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Delano R. Carter; Armando A. Rodriguez

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WEIGHTED $\mathcal{H}^\infty$ MIXED–SENSITIVITY MINIMIZATION FOR STABLE DISTRIBUTED PARAMETER PLANTS UNDER SAMPLED–DATA CONTROL[1,2]

DELANO R. CARTER AND ARMANDO A. RODRIGUEZ

This paper considers the problem of designing near-optimal finite-dimensional controllers for stable multiple-input multiple-output (MIMO) distributed parameter plants under sampled-data control. A weighted $\mathcal{H}^\infty$-style mixed-sensitivity measure which penalizes the control is used to define the notion of optimality. Controllers are generated by solving a "natural" finite-dimensional sampled-data optimization. *A priori* computable conditions are given on the approximants such that the resulting finite-dimensional controllers stabilize the sampled-data controlled distributed parameter plant and are near-optimal. The proof relies on the fact that the control input is appropriately penalized in the optimization. This technique also assumes and exploits the fact that the plant can be approximated uniformly by finite-dimensional systems. Moreover, it is shown how the optimal performance may be estimated to any desired degree of accuracy by solving a single finite-dimensional problem using a suitable finite-dimensional approximant. The constructions given are simple. Finally, it should be noted that no infinite-dimensional spectral factorizations are required. In short, the paper provides a straight forward control design approach for a large class of MIMO distributed parameter systems under sampled-data control.

1. INTRODUCTION

The problem of designing finite-dimensional controllers for infinite-dimensional systems, i. e. systems described by partial differential equations or with continuous-time delays, has received considerable attention. Some researchers have addressed the problem in a purely continuous time framework [10, 12, 14, 23, 25, 26, 30, 44], and references therein, others in a purely discrete-time framework [15, 32]. With the recent advances in sampled-data controller synthesis [1, 4, 6, 22, 24, 34, 36, 38, 40, 41, 43], and references therein, the problem has also been posed in the hybrid time framework which is encountered when performing sampled-data control of infinite-dimensional systems [5, 27, 33].

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[2] A version of this paper was presented at the 5th Mediterranean Conference on Control and Systems held in Paphos (Cyprus) on June 21–23, 1997.

This research is motivated by the following practical design problem:

Controller Synthesis Problem: Synthesize a finite-dimensional discrete-time controller for a stable MIMO distributed parameter plant such that the resultant sampled-data system closed-loop performance metric is near optimal with intersample behavior included.

Suppose a distributed parameter process is given, with some performance criterion (say in $\mathcal{H}^\infty$). If a finite-dimensional sampled-data controller is desired, then one can follow either one of the following two approaches:

1. A *Design/Approximate (Direct)* approach in which a controller is designed using infinite-dimensional sampled-data techniques. If the resulting discrete-time controller is infinite-dimensional, a finite-dimensional approximation is obtained. This approach is addressed in [33] and [5]. This approach will not be considered in this paper.
2. An *Approximate/Design (Indirect)* approach in which the plant is first approximated by a finite-dimensional model and then a finite-dimensional sampled-data controller is designed based on this model. This is the typical engineering approach. However, this approach generally comes with no *performance guarantees.* The key difficulty derives from discontinuity with respect to plant perturbations in the performance measure, even when the uniform topology is imposed.

The above problem naturally leads to studying the problem of designing finite-dimensional sampled-data controllers, for distributed parameter plants, that deliver near-optimal performance measured in $\mathcal{H}^\infty$ when the controllers are based on some continuous-time finite-dimensional plant approximation. The main objective then becomes to provide:

— *A priori* computable conditions on the approximants.

— A design method, based on the approximants, that delivers near-optimal performance.

In this paper, these objectives are achieved for stable MIMO distributed plants subject to an $\mathcal{H}^\infty$ mixed-sensitivity performance measure.

A rigorous treatment of the *Approximate/Design* approach is presented for a weighted $\mathcal{H}^\infty$ mixed-sensitivity performance criterion in which the control is penalized. The theory allows a large class of MIMO distributed parameter plants to be considered, including, for example a MIMO version of the Callier–Desoer class [3]. The problem solution shows that: (1) Given an "appropriate" finite-dimensional approximant for a MIMO distributed parameter plant, one can solve a "natural" finite-dimensional sampled-data problem in order to obtain a near-optimal finite-dimensional discrete-time controller. (2) The optimal performance can be estimated to a given tolerance by solving a single finite-dimensional sampled-data optimization based on an *a priori* determined finite-dimensional plant approximant.

By directly addressing *performance based approximation*, this study will hopefully shed light on the limitations of certain performance measures when only partial information is known about the plant.

The remainder of this paper is organized as follows. Section 2 contains notation and mathematical preliminaries. Section 3 contains a precise statement of two fundamental problems to be addressed in this paper. Section 4 presents the solution to the $\mathcal{H}^\infty$ mixed-sensitivity problems defined in Section 3. Section 5 presents a numerical application of the methodology outlined in Section 4. Finally, Section 6 summarizes the paper and presents directions for future research.

2. NOTATION AND MATHEMATICAL PRELIMINARIES

This section will establish notation and results required throughout the paper. Our primary references are [6], [39] and [9].

$\boldsymbol{C}$, $I\!R$ and $\boldsymbol{Z}$	Complex, real, and integer numbers, respectively.
$\boldsymbol{C}_+, \bar{\boldsymbol{C}}_+$ and $jI\!R$	Open, closed right half complex plane, imaginary axis.
$I\!D$ and $\bar{I\!D}$	Open and closed unit disc in complex plane.
$I\!R_+$ and $\boldsymbol{Z}_+$	Non-negative real and integer numbers.
$\sigma_{\max}(M)$	Maximum singular value of the matrix M.
$\mathcal{L}^2_m \stackrel{\text{def}}{=} \mathcal{L}^2(I\!R_+, \boldsymbol{C}_m)$	Lebesgue space of square integrable m-dimensional functions with support on $I\!R_+$.
$\ell^2_m \stackrel{\text{def}}{=} \ell^2(\boldsymbol{Z}_+, \boldsymbol{C}_m)$	Lebesgue space of square summable m-dimensional sequences with support on $\boldsymbol{Z}_+$.
$\mathcal{H}^\infty(\boldsymbol{C}_+)$	Hardy space of matrix-valued functions which are analytic and essentially bounded in $I\!R_+$.
$\mathcal{H}^\infty(I\!D)$	Hardy space of matrix-valued functions which are analytic and essentially bounded in $I\!D$.
$\mathcal{A}_{I\!R}$	Subspace of $\mathcal{H}^\infty(\boldsymbol{C}_+)$ consisting of functions continuous on $\bar{\boldsymbol{C}}^+ \cup \{\infty\}$ with real coefficients.
$I\!R\mathcal{H}^\infty(\boldsymbol{C}_+)$	Subspace of $\mathcal{H}^\infty(\boldsymbol{C}_+)$ consisting of real-rational functions.
$I\!R\mathcal{H}^\infty(I\!D)$	Subspace of $\mathcal{H}^\infty(I\!D)$ consisting of real-rational functions.
$\mathcal{L}(\mathcal{H})$	Space of bounded linear operators on the Hilbert space $\mathcal{H}$.
$\mathcal{Z}^{-1}[\mathcal{H}^\infty(I\!D)]$	Set of causal, linear, shift-invariant operators on ℓ^2_m.
$\mathcal{K}^2_m \stackrel{\text{def}}{=} \mathcal{L}^2([0,h), \boldsymbol{C}_m)$	Lebesgue space of square integrable m-dimensional functions with support on $[0,h)$.
$\boldsymbol{\ell}^2_m \stackrel{\text{def}}{=} \ell^2(\boldsymbol{Z}_+, \mathcal{K}^2_m)$	Lebesgue space of $\mathcal{K}^2_m$-valued square summable sequences with support on $\boldsymbol{Z}_+$.
$\mathcal{H}^2 \stackrel{\text{def}}{=} \mathcal{H}^2(\boldsymbol{C}_+)$	Hardy space of functions which are Laplace transforms of functions.
$\boldsymbol{\mathcal{H}}^\infty \stackrel{\text{def}}{=} \mathcal{H}^\infty(I\!D, \boldsymbol{\mathcal{L}}(\mathcal{K}^2_m))$	Hardy space of $\boldsymbol{\mathcal{L}}(\mathcal{K}^2)$-valued functions which are analytic and essentially bounded in $I\!D$.
$\mathcal{A}$	Subspace of $\boldsymbol{\mathcal{H}}^\infty$ consisting of the functions continuous on the boundary of the unit disk.
$I\!R\mathcal{A}$	Subspace of $\mathcal{A}$ consisting of lifted $I\!R\mathcal{H}^\infty$ functions.
$\boldsymbol{\mathcal{H}}^2 \stackrel{\text{def}}{=} \mathcal{H}^2(I\!D, \mathcal{K}^2_m)$	Hardy space of functions which are Z-transforms of functions in $\ell^2(\boldsymbol{Z}_+, \mathcal{K}^2)$.

$\underline{\mathcal{H}}^{\infty}(\mathbb{D})$	Image of $\mathcal{H}^{\infty}(\mathbb{D})$ in $\boldsymbol{\mathcal{H}}^{\infty}$ under the mapping $\mathcal{Q}$.
$\underline{\mathbb{R}\mathcal{H}}^{\infty}(\mathbb{D})$	Image of $\mathbb{R}\mathcal{H}^{\infty}(\mathbb{D})$ in $\underline{\mathcal{H}}^{\infty}(\mathbb{D})$ under the mapping $\mathcal{Q}$.
g, G	Signals and operators represented by lowercase and uppercase variables, respectively.
$\underline{(\cdot)}$	Lifting of signal or system.
$\widehat{(\cdot)}$	Laplace or Z-transform of $(\cdot)$.
$\mathcal{X}_{[0,\Omega]}$	Characteristic function of the interval $[0,\Omega]$ where $\Omega \in \mathbb{R}_{+}$; i.e. $\mathcal{X}_{\Omega}(j\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & \omega \in [0,\Omega]; \\ 0 & \text{elsewhere.} \end{cases}$
$\lfloor x \rfloor$	Floor of $x \in \mathbb{R}_{+}$; i.e. $\lfloor x \rfloor \stackrel{\text{def}}{=} k, \quad k \in \mathbb{Z}_{+}$ and $k \leq x < k+1$.

The standard function spaces listed above are endowed with their "natural" norms. The Hilbert space $\mathcal{K}_m^2$ inner product is $\langle \psi, \phi \rangle_{\mathcal{K}_m^2} \stackrel{\text{def}}{=} \int_0^h \langle \psi, \phi \rangle_2 \, dt$, where $\langle \cdot, \cdot \rangle_2$ is the Euclidean inner product on $\boldsymbol{C}_m$. The Hilbert space $\boldsymbol{\ell}_m^2$ inner product is $\langle \underline{u}, \underline{v} \rangle_{\boldsymbol{\ell}_m^2} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \langle u_k, v_k \rangle_{\mathcal{K}_m^2}$. The Banach algebra $\boldsymbol{\mathcal{H}}^{\infty}$ norm is $\|\widehat{G}\|_{\infty} \stackrel{\text{def}}{=} \sup_{|z|<1} \|\widehat{G}(z)\|_{\mathcal{K}^2 \to \mathcal{K}^2}$.

It is a fact that $\mathcal{H}^{\infty}$ and $\boldsymbol{\mathcal{H}}^{\infty}$ functions can be unitarily extended to have support almost everywhere on the imaginary axis [9]. When dealing with such functions, no distinction need be made between the function and its extension. Given this, the norms of such functions can be computed from their values on the imaginary axis. For $\widehat{\underline{F}} \in \boldsymbol{\mathcal{H}}^{\infty}$, for example, the norm becomes $\|\widehat{\underline{F}}\|_{\infty} = \operatorname{ess\,sup}_{\theta \in [-\pi,\pi]} \|\widehat{\underline{F}}(e^{j\theta})\|_{\mathcal{K}^2 \to \mathcal{K}^2}$, where ess sup denotes the essential supremum with respect to Lebesgue's measure [35]. The dependence on the dimension m of the above spaces will be suppressed in what follows unless the space dimension is of particular interest.

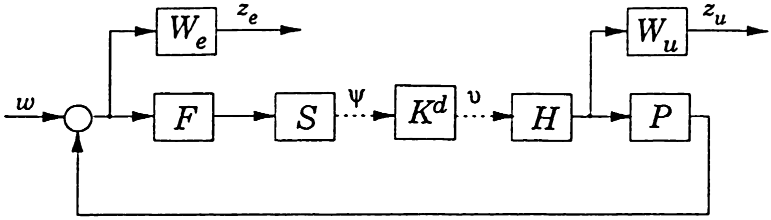

Fig. 1. Infinite-dimensional sampled-data feedback loop.

The sampled-data setting studied in this paper is depicted in Figure 1 where the solid lines represent continuous-time signals and the dashed lines represent discrete-time signals. The symbols used in this figure have the following interpretations:

F	strictly causal anti-aliasing filter in $\mathbb{R}\mathcal{H}^\infty(\boldsymbol{C}_+)$
P	MIMO stable infinite-dimensional plant
W_e	sensitivity weighting filter
W_u	control sensitivity weighting filter
K^d	MIMO discrete-time controller, possibly infinite-dimensional

Other definitions follow.

Definition 2.1. (Sample and Hold Operators.) Throughout the sequel, the *sample period* will be some positive real number and denoted by h. The sample S and hold H operators are defined respectively as $(Su)[k] \stackrel{\text{def}}{=} u(kh)$, $k \in \boldsymbol{Z}_+$ and $(He_d)(t) \stackrel{\text{def}}{=} e_d[k]$, $t \in [kh,(k+1)h)$ for each function $u : [0,\infty) \to \boldsymbol{C}_m$ and every sequence $e_d : \boldsymbol{Z}_+ \to \boldsymbol{C}_m$.

Comment 2.1. (Periodicity of Sampled–Data Systems.) The interconnection of continuous-time LTI operators and discrete-time LTI operators via h -synchronous sample and hold operators S and H results in an h-periodic operator.

Comment 2.2. (Anti–Aliasing Filter.) Throughout this paper F will denote an anti-aliasing filter. Moreover, it will be assumed that F is finite-dimensional, linear time-invariant, stable, and strictly causal. This assumption is made in order to ensure that $SF \in \mathcal{L}(\mathcal{L}^2, \ell^2)$ [6].

Definition 2.2. (Continuous–Time Lift Operator.) The continuous-time lift operator acts between the following spaces

$$L : \mathcal{L}^2 \to \boldsymbol{\ell}^2. \tag{1}$$

For every $u \in \mathcal{L}^2$, the lift operator produces

$$\underline{u} \stackrel{\text{def}}{=} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix} \stackrel{\text{def}}{=} Lu \in \boldsymbol{\ell}^2 \tag{2}$$

where

$$(u_k)(\tau) \stackrel{\text{def}}{=} u(\tau + kh) \quad \text{for } \tau \in [0,h), k \in \boldsymbol{Z}_+. \tag{3}$$

Note that each $u_k \in \mathcal{K}^2$. The lift operator has an inverse $L^{-1} : \boldsymbol{\ell}^2 \to \mathcal{L}^2$ defined as

$$u(t) \stackrel{\text{def}}{=} u_k(t - kh), \quad \text{for } t \in \mathbb{R}_+,\ k = \left\lfloor \frac{t}{h} \right\rfloor. \tag{4}$$

Proposition 2.1. (Lift Operator Isomorphism.) The continuous-time lift operator is an isomorphism between the spaces $\mathcal{L}^2$ and ℓ^2. It follows that if M is a bounded linear operator on $\mathcal{L}^2$, then

$$\underline{M} \stackrel{\text{def}}{=} LML^{-1}. \tag{5}$$

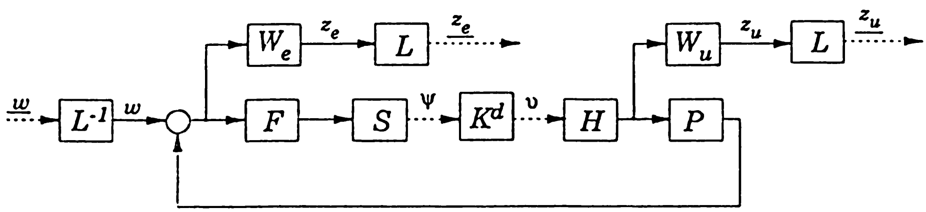

Fig. 2. Sampled-data infinite-dimensional plant lifting.

Proposition 2.2. (h-Periodic Operator Lifting to Time-Invariant Operator.) If M is an h-periodic operator on $\mathcal{L}^2$, the lifted operator $\underline{M}$ is a linear time-invariant operator on ℓ^2.

Proposition 2.2 follows from the fact that L intertwines the unilateral shift $\underline{U}$ on ℓ^2 and the delay operator D_h on $\mathcal{L}^2$ (i.e. $\underline{U}L = LD_h$). The lifted sampled-data system is depicted in Figure 2.

Proposition 2.3. (Rational Approximation.) The stable, proper, real rational transfer functions, $I\!R\mathcal{H}^\infty(\boldsymbol{C}_+)$, are dense in $\mathcal{A}_{I\!R}$. The subspace $\mathcal{A}_{I\!R}$ is precisely the set of $\mathcal{H}^\infty(\boldsymbol{C}_+)$ functions which are uniformly approximable by $I\!R\mathcal{H}^\infty(\boldsymbol{C}_+)$ functions.

Definiton 2.3. (The Gelfand or z-Transform.) The z-transform $\mathcal{Z} : \ell^2 \to \mathcal{H}^2$ is defined by $(\mathcal{Z}u)(z) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} u[k]z^k$, where $z \in I\!D$. The inverse mapping $\mathcal{Z}^{-1} : \mathcal{H}^2 \to \ell^2$ is well-defined.

Proposition 2.4. (Half-Plane Algebra Isometric Isomorphism.) For every $\widehat{P} \in \mathcal{A}_{I\!R}$, there exists a unique $\underline{\widehat{P}} \in \mathcal{A}$ such that $\mathcal{Z}^{-1}\Theta_{\underline{\widehat{P}}}\mathcal{Z} = \underline{P}$, where $\Theta_{\underline{\widehat{P}}}$ is the multiplication operator on $\mathcal{H}^2$ induced by $\underline{\widehat{P}}\check{\text{N}}$. Moreover, $\|\widehat{P}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} = \|\underline{\widehat{P}}\|_{\mathcal{H}^\infty}$.

Comment 2.3. (Stability.) Throughout the paper, the term stability or internal stability will be used to mean $\mathcal{L}^2$ finite-gain stability as defined in [6, pp. 247–257].

Proposition 2.5. (Stabilization.) Given $\widehat{P} \in \mathcal{H}^\infty(\boldsymbol{C}_+)$, a strictly causal $\widehat{F} \in I\!R\mathcal{H}^\infty(\boldsymbol{C}_+)$ and assuming nonpathological sampling [34], the set of proper discrete-time controllers which internally stabilize P in the sampled-data setting is [11, pp. 83–86]

$$S(P) \stackrel{\text{def}}{=} \{K^d(P, Q^d) \stackrel{\text{def}}{=} -Q^d(I - SFPHQ^d)^{-1} \mid Q^d \in \mathcal{Z}^{-1}[\mathcal{H}^\infty(I\!D)]\}. \tag{6}$$

Let

$$\widehat{\underline{K}} \stackrel{\text{def}}{=} LH\widehat{K}^d S\widehat{F}L^{-1}. \tag{7}$$

The set of lifted discrete-time controllers which internally stabilize $\widehat{\underline{P}} \in \mathcal{H}^\infty$ (cf. (5)) is

$$S(\widehat{\underline{P}}) \stackrel{\text{def}}{=} \{\widehat{\underline{K}}(\widehat{\underline{P}}, \widehat{\underline{Q}}) \stackrel{\text{def}}{=} -\widehat{\underline{Q}}(I - \widehat{\underline{P}}\widehat{\underline{Q}})^{-1} \mid \widehat{\underline{Q}}(Q^d) \in \underline{\mathcal{H}}^\infty(\mathbb{D})\} \tag{8}$$

where $\mathcal{Q} : \mathcal{H}^\infty(\mathbb{D}) \to \underline{\mathcal{H}}^\infty(\mathbb{D}) \stackrel{\text{def}}{=} LH\mathcal{H}^\infty(\mathbb{D})SFL^{-1} \subset \mathcal{H}^\infty$ and

$$\mathcal{Q}(\widehat{Q}^d) = LH\widehat{Q}^d S\widehat{F}L^{-1} \stackrel{\text{def}}{=} \widehat{\underline{Q}}(Q^d). \tag{9}$$

The pair $(P, HK^d(P_n, Q_n^d)SF)$ is stable if and only if the pair $(\widehat{\underline{P}}, \widehat{\underline{K}}(\widehat{\underline{P}_n}, \widehat{\underline{Q}_n}))$ is stable.

The utility of the above parameterization with respect to control law optimization is twofold. First, it provides a simple characterization of $S(\widehat{\underline{P}})$ - rather than optimizing over $\widehat{\underline{K}} \in S(\widehat{\underline{P}})$, one can optimize over $\widehat{\underline{Q}} \in \underline{\mathcal{H}}^\infty(\mathbb{D})$. Second, it permits one to transform optimization problems which depend in a linear fractional manner on $\widehat{\underline{K}}$, into convex optimization problems which depend affinely on $\widehat{\underline{Q}}$ [13], [42].

3. STATEMENT OF FUNDAMENTAL PROBLEMS

In this section two fundamental problems associated with the proposed *Approximate/Design* approach are precisely defined. Basic assumptions and definitions which will be used throughout the paper are now stated.

Throughout the paper, focus is placed exclusively on MIMO $\mathcal{L}^2$ finite-gain stable plants. Unstable MIMO plants will be treated in future work. Let $\widehat{P}(s) \in \mathcal{H}^\infty(\boldsymbol{C}_+)$ denote a stable MIMO transfer function matrix for a distributed parameter plant. Also, let $\{\widehat{P}_n(s)\}_{n=1}^{\infty} \subset \mathbb{R}\mathcal{H}^\infty(\boldsymbol{C}_+)$ denote a sequence of stable MIMO finite-dimensional approximants for $\widehat{P}$; the sense of which is a key issue and is to be made precise in subsequent sections. To do this, a performance measure is needed.

The weighted mixed-sensitivity performance measure is utilized which frequency weights the loop sensitivity and the control sensitivity. The closed loop *mixed sensitivity performance measure* associated with the periodically time-varying sampled-data system is defined as

Definition 3.2. (Mixed–Sensitivity.) Suppose $W_e, W_u, F, G, M \in \mathcal{L}(\mathcal{L}^2)$ and $V \in \mathcal{L}(\ell^2)$ are causal and LTI such that $HK^d(G,V)SF$ internally stabilizes M. The *mixed-sensitivity* of the pair $(M, HK^d(G,V)SF)$, denoted J_{mix}, is defined as the map $J_{\text{mix}}(\cdot,\cdot) : \mathcal{L}(\mathcal{L}^2) \times \mathcal{L}(\mathcal{L}^2) \times \mathcal{L}(\ell^2) \to \mathbb{R}_+$ where

$$\begin{aligned} & J_{\text{mix}}(M, K^d(G,V)) \\ \stackrel{\text{def}}{=} & \left\| \begin{pmatrix} W_e \\ W_u HK^d(G,V)SF \end{pmatrix} (I - MHK^d(G,V)SF)^{-1} \right\|_{\mathcal{L}^2 \to \mathcal{L}^2}. \end{aligned} \tag{10}$$

Given this, the *optimal performance* for the distributed parameter plant P with respect to the measure J_{mix} is defined as (see Figure 3).

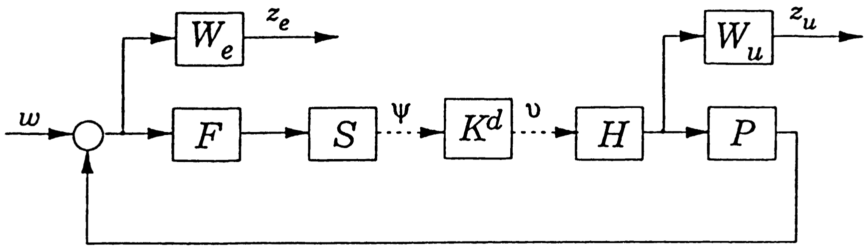

Fig. 3. Infinite-dimensional sampled-data feedback loop.

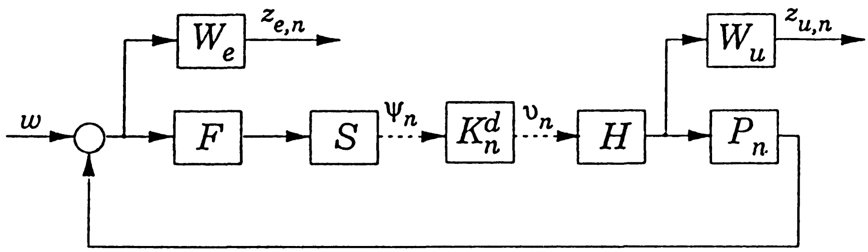

Fig. 4. Purely finite-dimensional sampled-data feedback loop.

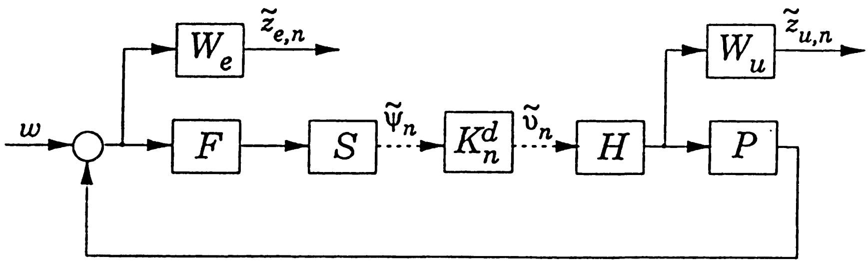

Fig. 5. Actual near-optimal sampled-data feedback loop.

Definition 3.2. (Optimal Performance.)

$$\mu_{\text{opt}} \stackrel{\text{def}}{=} \inf_{Q^d \in \mathcal{Z}^{-1}[\mathcal{H}^\infty(\mathbb{D})]} J_{\text{mix}}(P, K^d(P, Q^d)) \tag{11}$$

This optimization problem is the central problem being considered. Direct approaches for solving this problem have been proposed by various researchers, e. g. [5, 33]. The approach taken in this paper requires that P be approximated by a finite-dimensional system P_n. This motivates the following finite-dimensional optimization problem.

Definition 3.3. (Expected Performance.)

$$\mu_n \stackrel{\text{def}}{=} \inf_{Q^d \in \mathcal{Z}^{-1}[\mathbb{R}\mathcal{H}^\infty(\mathbb{D})]} J_{\text{mix}}(P_n, K^d(P_n, Q^d)). \tag{12}$$

In the context of this work, μ_n will be referred to as the *expected performance.* This terminology for μ_n is motivated by the fact that the numbers μ_n are typically used to guide engineers during the design process.

Let Q_n^d denote any optimal or near-optimal solution to the problem in Definition 3.3. By the parameterization given in Proposition 2.5, it follows that Q_n^d generates an internally stabilizing compensator K_n^d for P_n (see Figure 4). This compensator is given by:

$$K_n^d \stackrel{\text{def}}{=} K^d(P_n, Q_n^d) = -Q_n^d(I - SFP_nHQ_n^d)^{-1}. \tag{13}$$

Because in general, K_n^d may not be near-optimal with respect to μ_{opt} as defined in Definition 3.2, and in fact not even stabilizing for P, care must be taken. These issues motivate the following question: Under what conditions on the performance measure J_{mix} and the approximants $\{P_n(s)\}_{n=1}^\infty$, can one ensure that K_n^d generates a stabilizing sampled-data controller which delivers near-optimal performance for the MIMO distributed parameter plant P_n? This question leads one to naturally consider the feedback system obtained by substituting the finite-dimensional controller K_n^d into a closed loop sampled-data system with the distributed plant P (see Figure 5). Assuming that internal stability can be shown [6], this then motivates the following "natural" definition for the *actual performance.*

Definition 3.4. (Actual Performance.)

$$\tilde{\mu}_n \stackrel{\text{def}}{=} J_{\text{mix}}(P, K_n^d). \tag{14}$$

Given the above discussion, the *Approximate/Design Weighted $\mathcal{H}^\infty$ Mixed-Sensitivity Problem* is defined as follows.

Problem 3.1. (Approximate/Design.) Find conditions on the performance measure J_{mix} and the approximants $\{P_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \tilde{\mu}_n = \mu_{\text{opt}}$.

In practice, one would like to be able to compute μ_{opt} using finite-dimensional algorithms. With an ultimate intention of providing such algorithms, the following "purely" finite-dimensional problem is considered.

Problem 3.2. (Purely Finite-Dimensional.) Find conditions on the performance measure J_{mix} and the approximants $\{P_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \mu_n = \mu_{\text{opt}}$.
In the context of this work, this problem will be referred to as the *Purely Finite-Dimensional Weighted $\mathcal{H}^\infty$ Mixed-Sensitivity Problem.*

Solutions to Problems 3.1 – 3.2 will be presented in Section 4.

4. SOLUTION TO DISTRIBUTED PARAMETER SAMPLED-DATA CONTROL PROBLEMS

In this section, solutions are provided to the $\mathcal{H}^\infty$ *Approximate/Design Problem* and the $\mathcal{H}^\infty$ *Purely Finite-Dimensional Problem*. To address these problems, the following assumption on the weighting functions $\widehat{W}_e$ and $\widehat{W}_u$ will be made.

Assumption 4.1. (Mixed-Sensitivity Weightings.)

$$\widehat{W}_e, \widehat{W}_u, \widehat{W}_u^{-1} \in \mathbb{R}\mathcal{H}^\infty(\boldsymbol{C}_+).$$

The above implies that $\widehat{W}_e$ and $\widehat{W}_u$ are real-rational transfer function matrices with no poles in the extended closed right half plane. In addition, the filter $\widehat{W}_u$ has no zeros in the extended closed right half plane. In what follows, the invertibility of $\widehat{W}_u$ in $\mathcal{H}^\infty(\boldsymbol{C}_+)$ will be critical.

Throughout this section, it will be assumed that the approximants $\{\widehat{P}_n\}_{n=1}^\infty$ have been constructed as follows.

Construction 4.1. (Finite-Dimensional Approximants: $\{\widehat{P}_n\}_{n=1}^\infty$.)
Let $\{\widehat{P}_n\}_{n=1}^\infty$ denote a sequence of $\mathbb{R}\mathcal{H}^\infty(\boldsymbol{C}_+)$ matrix-valued functions such that

$$\lim_{n\to\infty} \|\widehat{P}_n - \widehat{P}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} = 0. \tag{15}$$

More specifically, suppose that one chooses a desired performance tolerance $\varepsilon_d > 0$, however small. Let $\varepsilon \in [0,1)$ satisfy the inequality

$$\varepsilon \le \frac{\varepsilon_d}{\|\widehat{W}_e\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} + 3 + \varepsilon_d}. \tag{16}$$

Define the (a *priori* known) quantity

$$B \stackrel{\text{def}}{=} B(\varepsilon, \widehat{W}_e, \widehat{W}_u) \stackrel{\text{def}}{=} \|\widehat{W}_u^{-1}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} \ \ (\|\widehat{W}_e\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} + \varepsilon). \tag{17}$$

Given this, choose $N \in \boldsymbol{Z}_+$ such that

$$\|\widehat{P}_n - \widehat{P}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} < \delta \stackrel{\text{def}}{=} \delta(\varepsilon, \widehat{W}_e, \widehat{W}_u) \stackrel{\text{def}}{=} \min\left\{\frac{\varepsilon}{\|\widehat{W}_e\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)}B}, \frac{\varepsilon}{B}\right\} \tag{18}$$

for all $n \ge N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$.

Comment 4.1. (Approximants and Desired Performance Tolerance.)
In what follows, it will be shown that given $\varepsilon_d > 0$, however small,

$$\mu_{\text{opt}} - 2\varepsilon \le \mu_n \le \mu_{\text{opt}} + 2\varepsilon \tag{19}$$

$$\mu_{\text{opt}} \leq \tilde{\mu}_n \leq \mu_{\text{opt}} + \varepsilon_d, \tag{20}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. These facts will be made clear in Theorems 4.1 – 4.2. Hence, it is ε_d which is the *actual performance tolerance* and not ε. Throughout the paper, however, it is convenient to work with the intermediate performance tolerance ε. From (19), it follows that ε determines how well μ_n approximates μ_{opt}.

It should be pointed out that the above approximation may be carried out on the basis of frequency response data. This makes the condition given in the construction practically appealing. Conditions on the plant $\widehat{P}$ under which such an approximating sequence $\{\widehat{P}_n\}_{n=1}^{\infty}$ exists can be inferred from Proposition 2.3. It should be noted, however, that not all distributed systems can be uniformly approximated by real-rational systems.

Comment 4.2. (Qualitative Trade-offs.) An interesting qualitative interpretation of Equation (18) is now given. For purposes of discussion, and without loss of generality, it can be assumed that N is a decreasing function of $\delta \stackrel{\text{def}}{=} \delta(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. Consequently, the smaller $\delta \stackrel{\text{def}}{=} \delta(\varepsilon, \widehat{W}_e, \widehat{W}_u)$ is, the larger N will be. From this, it then follows that:

(1) The smaller ε or ε_d, the larger N will be. This is natural to expect. Optimality, or near-optimality, comes at a price, namely model complexity.

(2) The larger $\|\widehat{W}_e\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)}$, the larger N will be. One typically selects a large $\|\widehat{W}_e\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)}$ in order to obtain a high level of performance. This suggests that a trade-off must be made between performance and model simplicity.

(3) The smaller $\|\widehat{W}_u\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)}$, the larger $\|\widehat{W}_u^{-1}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)}$ and hence the larger N will be. This follows from the inequality:

$$1 = \|\widehat{W}_u\widehat{W}_u^{-1}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} \leq \|\widehat{W}_u\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)}\|\widehat{W}_u^{-1}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} \tag{21}$$

and Equation (18). A small value of $\|\widehat{W}_u\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)}$ is typically selected in order to achieve larger stability margins; e. g. gain and phase margins. This suggests that a trade-off must be made between stability robustness and model simplicity. The above qualitative observations are consistent with practical heuristics.

The solution to Problems 3.1 – 3.2 are facilitated through continuous-time lifting of the sampled-data systems depicted in Figures 3, 4, and 5. The lifted sampled-data systems are time-invariant (Proposition 2.2) due to the periodic nature of the original system (Comment 2.1). However, the input and output spaces of the lifted systems are infinite-dimensional. The lifted systems are illustrated in Figures 6, 7, and 8. The following operators result from lifting the sampled-data systems depicted in Figures 3, 4, and 5:

$$\underline{W_e} = LW_eL^{-1} \tag{22}$$
$$\underline{W_u} = LW_uL^{-1} \tag{23}$$
$$\underline{P} = LPL^{-1} \tag{24}$$

$$\underline{K} = LKL^{-1},\ K \stackrel{\text{def}}{=} HK^dSF \tag{25}$$
$$\underline{P_n} = LP_nL^{-1} \tag{26}$$
$$\underline{K_n} = LK_nL^{-1},\ K_n \stackrel{\text{def}}{=} HK_n^dSF. \tag{27}$$

The lifted sampled-data systems are causal, linear, and shift-invariant and hence they have a frequency domain representation via the z-transform given in Definition 2.3. Invoking the isomorphism which exists for the z-transform and the lift operator (Proposition 2.1) permits the equivalent frequency domain descriptions of the performance metrics given in Definitions 3.2, 3.3, and 3.4 to be established. The frequency domain performance metrics are respectively (see also Figures 6, 7, and 8)

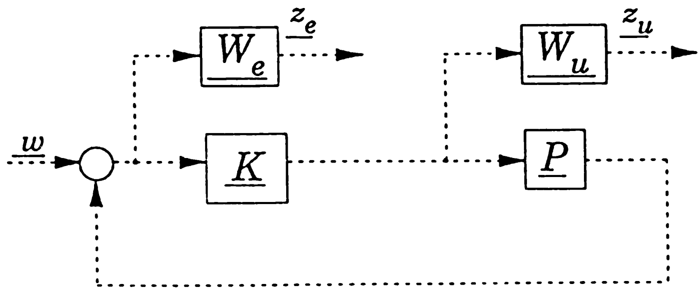

Fig. 6. Lifted infinite-dimensional sampled-data feedback loop.

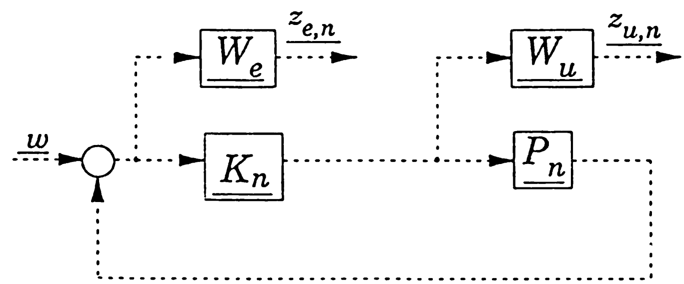

Fig. 7. Lifted purely finite-dimensional sampled-data feedback loop.

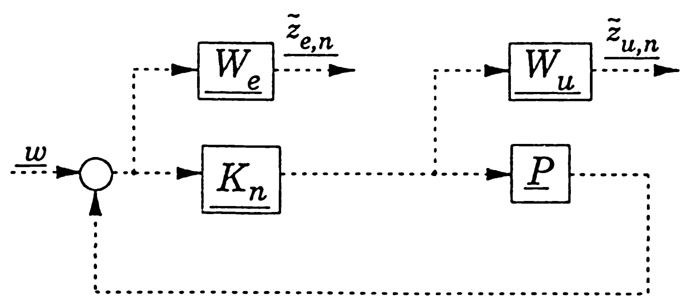

Fig. 8. Lifted purely finite-dimensional sampled-data feedback loop.

$$\mu_{\text{opt}} = \inf_{\underline{\hat{Q}} \in \underline{\mathcal{H}}^\infty(\mathbb{D})} \left\| \begin{pmatrix} \underline{\widehat{W}_e} \\ \underline{\widehat{W}_u}\, \underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}}) \end{pmatrix} \left(I - \underline{\hat{P}}\, \underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}})\right)^{-1} \right\|_{\mathcal{H}^\infty} \tag{28}$$

$$\mu_n = \inf_{\underline{\hat{Q}} \in \underline{\mathbb{R}\mathcal{H}}^\infty(\mathbb{D})} \left\| \begin{pmatrix} \underline{\widehat{W}_e} \\ \underline{\widehat{W}_u}\, \underline{\hat{K}}(\underline{\hat{P}_n}, \underline{\hat{Q}}) \end{pmatrix} \left(I - \underline{\hat{P}_n}\, \underline{\hat{K}}(\underline{\hat{P}_n}, \underline{\hat{Q}})\right)^{-1} \right\|_{\mathcal{H}^\infty} \tag{29}$$

$$\tilde{\mu}_n = \left\| \begin{pmatrix} \underline{\widehat{W}_e} \\ \underline{\widehat{W}_u}\, \underline{\hat{K}}(\underline{\hat{P}_n}, \underline{\hat{Q}_n}) \end{pmatrix} \left(I - \underline{\hat{P}}\, \underline{\hat{K}}(\underline{\hat{P}_n}, \underline{\hat{Q}_n})\right)^{-1} \right\|_{\mathcal{H}^\infty} \tag{30}$$

where the symbols are as defined in Proposition 2.5.

Lemma 4.1. (Lifted System Parameter Properties.) The lifted weighting filters

$$\underline{\widehat{W}_e}, \underline{\widehat{W}_u}, \underline{\widehat{W}_u}^{-1} \in \mathcal{A}. \tag{31}$$

The lifted approximants $\underline{\hat{P}_n}$ and the lifted plant $\underline{\hat{P}}$ satisfy

$$\|\underline{\hat{P}_n} - \underline{\hat{P}}\|_{\mathcal{H}^\infty} < \delta \tag{32}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\epsilon, \widehat{W}_e, \widehat{W}_u)$.

P r o o f . This follows from Assumption 4.1, Construction 4.1, and Proposition 2.4. □

Given our notion of stability, Comment 2.3, the following extended result from the algebraic control literature will be heavily exploited [45].

Given that $\underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}}) \stackrel{\text{def}}{=} -\underline{\hat{Q}}(I - \underline{\hat{P}}\underline{\hat{Q}})^{-1}$ (cf. Proposition 2.5, it follows from Equation (28) that the *optimal performance* μ_{opt} is given by the following expression:

$$\begin{aligned} \mu_{\text{opt}} &= \inf_{\underline{\hat{Q}} \in \underline{\mathcal{H}}^\infty(\mathbb{D})} \left\| \begin{pmatrix} \underline{\widehat{W}_e} \\ \underline{\widehat{W}_u}\, \underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}}) \end{pmatrix} \left(I - \underline{\hat{P}}\, \underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}})\right)^{-1} \right\|_{\mathcal{H}^\infty} \\ &= \inf_{\underline{\hat{Q}} \in \underline{\mathcal{H}}^\infty(\mathbb{D})} \left\| \begin{matrix} \underline{\widehat{W}_e}(I - \underline{\hat{P}}\underline{\hat{Q}}) \\ \underline{\widehat{W}_u}\, \underline{\hat{Q}} \end{matrix} \right\|_{\mathcal{H}^\infty}. \end{aligned} \tag{33}$$

It should be emphasized that this expression defines an infinite-dimensional optimization problem. In this section, it is shown that this infinite-dimensional problem can be avoided entirely. Before proceeding, it should be noted that $\mu_{\text{opt}} \leq \|\underline{\widehat{W}_e}\|_{\mathcal{H}^\infty} = \|\widehat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)}$. Thus, although μ_{opt} is not known *a priori*, an *a priori* upper bound is immediately available.

Similarly, from Equation (29), it follows that the *expected performance* μ_n is given by the following expression:

$$\begin{aligned}
\mu_n &= \inf_{\underline{\hat{Q}} \in \underline{\mathbb{R}\mathcal{H}^\infty}(\mathbb{D})} \left\| \begin{pmatrix} \underline{\widehat{W}_e} \\ \underline{\widehat{W}_u}\,\underline{\hat{K}_n}(\underline{\hat{P}_n},\underline{\hat{Q}}) \end{pmatrix} \left(I - \underline{\hat{P}_n}\,\underline{\hat{K}_n}(\underline{\hat{P}_n},\underline{\hat{Q}})\right)^{-1} \right\|_{\mathcal{H}^\infty} \\
&= \inf_{\underline{\hat{Q}} \in \underline{\mathbb{R}\mathcal{H}^\infty}(\mathbb{D})} \left\| \begin{matrix} \underline{\widehat{W}_e}(I - \underline{\hat{P}_n}\,\underline{\hat{Q}}) \\ \underline{\widehat{W}_u}\,\underline{\hat{Q}} \end{matrix} \right\|_{\mathcal{H}^\infty}. \qquad (34)
\end{aligned}$$

It should be noted that

$$\mu_n \leq \|\underline{\widehat{W}_e}\|_{\mathcal{H}^\infty} = \|\widehat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)}, \quad \forall n \in \mathbb{Z}_+. \qquad (35)$$

This shows that $\{\mu_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of real numbers.

In what follows, let $\mathcal{Q}(\hat{Q}_o^d) = \underline{\hat{Q}_o} \in \underline{\mathcal{H}}^\infty(\mathbb{D})$ satisfy the following inequality:

$$\left\| \begin{matrix} \underline{\widehat{W}_e}(I - \underline{\hat{P}}\underline{\hat{Q}_o}) \\ \underline{\widehat{W}_u}\,\underline{\hat{Q}_o} \end{matrix} \right\|_{\mathcal{H}^\infty} \leq \mu_{\text{opt}} + \varepsilon. \qquad (36)$$

A fundamental premise of this paper is that $\underline{\hat{Q}_o} \in \underline{\mathcal{H}}^\infty(\mathbb{D})$, and hence $\hat{Q}_o^d \in \mathcal{H}^\infty(\mathbb{D})$, is unknown. Although $\underline{\hat{Q}_o}$ is unknown, one can still obtain an *a priori* bound for its $\mathcal{H}^\infty$ norm. With this bound, an upper-semicontinuity result can be obtained as follows.

Proposition 4.1. (Upper-semicontinuity.) Given Assumption 4.1, it follows that

$$\|\underline{\hat{Q}_o}\|_{\mathcal{H}^\infty} \leq B \qquad (37)$$

and

$$\mu_n \leq \mu_{\text{opt}} + 2\varepsilon \qquad (38)$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. Moreover,

$$\lim_{n\to\infty} \inf_{k \geq n} \mu_k \leq \lim_{n\to\infty} \sup_{k\geq n} \mu_k \leq \mu_{\text{opt}}. \qquad (39)$$

P r o o f. Since $\underline{\widehat{W}_u}^{-1} \in \mathcal{H}^\infty$ and $\mu_{\text{opt}} \leq \|\underline{\widehat{W}_e}\|_{\mathcal{H}^\infty}$, it follows that

$$\begin{aligned}
\|\underline{\hat{Q}_o}\|_{\mathcal{H}^\infty} &\leq \|\underline{\widehat{W}_u}^{-1}\|_{\mathcal{H}^\infty} \|\underline{\widehat{W}_u}\,\underline{\hat{Q}_o}\|_{\mathcal{H}^\infty} \leq \|\underline{\widehat{W}_u}^{-1}\|_{\mathcal{H}^\infty}(\mu_{\text{opt}} + \varepsilon) \\
&\leq \|\underline{W_u}^{-1}\|_{\mathcal{H}^\infty}\left(\|\underline{\widehat{W}_e}\|_{\mathcal{H}^\infty} + \varepsilon\right). \qquad (40)
\end{aligned}$$

However,

$$\|\underline{W_u}^{-1}\|_{\mathcal{H}^\infty}(\|\underline{\widehat{W}_e}\|_{\mathcal{H}^\infty} + \varepsilon) = \|\widehat{W}_u^{-1}\|_{\mathcal{H}^\infty(\mathbb{C}_+)}(\|\widehat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} + \varepsilon) \stackrel{\text{def}}{=} B. \qquad (41)$$

One should note that to obtain this bound, the invertibility condition on $\widehat{W}_u$ was critical. Now, consider the following inequality:

$$\mu_n \leq \left\| \begin{array}{c} \underline{\widehat{W}_e(I-\widehat{P}_n\widehat{Q}_o)} \\ \underline{\widehat{W}_u\widehat{Q}_o} \end{array} \right\|_{\mathcal{H}^\infty} \leq \left\| \begin{array}{c} \underline{\widehat{W}_e(I-\widehat{P}\widehat{Q}_o)} \\ \underline{\widehat{W}_u\widehat{Q}_o} \end{array} \right\|_{\mathcal{H}^\infty} + \left\| \underline{\widehat{W}_e}(\underline{\widehat{P}_n}-\underline{\widehat{P}})\underline{\widehat{Q}_o} \right\|_{\mathcal{H}^\infty}. \tag{42}$$

Using the near-optimality of $\underline{\widehat{Q}_o}$ [see Equation (36)] and the bound for $\underline{\widehat{Q}_o}$ obtained in (37) yields

$$\mu_n \leq \mu_{\text{opt}} + \varepsilon + B\|\underline{\widehat{W}_e}\|_{\mathcal{H}^\infty}\|\underline{\widehat{P}_n}-\underline{\widehat{P}}\|_{\mathcal{H}^\infty}. \tag{43}$$

The proof then follows from the construction given for $\widehat{P}_n$ [see Equation (18)] and Lemma 4.1. □

Comment 4.3. (Upper-semicontinuity at $\underline{\widehat{P}}$.) The above proposition shows that the function $\mu(\underline{\widehat{P}})$ is upper-semicontinuous [35, pp. 48–50] at $\underline{\widehat{P}}$ in the uniform topology on $\mathcal{H}^\infty$ (see [37]); i. e.

$$\lim_{n\to\infty}\sup_{k\geq n}\mu_k = \lim_{n\to\infty}\sup_{k\geq n}\mu(\underline{\widehat{P}_k}) \leq \mu_{\text{opt}} \stackrel{\text{def}}{=} \mu(\underline{\widehat{P}}). \tag{44}$$

From the proof of the proposition, one sees that the upper-semicontinuity follows immediately since $\underline{\widehat{Q}_o} \in \mathcal{H}^\infty$ is a fixed element of $\mathcal{H}^\infty$ and $\underline{\widehat{P}_n}$ uniformly approximates $\underline{\widehat{P}}$. However, to provide the a *priori* estimate given in Proposition 4.1 (i. e. determine $N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$ a *priori*), the upper bound for $\underline{\widehat{Q}_o}$ obtained in (37) was exploited. This upperbound for $\underline{\widehat{Q}_o}$ was obtained by taking advantage of the invertibility of $\widehat{W}_u$ in $\mathcal{H}^\infty(\boldsymbol{C}_+)$.

Comment 4.4. (Upper-semicontinuity at $\widehat{P}$.) Comment 4.3 is directly applicable to $\widehat{P}$ by Proposition 2.4. That is,

$$\lim_{n\to\infty}\sup_{k\geq n}\mu_k = \lim_{n\to\infty}\sup_{k\geq n}\mu(\widehat{P}_k) \leq \mu_{\text{opt}} \stackrel{\text{def}}{=} \mu(\widehat{P}). \tag{45}$$

Proceeding as above, let $\mathcal{Q}(\widehat{Q}_n^d) = \underline{\widehat{Q}_n} \in \underline{\mathcal{H}}^\infty(\mathbb{D})$ satisfy the following inequality:

$$\left\| \begin{array}{c} \underline{\widehat{W}_e(I-\widehat{P}_n\widehat{Q}_n)} \\ \underline{\widehat{W}_u\widehat{Q}_n} \end{array} \right\|_{\mathcal{H}^\infty} \leq \mu_n + \varepsilon. \tag{46}$$

The bisection search method described in [6, pp. 336–345] and [2] can be used to compute $\widehat{Q}_n^d$. It is now shown that $\{\underline{\widehat{Q}_n}\}_{n=1}^\infty$ is a uniformly bounded sequence of operator-valued functions in $\mathcal{H}^\infty$. The bound is then used to obtain a lower-semicontinuity result.

Proposition 4.2. (Lower-semicontinuity.) Given Assumption 4.1, it follows that

$$\|\underline{\widehat{Q}_n}\|_{\mathcal{H}^\infty} \leq B \tag{47}$$

for all $n \in \boldsymbol{Z}_+$ and

$$\mu_{\mathrm{opt}} \leq \mu_n + 2\varepsilon \tag{48}$$

for all $n \geq N \stackrel{\mathrm{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. Moreover,

$$\mu_{\mathrm{opt}} \leq \lim_{n\to\infty} \inf_{k \geq n} \mu_k \leq \lim_{n\to\infty} \sup_{k\geq n} \mu_k. \tag{49}$$

Proof. Since $\underline{\widehat{W}_u}^{-1} \in \mathcal{H}^\infty$ and $\mu_n \leq \|\underline{\widehat{W}_e}\|_{\mathcal{H}^\infty}$ for all $n \in \boldsymbol{Z}_+$, it follows that

$$\begin{aligned} \|\underline{\widehat{Q}_n}\|_{\mathcal{H}^\infty} &\leq \|\underline{\widehat{W}_u}^{-1}\|_{\mathcal{H}^\infty} \|\underline{\widehat{W}_u}\,\underline{\widehat{Q}_n}\|_{\mathcal{H}^\infty} \leq \|\underline{\widehat{W}_u}^{-1}\|_{\mathcal{H}^\infty}(\mu_n + \varepsilon) \\ &\leq \|\underline{\widehat{W}_u}^{-1}\|_{\mathcal{H}^\infty}(\|\underline{\widehat{W}_e}\|_{\mathcal{H}^\infty} + \varepsilon). \end{aligned} \tag{50}$$

However,

$$\|\underline{\widehat{W}_u}^{-1}\|_{\mathcal{H}^\infty}(\|\underline{\widehat{W}_e}\|_{\mathcal{H}^\infty} + \varepsilon) = \|\widehat{W}_u^{-1}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)}(\|\widehat{W}_e\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} + \varepsilon). \stackrel{\mathrm{def}}{=} B \tag{51}$$

To obtain this uniform bound for $\{\underline{\widehat{Q}_n}\}_{n=1}^\infty$, the invertibility of $\widehat{W}_u$ in $\mathcal{H}^\infty(\boldsymbol{C}_+)$ was, once again, the key. To complete the proof, consider the following inequality:

$$\mu_{\mathrm{opt}} \leq \left\| \begin{matrix} \underline{\widehat{W}_e}(I - \underline{\widehat{P}}\,\underline{\widehat{Q}_n}) \\ \underline{\widehat{W}_u}\,\underline{\widehat{Q}_n} \end{matrix} \right\|_{\mathcal{H}^\infty} \leq \left\| \begin{matrix} \underline{\widehat{W}_e}(I - \underline{\widehat{P}_n}\,\underline{\widehat{Q}_n}) \\ \underline{\widehat{W}_u}\,\underline{\widehat{Q}_n} \end{matrix} \right\|_{\mathcal{H}^\infty} + \left\| \underline{\widehat{W}_e}(\underline{\widehat{P}_n} - \underline{\widehat{P}})\underline{\widehat{Q}_n} \right\|_{\mathcal{H}^\infty} \tag{52}$$

Using the near-optimality of $\underline{\widehat{Q}_n}$ [see Equation (46)] and the uniform bound for $\underline{\widehat{Q}_n}$ obtained in (47) yields

$$\mu_{\mathrm{opt}} \leq \mu_n + \varepsilon + B\|\underline{\widehat{W}_e}\|_{\mathcal{H}^\infty} \|\underline{\widehat{P}_n} - \underline{\widehat{P}}\|_{\mathcal{H}^\infty} \tag{53}$$

The proof then follows from the construction of $\widehat{P}_n$ [see Equation (18)] and Lemma 4.1. □

Comment 4.5. (Lower-semicontinuity at $\underline{\widehat{P}}$.) The above proposition shows that the function $\mu(\underline{\widehat{P}})$ is lower-semicontinuous [35, pp. 48–50] at $\underline{\widehat{P}}$ in the uniform topology on $\mathcal{H}^\infty$ (see [37]); i.e.

$$\mu_{\mathrm{opt}} \stackrel{\mathrm{def}}{=} \mu(\underline{\widehat{P}}) \leq \lim_{n\to\infty} \inf_{k\geq n} \mu_k = \lim_{n\to\infty} \inf_{k \geq n} \mu(\underline{\widehat{P}_k}). \tag{54}$$

From the proof of the proposition, one sees that the lower-semicontinuity does not follow immediately; i.e. not until the invertibility of $\widehat{W}_u$ in $\mathbb{R}\mathcal{H}^\infty(\boldsymbol{C}_+)$ is exploited to show that the sequence $\{\underline{\widehat{Q}_n}\}_{n=1}^\infty$ is in fact uniformly bounded in $\mathcal{H}^\infty$.

Comment 4.6. (Lower-semicontinuity at $\widehat{P}$.) Comment 4.5 is directly applicable to $\widehat{P}$ by Proposition 2.4. That is,

$$\mu_{\text{opt}} \stackrel{\text{def}}{=} \mu(\widehat{P}) \leq \lim_{n\to\infty} \inf_{k\geq n} \mu_k = \lim_{n\to\infty} \inf_{k\geq n} \mu(\widehat{P}_k). \tag{55}$$

From Propositions 4.1 and 4.2, one obtains the following theorem which shows that the *expected performance* μ_n approaches the *optimal performance* μ_{opt} as the fidelity of the approximants is improved.

Theorem 4.1. (Solution to Purely Finite-Dimensional Problem.) Given the Assumption 4.1, it follows that

$$|\mu_n - \mu_{\text{opt}}| \leq 2\varepsilon \tag{56}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. Moreover,

$$\lim_{n\to\infty} \mu_n = \mu_{\text{opt}}. \tag{57}$$

Comment 4.7. (Continuity at $\widehat{P}$ and Estimating μ_{opt}.) This theorem shows that when $\widehat{W}_u$ is invertible in $\mathcal{H}^\infty(\boldsymbol{C}_+)$, then the function $\mu(\widehat{P}) \stackrel{\text{def}}{=} \mu_{\text{opt}}$ is continuous [35, pp. 48–50] at $\widehat{P}$ in the uniform topology on $\mathcal{H}^\infty(\boldsymbol{C}_+)$ (see [37]); i. e.

$$\lim_{n\to\infty} \mu(\widehat{P}_n) = \mu(\widehat{P}). \tag{58}$$

Also, because N can be determined *a priori*, it follows that one can estimate the *optimal performance* μ_{opt}, to any *a priori* tolerance. This can be done by determining N *a priori* in accordance with Equation (18), and solving the finite-dimensional sampled-data problem associated with μ_N and $\widehat{P}_N$ for a near-optimal $\widehat{Q}_N^d \in \mathbb{R}\mathcal{H}^\infty(\mathbb{D})$. As stated earlier, this can be done by using the well known bisection search method described in [6, pp. 336–345] and [2]. Consequently, the theorem provides a solution to the *Purely Finite-Dimensional $\mathcal{H}^\infty$ Mixed-Sensitivity Problem.*

The compensator generated by $\widehat{Q}_n^d \in \mathcal{H}^\infty(\mathbb{D})$ is finite-dimensional and is given by $K_n^d \stackrel{\text{def}}{=} \widehat{K}^d(P_n, Q_n^d) \stackrel{\text{def}}{=} -Q_n^d(I - SFP_nHQ_n^d)^{-1}$. It is now shown that this compensator "internally" stabilizes the plant $\widehat{P}$ in the sampled-data setting for all except possibly a finite number of n. Proposition 2.5 will be invoked to establish this fact.

Proposition 4.3. (Stability of Actual Closed Loop Operator: $(\underline{\widehat{P}}, \underline{\widehat{K}_n})$.) Given Assumption 4.1, it follows that

$$\|(\underline{\widehat{P}_n} - \underline{\widehat{P}})\underline{\widehat{Q}_n}\|_{\mathcal{H}^\infty} < 1 \tag{59}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. Moreover,

$$\lim_{n\to\infty} \|(\underline{\widehat{P}_n} - \underline{\widehat{P}})\underline{\widehat{Q}_n}\|_{\mathcal{H}^\infty} = 0 \tag{60}$$

and the operator $\underline{\widehat{K}_n} \stackrel{\text{def}}{=} \underline{\widehat{K}}(\underline{\widehat{P}_n}, \underline{\widehat{Q}_n}) = -\underline{\widehat{Q}_n}(I - \underline{\widehat{P}_n}\,\underline{\widehat{Q}_n})^{-1}$ internally stabilizes the MIMO distributed parameter operator $\underline{\widehat{P}}$ for all except possibly a finite number of n.

Proof. Using the uniform bound obtained for $\underline{\widehat{Q}_n}$ in (47), one obtains $\|(\underline{\widehat{P}_n} - \underline{\widehat{P}})\underline{\widehat{Q}_n}\|_{\mathcal{H}^\infty} \leq B\|(\underline{\widehat{P}_n} - \underline{\widehat{P}})\|_{\mathcal{H}^\infty}$. The proof of this proposition then follows from the construction of $\widehat{P}_n$ [see Equation (18)], Lemma 4.1, and the small gain theorem [8]. □

Given that $\underline{\widehat{K}_n} \stackrel{\text{def}}{=} \underline{\widehat{K}}(\underline{\widehat{P}_n}, \underline{\widehat{Q}_n}) \stackrel{\text{def}}{=} -\underline{\widehat{Q}_n}(I - \underline{\widehat{P}_n}\,\underline{\widehat{Q}_n})^{-1}$ stabilizes $\underline{\widehat{P}}$ for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$, it follows from Equation (30) that the *actual performance*, $\tilde{\mu}_n$ is well defined and given by:

$$\begin{aligned}
\tilde{\mu}_n &= \left\| \begin{pmatrix} \widehat{W}_e \\ \underline{\widehat{W}_u}\,\underline{\widehat{K}_n} \end{pmatrix} \left(I - \underline{\widehat{P}}\,\underline{\widehat{K}_n}\right)^{-1} \right\|_{\mathcal{H}^\infty} \\
&= \left\| \begin{pmatrix} \underline{\widehat{W}_e}(I - \underline{\widehat{P}_n\widehat{Q}_n}) \\ \underline{\widehat{W}_u}\,\underline{\widehat{Q}_n} \end{pmatrix} \left(I - (\underline{\widehat{P}_n} - \underline{\widehat{P}})\underline{\widehat{Q}_n}\right)^{-1} \right\|_{\mathcal{H}^\infty}
\end{aligned} \tag{61}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$.

Given this, the following theorem provides a solution to the $\mathcal{H}^\infty$ *Approximate/Design Mixed-Sensitivity Problem.*

Theorem 4.2. (Solution to $\mathcal{H}^\infty$ Approximate/Design Mixed-Sensitivity Problem.) Given Assumption 4.1, it follows that

$$\mu_{\text{opt}} \leq \tilde{\mu}_n \leq \mu_{\text{opt}} + \varepsilon_d \tag{62}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. Moreover,

$$\lim_{n \to \infty} \tilde{\mu}_n = \mu_{\text{opt}}. \tag{63}$$

Proof. From Equation (61), one obtains the following inequality:

$$\tilde{\mu}_n \leq \left\| \begin{matrix} \underline{\widehat{W}_e}(I - \underline{\widehat{P}_n\widehat{Q}_n}) \\ \underline{\widehat{W}_u}\,\underline{\widehat{Q}_n} \end{matrix} \right\|_{\mathcal{H}^\infty} \frac{1}{1 - \|(\underline{\widehat{P}_n} - \underline{\widehat{P}})\underline{\widehat{Q}_n}\|_{\mathcal{H}^\infty}}. \tag{64}$$

Since $\underline{\widehat{K}_n} \stackrel{\text{def}}{=} -\underline{\widehat{Q}_n}(I - \underline{\widehat{P}_n}\,\underline{\widehat{Q}_n})^{-1}$ stabilizes $\underline{\widehat{P}}$ for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$, it follows that

$$\mu_{\text{opt}} \leq \tilde{\mu}_n \tag{65}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. Since $\underline{\widehat{Q}_n}$ satisfies the inequality (46), it follows from Theorem 4.1 that

$$\left\| \begin{matrix} \underline{\widehat{W}_e}(I - \underline{\widehat{P}_n\widehat{Q}_n}) \\ \underline{\widehat{W}_u}\,\underline{\widehat{Q}_n} \end{matrix} \right\|_{\mathcal{H}^\infty} \leq \mu_n + \varepsilon \leq \mu_{\text{opt}} + 3\varepsilon \tag{66}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. This, then yields

$$\mu_{\text{opt}} \leq \tilde{\mu}_n \leq \frac{\mu_{\text{opt}} + 3\varepsilon}{1 - \|(\underline{\widehat{P}_n} - \underline{\widehat{P}})\underline{\widehat{Q}_n}\|_{\mathcal{H}^\infty}} \tag{67}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. Using the uniform bound for $\underline{\widehat{Q}_n}$ obtained in (47) yields

$$\mu_{\text{opt}} \leq \tilde{\mu}_n \leq \frac{\mu_{\text{opt}} + 3\varepsilon}{1 - B\|(\underline{\widehat{P}_n} - \underline{\widehat{P}})\|_{\mathcal{H}^\infty}} \tag{68}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. The proof of the theorem then follows from the the construction of $\widehat{P}_n$ [see Equation (18)], Lemma 4.1 and

$$\begin{aligned} \frac{\mu_{\text{opt}} + 3\varepsilon}{1 - \varepsilon} &\leq \mu_{\text{opt}} + (\mu_{\text{opt}} + 3)\frac{\varepsilon}{1 - \varepsilon} \\ &\leq \mu_{\text{opt}} + (\|\widehat{W}_e\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} + 3)\frac{\varepsilon}{1 - \varepsilon} \\ &\leq \mu_{\text{opt}} + \varepsilon_d. \end{aligned}$$

□

Comment 4.8. (Solution to Mixed–Sensitivity Problems: Issues.) Given the previous theorems, some comments are in order. First, it is important to note that no infinite-dimensional spectral factorization is required. Also, the optimal performance need not be known *a priori* in order to construct near-optimal finite-dimensional controllers. Moreover, in this paper, Construction 4.1 provides precise *a priori* conditions on the approximants so that the resulting finite-dimensional controllers deliver near-optimal performance, in a weighted $\mathcal{H}^\infty$ mixed-sensitivity sense, for the sampled-data controlled MIMO distributed parameter plant. A consequence of this, is that the *optimal performance* can be determined to within an *a priori* specified tolerance. Finally, it should also be stated that robust controllers with respect to normalized coprime factor perturbations (see [29]) can be accommodated within the framework presented in this section.

Summary of Design Methodology. The proposed "indirect" procedure for synthesizing a finite-dimensional sampled-data controller for a MIMO stable infinite-dimensional plant is as follows:

1. Start with the sampled-data system with infinite-dimensional plant, P, and performance measure, μ_{opt}, which takes into account intersample behavior (Figure 3 and Equation (11)). A specified performance criterion is that the actual performance be near-optimal (Equation (62)). The actual performance is defined as the performance achieved from the sampled-data system using the infinite-dimensional plant and a finite-dimensional discrete-time controller, K_n^d (Figure 5).
2. Approximate the infinite-dimensional plant with finite-dimensional $\mathbb{R}\mathcal{H}^\infty(\boldsymbol{C}_+)$ approximants of *a priori* determinable order based on ε_d and the weighting filters. Use this approximant in place of the infinite-dimensional plant in the sampled-data set-up (Figure 4).

3. Lift the resultant finite-dimensional sampled-data system (Figure 9).
4. In Figure 9, $\mathcal{H}^\infty$-discretize [6, pp. 317–320] the operator which maps $\begin{bmatrix} \underline{w} \\ v_n \end{bmatrix} \to \begin{bmatrix} \underline{z_n} \\ \psi_n \end{bmatrix}$.
5. Synthesize a finite-dimensional discrete-time controller, K_n^d, using "natural" discrete-time $\mathcal{H}^\infty(\mathbb{D})$ design algorithms based on the discretized finite-dimensional operator.
6. The synthesized finite-dimensional discrete-time controller results in stable closed loop performance for the original infinite-dimensional sampled-data system with guaranteed performance measure satisfying Equation (62).

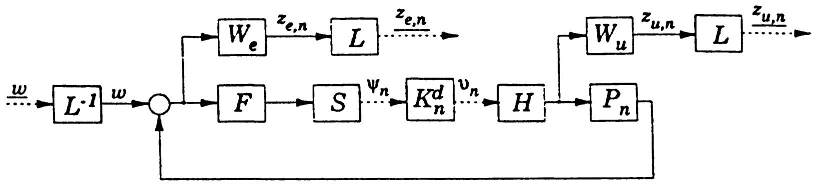

Fig. 9. Sampled-data feedback loop input/output lifting.

This completes the discussion of the two fundamental $\mathcal{H}^\infty$ *Mixed-Sensitivity Problems* considered in this paper. Section 5 presents a numerical example of the finite-dimensional sampled-data controller synthesis methodology.

5. NUMERICAL EXAMPLE

This section presents a numerical example of the results presented in Section 4. First, an application of Theorem 4.1 is performed where the infinite-dimensional sampled-data performance measure $\mu_{\rm opt}$ is estimated using finite-dimensional sampled-data techniques. Then an application of Theorem 4.2 is performed where a finite-dimensional discrete-time controller is synthesized which guarantees near-optimal performance for the infinite-dimensional sampled-data system. The controller design methodology enumerated in Section 4 will be applied here.

The SISO infinite-dimensional plant used for this example is

$$\widehat{P}(s) = \frac{e^{-s}}{s+1}. \tag{69}$$

The sample rate $T_s = 0.3$ seconds and the anti-aliasing filter is

$$\widehat{F}(s) = \frac{1}{\frac{0.3}{\pi}s + 1} \tag{70}$$

which has a pole at the Nyquist frequency. The weighting filters selected for this example are

$$\widehat{W_e}(s) = \frac{1}{\left(\frac{5}{2\pi}s + 1\right)^2} \tag{71}$$

$$\widehat{W_u}(s) = \frac{\left(\frac{5}{2\pi}s + 10^{-3}\right)}{\left(\frac{\pi}{75}s + 1\right)}. \tag{72}$$

The $\mathcal{H}^\infty$ norms for these filters are

$$||\widehat{W}_e||_{\mathcal{H}^\infty(\boldsymbol{C}_+)} = 1 \tag{73}$$

$$||\widehat{W}_u||_{\mathcal{H}^\infty(\boldsymbol{C}_+)} = 19. \tag{74}$$

The infinite-dimensional portion of the plant $\widehat{P}(s)$ in (69) is approximated by [31]

$$\widehat{P}_n^{dly}(s) = \frac{\widehat{N}_{\text{pn}}(s)}{\widehat{D}_{\text{pn}}(s)} \tag{75}$$

where

$$\widehat{D}_{\text{pn}}(s) = \sum_{k=0}^{n} \frac{(2n-k)!n!}{2n!k!(n-k)!} s^k \tag{76}$$

and

$$\widehat{N}_{\text{pn}}(s) = \widehat{D}_{\text{pn}}(-s). \tag{77}$$

This yields plant approximants of the form

$$\widehat{P}_n(s) = \frac{\widehat{N}_{\text{pn}}(s)}{\widehat{D}_{\text{pn}}(s)} \left(\frac{1}{s+1}\right). \tag{78}$$

These approximants uniformly approximate $\widehat{P}$ [20]; i.e.

$$\lim_{n\to\infty} ||\widehat{P}_n - \widehat{P}||_{\mathcal{H}^\infty(\boldsymbol{C}_+)} = 0. \tag{79}$$

The approximation error $|\widehat{P}_n - \widehat{P}|$ as a function of frequency is displayed in Figure 10. Figure 11 shows the plant approximation $\mathcal{H}^\infty(\boldsymbol{C}_+)$ norm error as a function of approximant order. Also displayed in this figure are the upper and lower error bounds for the approximants [16, p. 385]. The approximations asymptotically approaches the optimal convergence rate of $O(n^{-1})$ [20, p. 241].

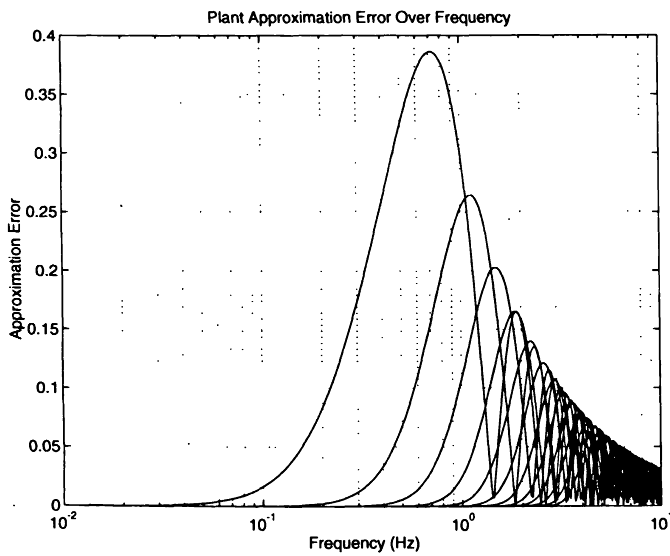

Fig. 10. Plant approximation error curves.

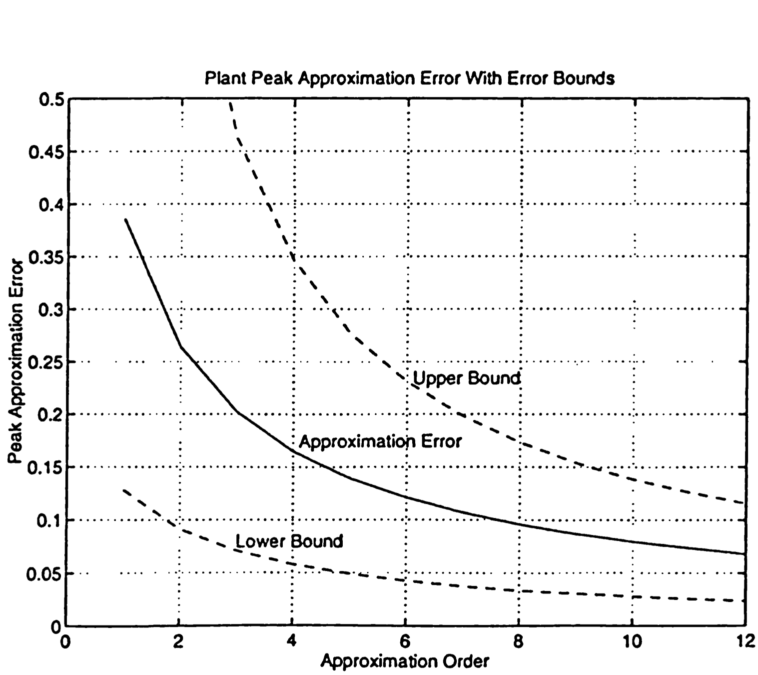

Fig. 11. $\mathcal{H}^\infty$ norm of plant approximation error.

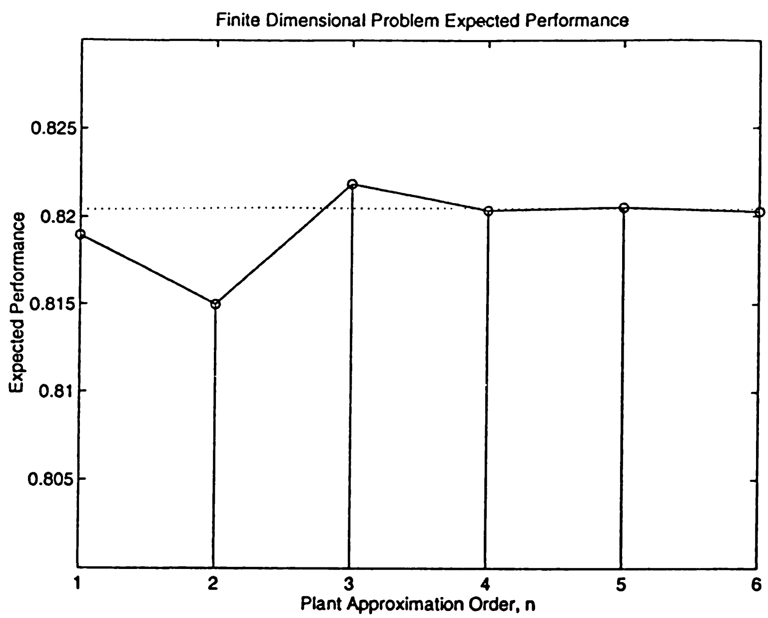

Fig. 12. Convergence of expected performance, μ_n.

Estimation of μ_{opt}. To estimate μ_{opt}, a sequence of finite-dimensional sampled-data runs were performed starting with a plant approximant order of one and incremented an order at a time until μ_n convergence was evident. By Theorem 4.1, μ_n converges to μ_{opt} as n increases. The computer results were generated using a script file written for MATLAB and the μ-Analysis and Synthesis Toolbox implementing algorithms from [6, pp. 309–345]. The μ_n-values were determined by calculating the closed-loop sampled-data system $\mathcal{H}^\infty$-norm iteratively. The convergence results are displayed in Figure 12.

These results yield a strong indication that the infinite-dimensional sampled-data system optimal performance μ_{opt} is around 0.82. To what degree of certainty do we have that the infinite-dimensional sampled-data system optimal performance is 0.82? This question is answered by applying the results of Theorem 4.1, Equation (56) which is now restated

$$|\mu_n - \mu_{\text{opt}}| \le 2\varepsilon \tag{80}$$

for all $n \ge N \stackrel{\text{def}}{=} N(\varepsilon, \widehat{W}_e, \widehat{W}_u)$. Given that the highest order approximant used to generate the results in Figure 12 is 6^{th} order, $N(\varepsilon, \widehat{W}_e, \widehat{W}_u) \le 6$ for Theorem 4.1 results to apply. Set $N(\varepsilon, \widehat{W}_e, \widehat{W}_u) = 6$. We determine the nearness of μ_6 to μ_{opt} using

$$|\mu_6 - \mu_{\text{opt}}| \le 2\varepsilon \tag{81}$$

and calculating ε. As stated in Construction 4.1, the plant approximant order must satisfy

$$||\widehat{P}_6 - \widehat{P}||_{\mathcal{H}^\infty(\boldsymbol{C}_+)} < \delta \stackrel{\text{def}}{=} \min\left\{\frac{\varepsilon}{||\widehat{W}_e||_{\mathcal{H}^\infty(\boldsymbol{C}_+)}B}, \frac{\varepsilon}{B}\right\}, \quad \forall n \ge N(\varepsilon, \widehat{W}_e, \widehat{W}_u) = 6 \tag{82}$$

where

$$\begin{aligned} B &= ||\widehat{W}_u^{-1}||_{\mathcal{H}^\infty(\boldsymbol{C}_+)} \left(||\widehat{W}_e||_{\mathcal{H}^\infty(\boldsymbol{C}_+)} + \varepsilon\right) \\ &= (19^{-1})(1+\varepsilon) \end{aligned} \tag{83}$$

and

$$||\widehat{P}_6 - \widehat{P}||_{\mathcal{H}^\infty(\boldsymbol{C}_+)} = 0.1211 \tag{84}$$

by Figure 11. The minimum ε value which satisfies these relations is

$$\varepsilon = 0.0064. \tag{85}$$

Given this, it follows that

$$|\mu_6 - \mu_{\text{opt}}| \le 2\varepsilon = 0.0128. \tag{86}$$

If μ_{opt} is near 0.82, then our error in approximating μ_{opt} by μ_6 is on the order of 1.6 %.

Near–Optimal Controller Synthesis. Now a finite-dimensional discrete-time controller is synthesized which guarantees near-optimal performance for the infinite-dimensional sampled-data system.

In the first step of the design methodology enumerated in Section 4, the weighting filters and the desired performance tolerance are specified. The weighting filters have already been specified in Equations (71) and (72). Using our μ_{opt} estimation results, we'll specify $\varepsilon_d = 0.08$. This is to attain a small deviation between the actual $\tilde{\mu}_n$ and optimal μ_{opt} performances. From Equation (16), the desired performance tolerance of $\varepsilon_d = 0.08$ results in

$$\varepsilon \leq \frac{\varepsilon_d}{\|W_e\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} + 3 + \varepsilon_d} = \frac{0.08}{1+3+0.08} = \tfrac{1}{51}. \tag{87}$$

The second step requires knowledge or determination of the form of $\mathbb{R}\mathcal{H}^\infty(\boldsymbol{C}_+)$ plant approximants. These were defined in Equation (78).

With the choice of design parameters stated (i.e. weighting filters, desired performance tolerance, and the form of $\mathbb{R}\mathcal{H}^\infty(\boldsymbol{C}_+)$ plant approximants), the plant approximant order is determinable. As stated in Construction 4.1, the plant approximant order must satisfy

$$\|\widehat{P}_n - \widehat{P}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} < \delta \stackrel{\text{def}}{=} \min\left\{\frac{\varepsilon}{\|\widehat{W}_e\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)}B}, \frac{\varepsilon}{B}\right\}, \quad \forall\, n \geq N(\varepsilon, \widehat{W}_e, \widehat{W}_u) \tag{88}$$

where

$$\begin{aligned} B &= \|\widehat{W}_u^{-1}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)}\ \left(\|\widehat{W}_e\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} + \varepsilon\right) \\ &= (19^{-1})\left(1 + \frac{1}{51}\right) \\ &= 53.69 \times 10^{-3}. \end{aligned} \tag{89}$$

This results in

$$\|\widehat{P}_n - \widehat{P}\|_{\mathcal{H}^\infty(\boldsymbol{C}_+)} \;\leq\; \frac{1/51}{53.69 \times 10^{-3}} = 0.37.$$

To be within this system approximation error, Figure 11 indicates that the system approximant must be greater than first-order. A second order plant approximant is used in place of the infinite-dimensional plant to perform controller synthesis.

Controller synthesis methodology steps 3-6 were implemented in a MATLAB script file written for the μ-Analysis and Synthesis Toolbox. A MATLAB simulation was used to model the plant dynamics and to synthesize the finite-dimensional discrete-time controller with near-optimal performance. The second order approximant form is given by

$$\widehat{P}_2(s) = \frac{0.0833s^2 - 0.5s + 1}{(0.0833s^2 + 0.5s + 1)(s+1)}. \tag{90}$$

Script file execution resulted in a discrete-time controller which yielded closed-loop performance of $\mu_2 = 0.8149$. The resultant controller form is

$$\widehat{K}_2^d = 0.0368\frac{(z+0.8882)(z-0.5987)(z-0.7408)(z-0.0432)(z-0.0008)(z^2-0.7058z+0.1653)}{(z+0.8814)(z-0.7744)(z-0.6103)(z^2-0.6316z+0.1322)(z^2-1.2375z+0.4542)}.$$

Theorem 4.1 guarantees this to be within $\pm 2\varepsilon = \pm\frac{2}{51} = 0.039$ of the optimal performance, μ_{opt}. By Theorem 4.2, the actual performance guarantee is that $0 \le \tilde{\mu}_2 - \mu_{\text{opt}} \le 0.08$.

Controller Sequence Convergence. Given that $\widehat{P}_n \overset{n\to\infty}{\longrightarrow} \widehat{P}$, $\mu_n \overset{n\to\infty}{\longrightarrow} \mu_{\text{opt}}$ and $\tilde{\mu}_n \overset{n\to\infty}{\longrightarrow} \mu_{\text{opt}}$ one might expect that the sequence of discrete-time finite-dimensional controllers $\widehat{K}_n^d$ converges to some, possibly infinite-dimensional, near-optimal compensator $\widehat{K}_{\text{opt}}^d$. While such a result is in general difficult to prove, if one assumes that $\widehat{K}_{\text{opt}}^d$ is unique then advanced mathematical concepts such as the Arzela–Ascoli theorem [7, p. 175] maybe useful to arrive at such a result. The plot displayed in Figure 13 shows the frequency responses for $\widehat{K}_1^d, \widehat{K}_2^d, \ldots, \widehat{K}_6^d$. This plot suggests that the controllers $\widehat{K}_1^d, \widehat{K}_2^d, \ldots, \widehat{K}_6^d$ are in fact converging.

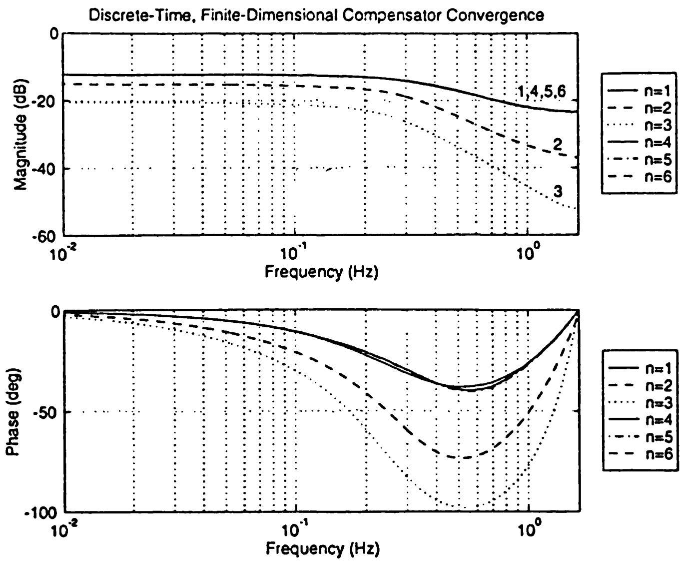

Fig. 13. Discrete-time, finite-dimensional controller convergence.

6. SUMMARY AND FUTURE RESEARCH

This paper presents a systematic methodology for synthesizing near-optimal finite-dimensional sampled-data controllers for a large class of continuous-time MIMO stable distributed parameter plants, based on finite-dimensional plant approximants. The criteria used to determine optimality is a weighted induced $\mathcal{L}^2$ mixed-sensitivity measure which penalizes both the sensitivity operator and a operator associated with the control. More specifically, it has been shown that given an "appropriate" finite-dimensional approximant for a distributed parameter plant, one can solve a single (a *priori* determinable) finite-dimensional sampled-data problem in order to obtain a near-optimal finite-dimensional discrete-time controller. The key technical requirements are that uniform plant approximants are available and that the control is penalized in a nonsingular manner. A numerical example demonstrating the controller synthesis methodology on a delay system was also presented. This example displayed the expected performance measure convergence. In addition, it has been shown that the optimal performance can be approximated to any arbitrary accuracy by solving a single (a *priori* determinable) finite-dimensional optimal sampled-data problem rather than a possibly infinite-dimensional eigenvalue/eigenfunction problem.

Issues to be resolved in future work includes extending these results to MIMO unstable plants, loop convergence properties, and approximation methods for controller order minimization. One step towards the controller order reduction problem is to use system approximants which converge rapidly to the infinite-dimensional system. For the system approximants implemented in this paper, the convergence rate is displayed Figure 11. In future work, we'll research implementing better system approximant schemes [17], [18], [19], [20], [21], and [28].

In summary, the approach presented here allows one to forego solving a "complex" infinite-dimensional sampled-data $\mathcal{H}^\infty$ problem and provides rigorous justification for some of the approximations that control engineers typically make in practice.

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REFERENCES

[1] B. A. Bamieh and J. B. Pearson: A general framework for linear periodic systems with applications to $\mathcal{H}^\infty$ sampled-data control. IEEE Trans. Automat. Control *37* (1992), 418–435.

[2] S.P. Boyd, V. Balakrishnan and P. Kabamba: A bisection method for computing the $\mathcal{H}^\infty$ norm of a transfer matrix and related problems. Math. Control Signals Systems *2* (1989), 207–219.

[3] F. M. Callier and C. A. Desoer: An algebra of transfer functions for distributed linear time-invariant systems. IEEE Trans. Circuits and Systems *25* (1978), 9, 651–662.

[4] M. W. Cantoni and K. Glover: A design framework for continuous-time systems under sampled-data control. In: Proceedings of the 35th Conference on Decision and Control, Kobe 1996, pp. 458–463.

[5] T. Chen and B. Francis: $\mathcal{H}^2$-optimal Sampled-Data Control. Technical Report No. 9001, Dept. Elect. Eng., Univ. Toronto, 1990.

[6] T. Chen and B. Francis: Optimal Sampled-Data Control Systems. Springer-Verlag, London – New York 1995.

[7] J. B. Conway: A Course in Functional Analysis. Springer-Verlag, Berlin 1990.

[8] C. A. Desoer and M. Vidyasagar: Feedback Systems: Input-Output Properties. Academic Press, NY 1975.

[9] G. E. Dullerud: Control of Uncertain Sampled-Data Systems. Birkhäuser, Boston 1996.

[10] D. S. Flamm and S. K. Mitter: Approximation of ideal compensators for delay systems. In: Linear Circuits, Systems and Signal Processing: Theory and Applications (C. I.

Byrnes, C. F. Martin and R. E. Saeks, eds.), Elsevier Science Publishers B. V., 1988, pp. 517–524.

[11] C. Foias, B. Francis, J. W. Helton, H. Kwakernaak and J. B. Pearson: $\mathcal{H}^\infty$-Control Theory. (Lecture Notes in Mathematics 1496.) Springer-Verlag, Berlin 1991.

[12] C. Foias, H. Özbay and A. Tannenbaum: Robust Control of Infinite Dimensional Systems. (Lecture Notes in Control and Information Sciences.) Springer–Verlag, Berlin 1996.

[13] B. A. Francis: A Course in Control Theory. Springer-Verlag, Berlin 1987.

[14] J. S. Gibson and A. Adamian: Approximation theory for linear-quadratic-gaussian optimal control of flexible structures. SIAM J. Control Optim. *29* (1991), 1, 1–37.

[15] J. S. Gibson and I. G. Rosen: Numerical approximation for the infinite-dimensional discrete-time optimal linear-quadratic regulator problem. SIAM J. Control Optim. *26* (1988), 2, 428–451.

[16] C. Glader, G. Högnäs, P. M. Mäkilä and H. T. Toivonen: Approximation of delay systems – a case study. Internat. J. Control *53* (1991), 2, 369–390.

[17] K. Glover: All optimal Hankel-norm approximations of linear multivariable systems and their $\mathcal{L}^\infty$-error bounds. Internat. J. Control *39* (1984), 6, 1115–1193.

[18] K. Glover, R. F. Curtain and J. R. Partington: Realisation and approximation of linear infinite-dimensional systems with error bounds. SIAM J. Control Optim. *26* (1988), 4, 863–898.

[19] K. Glover, J. Lam and J. R. Partington: Rational approximation of a class of infinite-dimensional systems. I: singular values of Hankel operators. Math. Control Signals Systems *4* (1990), 325–344.

[20] K. Glover, J. Lam and J. R. Partington: Rational approximation of a class of infinite-dimensional systems. II: optimal convergence rates of $\mathcal{L}^\infty$ approximants. Math. Control Signals Systems *4* (1991), 233–246.

[21] G. Gu, P. P. Khargonekar and E. B. Lee: Approximation of infinite-dimensional systems. IEEE Trans. Automat. Control *34* (1989), 6, 610–618.

[22] A. Ichikawa: The semigroup approach to $\mathcal{H}^2$ and $\mathcal{H}^\infty$-control for sampled-data systems with first-order hold. In: Proceedings of the 35th Conference on Decision and Control, Kobe 1996, pp. 452–457.

[23] K. Ito: Finite-dimensional compensators for infinite-dimensional systems via Galerkin-type approximation. SIAM J. Control Optim. *28* (1990), 6, 1251–1269.

[24] P. T. Kabamba and S. Hara: Worst-case analysis and design of sampled-data control systems. IEEE Trans. Automat. Control *38* (1993), 1337–1357.

[25] E. W. Kamen, P. P. Khargonekar and A. Tannenbaum: Stabilization of time delay systems with finite-dimensional compensators. IEEE Trans. Automat. Control *30* (1985), 75–78.

[26] K. Lenz, H. Ozbay, A. Tannenbaum, J. Turi and B. Morton: Robust control design for a flexible beam using a distributed parameter $\mathcal{H}^\infty$ method. In: CDC, Tampa 1989.

[27] H. Logemann and S. Townley: Adaptive low-gain sampled-data control of DPS. In: Proceedings of the 34th Conference on Decision and Control, New Orleans 1995, pp, 2946–2947.

[28] P. M. Mäkilä: Laguerre series approximation of infinite dimensional systems. Automatica *26* (1990), 6, 985–995.

[29] D. C. McFarlane and K. Glover: Robust Controller Design Using Normalized Coprime Factor Plant Descriptions. Springer–Verlag, Berlin 1990.

[30] A. A. Rodriguez and M. A. Dahleh: Weighted $\mathcal{H}^\infty$ optimization for stable infinite-dimensional systems using finite-dimensional techniques. In: Proceedings of the 29th IEEE CDC, Honolulu 1990, pp. 1814–1820.

[31] A. A. Rodriguez and M. A. Dahleh: On the computation of induced norms for non-compact Hankel operators arising from distributed control problems. Systems Control Lett. *19* (1992), 429–438.
[32] I. G. Rosen: Optimal discrete-time LQR problems for parabolic systems with unbounded input – approximation and convergence. Control Theory Adv. Tech. *5* (1989), 227–300.
[33] I. G. Rosen and C. Wang: On the continuous dependence with respect to sampling of the linear quadratic regulator problem for distributed parameter systems. SIAM J. Control Optim. *30* (1992), 4, 942–974.
[34] I. G. Rosen and C. Wang: On stabilizability and sampling for infinite dimensional systems. IEEE Trans. Automat. Control *37* (1992), 10, 1653–1656.
[35] H. L. Royden: Real Analysis. MacMillan Publishing Co, Inc, 1968.
[36] M. F. Sågfors and H. T. Toivonen: The sampled-data $\mathcal{H}^\infty$ problem: The equivalence of discretization-based methods and a Riccati equation solution. In: Proceedings of the 35th Conference on Decision and Control, Kobe 1996, pp. 428–433.
[37] M. Smith: Well-posedness of $\mathcal{H}^\infty$ optimal control problems. SIAM J. Control Optim. *28* (1990), 342–358.
[38] W. Sun, K. M. Nagpal and P. P. Khargonekar: $\mathcal{H}^\infty$ control and filtering for sampled-data systems. IEEE Trans. Automat. Control *38* (1993), 1162–1175.
[39] B. Sz.-Nagy and C. Foiaş: Harmonic Analysis of Operators on Hilbert Space. North-Holland, Amsterdam 1970.
[40] G. Tadmor: $\mathcal{H}^\infty$ optimal sampled-data control in continuous time systems. Internat. J. Control *56* (1992), 1, 99–141.
[41] H. T. Toivonen: Sampled-data control of continuous-time systems with an $\mathcal{H}^\infty$ optimality criterion. Automatica *28* (1992), 45–54.
[42] M. Vidyasagar: Control Systems Synthesis: A Factorization Approach. MIT Press, Cambrdige MA 1985.
[43] Y. Yamamoto: A function space approach to sampled-data control systems and tracking problem. IEEE Trans. Automat. Control *39* (1994), 703–713.
[44] D. Yue, Y. Liu and S. Xu: Finite-dimensional compensator for a class of uncertain distributed parameter systems. Internat. J. Systems Sci. *26* (1995), 12, 2383–2390.
[45] D. C. Youla, H. A. Jabr, and J. J. Bongiorno: Modern Wiener-Hopf design of optimal controllers. Part 2: The multivariable case. IEEE Trans. Automat. Control *21* (1976), 319–338.

Dr. Delano R. Carter and Dr. Armando A. Rodriguez, Associate Professor, Department of Electrical Engineering, Arizona State University, CSSE, Tempe, AZ 85287-7606. U.S.A.
e-mails: Delano.Carter@asu.edu, aar@asu.edu