sigma| - \delta)^{1-\rho}$, $|\sigma| > \delta$ with $\rho < 1$,
4. the set $|\sigma| \le \delta + |\dot\sigma|\tau(2\delta) + 0.5A_M\tau(2\delta)^2$ corresponding to Proposition 8;
5. the set family of the points accessible from the measurement point $(\sigma_1, \dot\sigma_1)$ with not more than one measurement on the way (Propositions 9, 11 in accordance with formula for τ) where $(\sigma_1, \dot\sigma_1)$ takes on values in the sets 1)−4). The set corresponding to Proposition 11 is

5a. $|\sigma - \sigma_1| \le \lambda\xi_1(k^{-1})\,|\dot\sigma_1|^{2\rho+1}$, $|\dot\sigma - \dot\sigma_1| \le \lambda\xi_2(k^{-1})\,|\dot\sigma_1|^{2\rho}$.

The set corresponding to Proposition 9 is

5b. $|\sigma - \sigma_1| \le 2\dot\sigma_1\tau_m + 2A_M\tau_m^2$, $|\dot\sigma - \dot\sigma_1| \le 2A_M\tau_m$.

After the transformation G_η the set $\dot\sigma^2/|\sigma| \in [k^{-1}, k]$ does not change. Set 1) transfers to $|\sigma| \le \eta^2\delta$. Set 2) transfers to

$$|\dot\sigma| < 2\eta^{2\rho-1}\delta/\lambda/(|\sigma| - \eta^2\delta)^\rho,\ |\sigma| > \eta^2\delta.$$

Set 3) transfers to

$$|\dot\sigma| \ge \eta^{2\rho-1}(1-\rho)^{1-\rho}(2\rho)^\rho\lambda^{-1}(|\sigma| - \eta^2\delta)^{1-\rho},\ |\sigma| > \eta^2\delta \text{ with } 0.5 \le \rho < 1.$$

Set 4) transfers to

$$|\sigma| \le \eta^2\delta + \eta|\dot\sigma|\tau(2\delta) + 0.5\eta^2A_M\tau(2\delta)^2.$$

Set 5a) transfers to

$$|\sigma - \sigma_1| \le \lambda\xi_1(k^{-1})\eta^{1-2\rho}\,|\dot\sigma_1|^{2\rho+1},\ |\dot\sigma - \dot\sigma_1| \le \lambda\xi_2(k^{-1})\eta^{1-2\rho}\,|\dot\sigma_1|^{2\rho},$$

set 5b) transfers to

$$|\sigma - \sigma_1| \le 2\eta\dot\sigma_1\tau_m + 2\eta^2A_M\tau_m^2,\ |\dot\sigma - \dot\sigma_1| \le 2\eta A_M\tau_m,$$

where $(\sigma_1, \dot\sigma_1)$ belongs now to the images of sets 1) and 2).

Consider two cases: a) $\delta \ll \tau_m^{2\rho+1}$, $\eta = \tau_m^{-1}$; and b) $\tau_m \ll \delta^{1/(2\rho+1)}$, $\eta = \delta^{-1/(2\rho+1)}$. It is easy to see that in both cases the image of the set Ω is bounded with $\rho \ge 0.5$. Apply Lemma 5 with images of E_σ, $E_{\dot\sigma}$ substituted for E_σ, $E_{\dot\sigma}$, and receive that there is a bounded set attracting in finite time. After the inverse transformation the desired evaluation of sliding accuracy is achieved in both cases. To get the general estimation, it is sufficient now to utilize the fact that the values of $\sup|\sigma|$ and $\sup|\dot\sigma|$ in the steady mode are monotonously increasing functions of δ.

□

6.7. On the other proofs

Theorem 1 may be considered as a particular case of Theorem 2. Theorem 3 is proved in a very similar way by the transformation G_η with $\eta = \delta^{-1/2}$.

Remark. As follows from (16), (17) and the description of set 5a), with $0 < \rho < 0.5$ the error domain obtained according to the above reasoning, fills all the plane after transformation G_η, $\eta \to \infty$. Hence, the applied method does not work here. Simulation shows that in that case sliding accuracy does not tend to zero when $\delta \to 0$, $\tau_m \to 0$.

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