Kybernetika

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Kybernetika, Vol. 34 (1998), No. 3, [317]--333

Persistent URL: `http://dml.cz/dmlcz/135210`

KYBERNETIKA — VOLUME 34 (1998), NUMBER 3, PAGES 317–333

SPECTRUM OF RANDOMLY SAMPLED MULTIVARIATE ARMA MODELS

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The paper is devoted to the spectrum of multivariate randomly sampled autoregressive moving-average (*ARMA*) models. We determine precisely the spectrum numerator coefficients of the randomly sampled *ARMA* models. We give results when the non-zero poles of the initial *ARMA* model are simple. We first prove the results when the probability generating function of the random sampling law is injective, then we precise the results when it is not injective.

1. INTRODUCTION

Let $X = (X_t)_{t\in\mathbb{Z}}$ be a discrete-time second order stationary process with zero-mean and values in $\mathbb{R}^k$, satisfying an autoregressive moving-average model. Suppose that the process is sampled by a random walk $T = (T_n)_{n\in\mathbb{Z}}$ with values in $\mathbb{Z}$, independent of X. Denote the randomly sampled process by $\widetilde{X} = (X_{T_n})_{n\in\mathbb{Z}}$. Let us consider the situation where the available data are only from the process $\widetilde{X}$. The problem is to recover the covariance properties of the original process X. According to Shapiro and Silverman [17], we know that the univalence of the sampling probability generating function is sufficient to allow unique recovering of the covariance function of X. Hence the study of the model structure of the process $\widetilde{X}$ arises. Robinson [15] proves that when X is an *ARMA* model, $\widetilde{X}$ is also an *ARMA*.

In a previous paper [12], we obtain the rational spectrum of the process $\widetilde{X}$, when X is a univariate *ARMA* model. We give matrix representations for the spectrum numerator coefficients of $\widetilde{X}$. The *AR* part is given in Robinson [15] in the univariate case, and in Kadi et al [11] in the multivariate case. A functional relation between the poles of X (the roots of the *AR* part) and those of $\widetilde{X}$ is derived. The problem of the zeros of $\widetilde{X}$ (the roots of the *MA* part) still arises in the multivariate case.

In the present paper, we examine the rational spectrum of $\widetilde{X}$ when X is a multivariate *ARMA* model. The spectrum numerator coefficients of $\widetilde{X}$ are expressed through block-matrices. The non-zero poles of the initial model are assumed to be simple.

Another interesting problem in random sampling situation is the estimation of the second order characteristics of the process X using directly the observations from the process $\widetilde{X}$. An extensive literature already exists for this statistical problem in the univariate case; see for instance Bloomfield [1], Brillinger [2], Dunsmuir [3,4,5], Dunsmuir and Robinson [6,7,8], Marshall [13], Parzen [14], Robinson [17], Toloi and Morettin [18].

The organization of the paper is as follows: In Section 2, we introduce some definitions and recall some results about randomly sampled multivariate *ARMA* models. In Section 3, we derive the spectrum of $\widetilde{X}$ for initial *ARMA* process with simple non-zero poles. We examine cases when the sampling probability generating function is injective and when it is non-injective. The numerator spectrum coefficients are given in terms of the initial *ARMA* model parameters and of the sampling distribution convolution law.

2. PRELIMINARIES

Let $\epsilon = (\epsilon_t)_{t\in\mathbb{Z}}$ be a zero-mean white noise, with values in $\mathbb{R}^k$ and Σ_ϵ its covariance matrix.

Let $X = (X_t)_{t\in\mathbb{Z}}$ be a zero-mean second-order stationary process with values in $\mathbb{R}^k$ satisfying the $ARMA(p,q)$ equation:

$$\sum_{j=0}^{p} \Phi_j X_{t-j} = \sum_{j=0}^{q} \Theta_j \epsilon_{t-j}, \quad \forall t \in \mathbb{Z}, \tag{1}$$

where Φ_j $(0 \leq j \leq p)$ and $\Theta_j = (0 \leq j \leq q)$ are the matrix coefficients with $\Phi_0 = \Theta_0 = I_k$.

Denote the AR matrix polynomial by $\Phi(z) = \sum_{j=0}^{p} \Phi_j z^{p-j}$, the MA matrix polynomial by $\Theta(z) = \sum_{j=0}^{q} \Theta_j z^{q-j}$, and, for any square matrix A, we write $|A|$ for the determinant of A and $com A$ for the matrix of cofactors of A. We will refer to the roots of $|\Phi(z)|$ as the poles of the model and to the roots of $|\Theta(z)|$ as the zeros of the model. Denote the spectrum of the process X by $\widehat{C}_X(z) = \sum_{h\in\mathbb{Z}} C_X(h) z^{-h}$, where $C_X(h) = E(X_0 {}^t X_h)$, ${}^t X$ is the transpose of X.

Let $||\cdot||$ denotes any of the norms on the $k \times k$ matrices with complex coefficients. The sequence $(C_X(h))$ is square summable, i.e.,

$$\sum_{h\in\mathbb{Z}} ||C_X(h)||^2 < \infty.$$

Consider now a sampling process $T = (T_n)_{n\in\mathbb{Z}}$ where the random variables $(T_{n+1} - T_n)_{n\in\mathbb{Z}}$ are mutually independent and identically distributed. Denote by L the distribution of $(T_{n+1} - T_n)$ and by $L_j = P(T_{n+1} - T_n = j)$. Let $\widehat{L}(z) = \sum_{j=\ell}^{\infty} L_j z^j$ be the probability generating function of L which is assumed to be defined in a domain including the unit disk; ℓ is the smallest integer such that $L_\ell \neq 0$. Denote by L^{*h} the convolution of the distribution function L with itself, h times.

The sampled process $\widetilde{X} = (X_n)_{n\in\mathbb{Z}}$ is defined by:

$$\widetilde{X}_n = X_{T_n}, \quad n \in \mathbb{Z}. \tag{2}$$

We assume the following assumptions:

$\mathcal{A}_1$) the poles of X are inside the unit circle;

$\mathcal{A}_2$) the zeros of X are inside or on the unit circle;

$\mathcal{A}_3$) Φ and Θ have no common left divisors;

$\mathcal{A}_4$) the matrix Φ_p is of full rank;

$\mathcal{A}_5$) $T_0 = 0$;

$\mathcal{A}_6$) the support of L is $\mathbb{N}^*$;

$\mathcal{A}_7$) the sampling process T is independent of X.

Let us now recall some results on randomly sampled multivariate *ARMA* models (see Kadi et al [11]):

i) The process $\widetilde{X}$ is an *ARMA*.

ii) Since X has a rational spectrum, there exists a in $]0,1[$ such that $\widehat{C}_X$ exists in the ring $]a,a^{-1}[$. Then, the spectrum of $\widetilde{X}$ exists for all z in the ring $]a,a^{-1}[$

$$\widehat{C}_{\widetilde{X}}(z) = \left[\frac{1}{2i\pi}\int_{C_\gamma}\left(\frac{{}^t\widehat{C}_X(x)}{1-z\widehat{L}(x)} + \frac{\widehat{C}_X(x)}{1-z^{-1}\widehat{L}(x)}\right)\frac{\mathrm{d}x}{x}\right] - C_X(0), \tag{3}$$

C_γ is the circle of radius γ with $a < \gamma < \min(|z|,|z|^{-1})$.

iii) There exists a representation of $\widetilde{X}$ whose poles are the non-zero images by $\widehat{L}$ of the non-zero poles of X, with fewer or the same multiplicity orders.

Before stating the next section, we need two technical lemmas which will be useful in the proofs. Denote by J the Jordan partitioned square matrice of finite dimension

$$J = \begin{pmatrix} 0_k & 0_k & 0_k & \dots & 0_k \\ I_k & 0_k & 0_k & \dots & 0_k \\ 0_k & I_k & 0_k & \dots & 0_k \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_k & 0_k & 0_k & I_k & 0_k \end{pmatrix}$$

where O_k and I_k have size $k \times k$. The dimension of J will be specified in every case. Note that the matrix J is nilpotent of order equal to its dimension.

We admit to set as notation

$$J^j = J^{-j}, \quad \text{if } j < 0.$$

Lemma 1. Let P and Q be two matrix polynomials of degree m, $P(z) = \sum_{j=0}^m A_j z^j$ and $Q(z) = \sum_{j=0}^m B_j z^j$, where the size of matrices (A_j) and (B_j) is $k \times k$. Then the matrix coefficients C_j of the expression $P(z)^t Q(z^{-1})$ are as

$$C_j = {}^tAJ^jB, \quad \text{if } j \geq 0$$

and

$$C_j = {}^tA^tJ^jB, \quad \text{if } j < 0$$

with $A = {}^t(A_0, A_1, \ldots, A_m)$ and $B = {}^t(B_0, B_1, \ldots, B_m)$.

Lemma 2. Let P and Q be two matrix polynomials of degree m, $P(z) = \sum_{j=0}^m A_j z^j$ and $Q(z) = \sum_{j=0}^m B_j z^j$, where the size of matrices (A_j) and (B_j) is $k \times k$. Then the matrix coefficients C_j of the expression ${}^tP(z^{-1})Q(z)$ are as

$$C_j = {}^tA^tJ^jB, \quad \text{if } j \geq 0$$

and

$$C_j = {}^tAJ^jB, \quad \text{if } j < 0$$

with $A = {}^t({}^tA_0, {}^tA_1, \ldots, {}^tA_m)$ and $B = {}^t({}^tB_0, {}^tB_1, \ldots, {}^tB_m)$.

3. SPECTRUM OF RANDOMLY SAMPLED *ARMA* MODELS

Let us introduce the following notations.

(r_j) are the simple non-zero poles of X, $|\Phi(x)| = \prod_{j=1}^{kp}(x - r_j)$. $\Phi_1(x) = x^p\Phi(x^{-1})$, $\Theta_1(x) = x^q\Theta(x^{-1})$, and $M(x) = {}^t\big(com\ \Phi(x)\big)\Theta(x)\Sigma_\epsilon{}^t\Theta_1(x)\big(com\ \Phi_1(x)\big)$. The elements of $M(x)$ are polynomials in the variable x.
Set $M = {}^t({}^tM_{q-p}, {}^tM_{q-p-1}, \ldots, {}^tM_0)$ when $q - p \geq 0$. M_j is the coefficient of x^j in the matrix polynomial $M(x)$.

$$R_j = \frac{M(r_j)}{r_j^{q-p+1}\prod_{\ell=1}^{kp}(1 - r_jr_\ell)\prod_{\ell\neq j}(r_j - r_\ell)} \quad \text{and} \quad R = {}^t({}^tR_1, {}^tR_2, \ldots, {}^tR_{kp}).$$

Set $\Psi = {}^t(\psi_0 I_k, \psi_1 I_k, \ldots, \psi_{q-p} I_k)$; ψ_j are the first coefficients of the series $\left[x^{kp}|\Phi(x)|\,|\Phi(x^{-1})|\right]^{-1}$.
$|\widetilde{\Phi}(z)| = \prod_{j=1}^{kp}(z - \widehat{L}(r_j)) = \sum_{i=0}^{kp}\widetilde{\phi}_i z^{kp-i}$ is the determinant of the AR characteristic polynomial of $\widetilde{X}$ and set $\widetilde{\phi} = {}^t(\widetilde{\phi}_0 I_k, \widetilde{\phi}_1 I_k, \ldots, \widetilde{\phi}_{kp} I_k)$.
$|\widetilde{\Phi}(z)|^{(j)} = \prod_{h\neq j}(z - \widehat{L}(r_h)) = \sum_{i=0}^{kp-1}\widetilde{\phi}_i^{(j)} z^{kp-1-i}$ with $\widetilde{\phi}_0^{(j)} = 1$, $\forall j \in \{1, 2, \ldots, kp\}$.

In fact, the polynomial $|\widetilde{\Phi}(z)|^{(j)}$ coincides with the determinant of the characteristic polynomial of the randomly sampled process $\widetilde{X}$ without the root $\widehat{L}(r_j)$.

Let

$$A = \begin{pmatrix} \widetilde{\phi}_0^{(1)}I_k & \widetilde{\phi}_0^{(2)}I_k & \ldots & \widetilde{\phi}_0^{(kp)}I_k \\ \widetilde{\phi}_1^{(1)}I_k & \widetilde{\phi}_1^{(2)}I_k & \ldots & \widetilde{\phi}_1^{(kp)}I_k \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{\phi}_{kp-1}^{(1)}I_k & \widetilde{\phi}_{kp-1}^{(2)}I_k & \ldots & \widetilde{\phi}_{kp-1}^{(kp)}I_k \\ 0_k & 0_k & \ldots & 0_k \end{pmatrix}$$

A is of dimension $(k(kp+1), k(kp))$. $\widetilde{\varphi}_i$ $(-kp \le i \le kp)$ is the coefficient of z^i in the product $|\widetilde{\Phi}(z)|\,|\widetilde{\Phi}(z^{-1})|$ and $\widetilde{\varphi}_i = \widetilde{\varphi}_{-i}$, $\forall i \in \{1, 2, \ldots, kp\}$.

Define $\Delta_L = \sum_{j=\ell}^{q-p} L_j J^j$. Our main result is as follows

Theorem. Assume that the poles of X are simple and that $(q-p) \ge 0$. Then the spectrum of the process $\widetilde{X}$ is

$$\widehat{C}_{\widetilde{X}}(z) = \frac{\sum_{j=-(kp+n)}^{kp+n} \widetilde{V}_j(\Phi, \Theta, L) z^j}{\prod_{j=1}^{kp}(1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))},$$

n is the highest integer such that $n\ell \le q-p$, and

$$\widetilde{V}_j(\Phi, \Theta, L) = \begin{cases} \widetilde{V}_j^{(AR)}(\Phi, \Theta, L) + \widetilde{\Gamma}_{j+n+kp}(\Phi, \Theta, L), & \forall\, j \in \{0, 1, \ldots, kp\} \\ \widetilde{\Gamma}_{j+n+kp}(\Phi, \Theta, L), & \forall\, j \in \{kp+1, \ldots, kp+n\} \end{cases}$$

where

$$\widetilde{V}_j^{(AR)} = {}^tR^tAJ^j\widetilde{\phi} + {}^t\widetilde{\phi}\,{}^tJ^jA \times R - \frac{1}{2}\left[\sum_{h=1}^{kp}({}^tR_h + R_h)\right]{}^t\widetilde{\phi}J^j\widetilde{\phi}, \quad \forall\, j \in \{0, 1, \ldots, kp\},$$

and

$$\widetilde{\Gamma}_j(\Phi, \Theta, L) = \sum_{h=0}^{j}({}^tM\Delta_L^{h-n}\Psi)\widetilde{\varphi}_{kp-(i-h)}, \quad \forall j \in \{0, 1, \cdots, kp+n\}.$$

The numerator spectrum coefficients satisfy

$${}^t\widetilde{V}_{-j} = \widetilde{V}_j, \ \forall\, j \in \{1, 2, \ldots, n+kp\}.$$

Proof. We proceed to the calculation of $\widehat{C}_{\widetilde{X}}$ (see formula (3)) by residues. As X is an $ARMA$ process,

$$\begin{aligned} \widehat{C}_X(x) &= \Phi^{-1}(x)\,\Theta(x)\Sigma_\epsilon{}^t\Theta\left(\frac{1}{x}\right){}^t\Phi^{-1}\left(\frac{1}{x}\right) \\ &= \frac{M(x)}{x^{q-p}\prod_{j=1}^{kp}(x - r_j)\,(1 - r_j x)}. \end{aligned}$$

We have then

$$\begin{aligned} \widehat{C}_{\widetilde{X}}(z) = \Bigg\{ \frac{1}{2i\pi}\int_{C_\gamma} &\left[x^{q-p+1}\prod_{j=1}^{kp}(x - r_j)(1 - r_j x)\right]^{-1} \\ &\cdot\left(\frac{M(x)}{1 - z^{-1}\widehat{L}(x)} + \frac{{}^tM(x)}{1 - z\widehat{L}(x)}\right)\mathrm{d}x\Bigg\} - C_X(0). \end{aligned} \tag{4}$$

For $|x| \leq \gamma$,we have $|\widehat{L}(x)| \leq \gamma < \min(|z|, |z|^{-1})$; therefore $1 - z\widehat{L}(x) \neq 0$ and $1 - z^{-1}\widehat{L}(x) \neq 0$. So the expression under the integral sign has as poles the roots r_j of $|\Phi(x)|$ and zero.

We have to compute the three terms in the right-hand side (RHS) of formula (4):

a)

– The residue at the simple pole r_j is the constant term in the expansion of

$$M(x)\left[x^{q-p+1}\prod_{\ell=1}^{kp}(1-r_\ell x)\prod_{\ell\neq j}(x-r_\ell)\left(1-\frac{\widehat{L}(x)}{z}\right)\right]^{-1}$$

in powers of $(x - r_j)$; thus it is

$$\frac{M(r_j)}{r_j^{q-p+1}\prod_{\ell=1}^{kp}(1-r_j r_\ell)\prod_{\ell\neq j}(r_j-r_\ell)}\times\frac{1}{1-\frac{\widehat{L}(r_j)}{z}}.$$

– The residue at the pole zero (this occurs when $q - p + 1 > 0$) is the coefficient of x^{q-p} in the expansion of

$$M(x)\left[\prod_{j=1}^{kp}(1-xr_j)(x-r_j)(1-\frac{\widehat{L}(x)}{z})\right]^{-1} \tag{5}$$

in powers of x.

Since $(1 - z^{-1}\widehat{L}(x))^{-1}$ may be expanded into:

$$\sum_{h=0}^{\infty}(z^{-1}\widehat{L}(x))^h,$$

and as $\left(\widehat{L}(x)\right)^h = \widehat{L}^{*h}(x) = \sum_{j=h\ell}^{\infty} L_j^{*h}x^j$, the regular part of the expansion of $\left(1 - z^{-1}\widehat{L}(x)\right)^{-1}$ in powers of x at order $(q - p)$ is

$$\sum_{h=0}^{n} z^{-h}\left(\sum_{j=h\ell}^{q-p} L_j^{*h}x^j\right),$$

where n is the highest integer such that $n\ell \leq (q - p)$. Now, to obtain the regular part in the expansion of (5), we need only to express

$$\left(\sum_{j=0}^{q-p}\psi_j x^j\right)\left[\sum_{h=0}^{n} z^{-h}\left(\sum_{j=h\ell}^{q-p} L_j^{*h}x^{j+q-p}\right)\right]\left(\sum_{j=0}^{q-p} M_{-j+(q-p)}x^{-j}\right).$$

We find that the residue at the pole zero is a matrix polynomial in z^{-1} of degree n, and using Lemma 1, the coefficient of z^{-h} is

$$\sum_{j=h\ell}^{q-p} L^{*h}\ {}^t\Psi\ {}^tJ^j\ M$$

but

$$\sum_{j=h\ell}^{q-p} L_j^{*h}\ {}^tJ^j = \left(\sum_{j=\ell}^{q-p} L_j\ {}^tJ^j\right)^h = {}^t\Delta_L^h.$$

Let us denote by $\left(\widetilde{V}_j^{(0)}(\Phi,\Theta,L)\right)_{-n\leq j<0}$ the coefficients in z^{-1} of the residue at zero.

b) In order to compute the second term in the RHS of formula (4), we need to replace $1/z$ by z and $M(x)$ by ${}^tM(x)$.

To derive $\left(\widetilde{V}_j^{(0)}(\Phi,\Theta,L)\right)$ for $0<j\leq n$, we apply Lemma 2. We obtain

$$\widetilde{V}_j^{(0)}(\Phi,\Theta,L) = {}^tM\Delta_L^j\Psi.$$

These coefficients satisfy

$${}^t\widetilde{V}_{-j}^{(0)}(\Phi,\Theta,L) = \widetilde{V}_j^{(0)}(\Phi,\Theta,L), \quad \forall j\in\{1,2,\ldots,n\}$$

and

$$\widetilde{V}_0^{(0)}(\Phi,\Theta,L) = {}^tM\Psi + {}^t\Psi M.$$

c) The last term $C_X(0)$ in the RHS of formula (4) is equal to

$$\frac{1}{4i\pi}\int_{C_\gamma}\left({}^t\widehat{C}_X(x)+\widehat{C}_X(x)\right)\frac{\mathrm{d}x}{x}$$

and the residue of this integral is $\frac{1}{2}\sum_{j=1}^{kp}(R_j+{}^tR_j)$.

It follows from **a**), **b**) and **c**) that the spectrum of the process $\widetilde{X}$ is

$$\sum_{j=1}^{kp}\frac{R_j}{1-z^{-1}\widehat{L}(r_j)}+\sum_{j=1}^{kp}\frac{{}^tR_j}{1-z\widehat{L}(r_j)}-\frac{1}{2}\sum_{j=1}^{kp}(R_j+{}^tR_j)+\sum_{j=-n}^{n}\widetilde{V}_j^{(0)}(\Phi,\Theta,L)z^j.$$

Now, we express the difference

$$\left[\widehat{C}_{\widetilde{X}}(z)-\sum_{j=-n}^{n}\widetilde{V}_j^{(0)}(\Phi,\Theta,L)z^j\right]. \tag{6}$$

After reduction to the same denominator, we obtain as numerator in the matrix expression (6)

$$\begin{aligned}&\left(\sum_{j=1}^{kp}{}^tR_j\prod_{h\neq j}(1-z\widehat{L}(r_h))\right)\prod_h(1-z^{-1}\widehat{L}(r_h)) \\ &\quad+\left(\sum_{j=1}^{kp}R_j\prod_{h\neq j}(1-z^{-1}\widehat{L}(r_h))\right)\prod_h(1-z\widehat{L}(r_h)) \\ &\quad-\frac{1}{2}\left[\sum_{j=1}^{kp}(R_j+{}^tR_j)\right]\prod_h(1-z\widehat{L}(r_h))(1-z^{-1}\widehat{L}(r_h)).\end{aligned} \tag{7}$$

The matrix polynomial $\sum_{j=1}^{kp} {}^tR_j \prod_{h\neq j}(1 - z\widehat{L}(r_h))$ is equal to

$$\sum_{j=1}^{kp} {}^tR_j \prod_{h\neq j}(1 - z\widehat{L}(r_h)) = \sum_{j=1}^{kp} {}^tR_j \left[z^{kp-1}|\widetilde{\Phi}(z^{-1})|^{(j)}\right] = \sum_{i=0}^{kp-1}\left(\sum_{j=1}^{kp} {}^tR_j\widetilde{\phi}_i^{(j)}\right) z^i.$$

Set:

$$\widetilde{R}_i = \sum_{j=1}^{kp} R_j\widetilde{\phi}_i^{(j)}, \quad \forall i \in \{0, 1, \ldots, kp-1\},$$

the partitioned matrix ${}^t({}^t\widetilde{R}_0, {}^t\widetilde{R}_1, \ldots, {}^t\widetilde{R}_{kp-1}, 0_k)$ may be written in a matrix form: $A \times R$ (The matrices A and R are introduced in the notations).

Now we have

$$\left(\sum_{i=0}^{kp-1} {}^t\widetilde{R}_i z^i\right)\prod_h\left(1 - z^{-1}\widehat{L}(r_h))\right) = \left(\sum_{i=0}^{kp-1} \widetilde{R}_i z^i\right)\left[z^{-kp}|\widetilde{\Phi}(z)|\right]$$

$$= \left(\sum_{i=0}^{kp-1} \widetilde{R}_i z^i\right)\left(\sum_{i=0}^{kp}\widetilde{\phi}_i z^{-i}\right) = \sum_{h=-kp}^{kp} \beta_h z^h$$

where by Lemma 1

$$\beta_h = \begin{cases} {}^tR^tAJ^h\widetilde{\phi}, & \text{if } h \geq 0 \\ {}^tR^tA\ {}^tJ^h\widetilde{\phi}, & \text{if } h < 0 \end{cases}$$

In the same way

$$\left[\sum_{j=1}^{kp} R_j \prod_{h\neq j}(1 - z^{-1}\widehat{L}(r_h))\right]\prod_h(1 - z\widehat{L}(r_h))$$

$$= \left(\sum_{i=0}^{kp-1} \widetilde{R}_i z^{-i}\right)\left(\sum_{i=0}^{kp}\widetilde{\phi}_i z^i\right) = \sum_{h=-kp}^{kp} \beta'_h z^h,$$

where

$$\beta'_h = \begin{cases} {}^t\widetilde{\phi}J^hA \times R, & \text{if } h \geq 0 \\ {}^t\widetilde{\phi}\ {}^tJ^hA \times R, & \text{if } h < 0. \end{cases}$$

and

$$\sum_{j=1}^{kp} {}^tR_j \prod_h(1 - z\widehat{L}(r_h))(1 - z^{-1}\widehat{L}(r_h)) = \left[\sum_{j=1}^{kp} {}^tR_j z^{kp}|\widetilde{\Phi}(z^{-1})|\right]\left(\sum_{i=0}^{kp}\widetilde{\phi}_i z^{-i}\right)$$

$$= \left[\sum_{i=0}^{kp}\left(\sum_{j=1}^{kp} {}^tR_j\right)\widetilde{\phi}_i z^i\right]\left(\sum_{i=0}^{kp}\widetilde{\phi}_i z^{-i}\right) = \sum_{h=-kp}^{kp} \beta''_h z^h$$

it follows from Lemma 1 that the coefficients β_h'' of this product are

$$\beta_h'' = \begin{cases} \sum_{j=1}^{kp} {}^tR_j {}^t\widetilde{\phi} J^h \widetilde{\phi}, & \text{if } h \geq 0 \\ \sum_{j=1}^{kp} {}^tR_j {}^t\widetilde{\phi}\, {}^tJ^h \widetilde{\phi}, & \text{if } h < 0. \end{cases}$$

Then it comes that

$$\widehat{C}_{\widetilde{X}}(z) - \sum_{j=-n}^{n} \widetilde{V}_j^{(0)}(\Phi, \Theta, L) z^j = \frac{\sum_{j=-kp}^{kp} \widetilde{V}_j^{(AR)}(\Phi, \Theta, L) z^j}{\prod_{j=1}^{kp}(1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))},$$

with

$$\widetilde{V}_j^{(AR)}(\Phi, \Theta, L) = {}^tR^tAJ^j\widetilde{\phi} + {}^t\widetilde{\phi}\,{}^tJ^jA \times R - \frac{1}{2}\left[\sum_{h=1}^{kp}({}^tR_h + R_h)\right] {}^t\widetilde{\phi}J^j\widetilde{\phi},$$
$$j \in \{0, 1, \ldots, kp\}.$$

The coefficients $\widetilde{V}_j^{(AR)}(\Phi, \Theta, L)$ satisfy

$${}^t\widetilde{V}_{-j}^{(AR)}(\Phi, \Theta, L) = \widetilde{V}_j^{(AR)}(\Phi, \Theta, L), \quad \forall j \in \{1, \ldots, kp\}.$$

This leads to

$$\widehat{C}_{\widetilde{X}}(z) = \frac{\sum_{j=-kp}^{kp} \widetilde{V}_j^{(AR)}(\Phi, \Theta, L) z^j + \left(\sum_{j=-n}^{n} \widetilde{V}_j^{(0)} z^j\right) \prod_{j=1}^{kp}(1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))}{\prod_{j=1}^{p}(1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))}$$

and

$$\begin{aligned} &\left(\sum_{j=-n}^{n} \widetilde{V}_j^{(0)} z^j\right) \prod_{j=1}^{kp}(1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j)) \\ &= \frac{1}{z^{n+kp}} \left(\sum_{i=0}^{2n} \widetilde{V}_{n-i}^{(0)} z^{2n-i}\right) \left(\sum_{i=0}^{2kp} \widetilde{\varphi}_{kp-i} z^{2kp-i}\right) \\ &= \frac{1}{z^{n+kp}} \sum_{i=0}^{2(n+kp)} \widetilde{\Gamma}_i z^i = \sum_{i=-(n+kp)}^{n+kp} \widetilde{\Gamma}_{i+n+kp} z^i \end{aligned}$$

with $\widetilde{\Gamma}_i = \sum_{h=0}^{i} \widetilde{V}_{h-n}^{(0)} \widetilde{\varphi}_{kp-(i-h)}$.

Finally, it comes that

$$\widehat{C}_{\widetilde{X}}(z) = \frac{\sum_{j=-(kp+n)}^{kp+n} \widetilde{V}_j(\Phi, \Theta, L) z^j}{\prod_{j=1}^{kp}(1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))}$$

with

$$\widetilde{V}_j(\Phi,\Theta,L) = \begin{cases} \widetilde{V}_j^{(AR)}(\Phi,\Theta,L) + \widetilde{\Gamma}_{j+n+kp}(\Phi,\Theta,L), & \text{if } j \in \{0,1,\ldots,kp\} \\ \widetilde{\Gamma}_{j+n+kp}(\Phi,\Theta,L), & \text{if } j \in \{kp+1,\ldots,kp+n\}. \end{cases}$$

This concludes the proof of the theorem. □

Remark 1. The matrix J which we use to define Δ_L is of dimension $k(q-p+1) \times k(q-p+1)$ while it is of dimension $k(kp+1) \times k(kp+1)$ elsewhere in this theorem. We keep the notation J for the same type of matrices.

Remark 2. The initial model X has no zero pole owing to assumption $\mathcal{A}_4$. There is no further difficulty to replace this assumption by weaker one: the matrix (Φ_p, Θ_q) is of full rank. In this case, we have to consider zero as a possible pole of X with multiplicity s_0.

Remark 3. When $(q-p) < 0$, there is no pole at 0 and the spectrum of $\widetilde{X}$ is simply given by:

$$\begin{aligned} \widehat{C}_{\widetilde{X}}(z) &= \sum_{j=1}^{kp} \frac{R_j}{1 - z^{-1}\widehat{L}(r_j)} + \sum_{j=1}^{kp} \frac{{}^tR_j}{1 - z\widehat{L}(r_j)} - \frac{1}{2}\sum_{j=1}^{kp}(R_j + {}^tR_j) \\ &= \frac{\sum_{j=-kp}^{kp} \widetilde{V}_j(\Phi,0,L)z^j}{\prod_{j=1}^{kp}(1 - z\widehat{L}(r_j))(1 - z^{-1}\widehat{L}(r_j))} \end{aligned}$$

where

$$\begin{aligned} \widetilde{V}_j(\Phi,0,L) &= {}^tR\,{}^tAJ^j\widetilde{\phi} + {}^t\widetilde{\phi}\,{}^tJ^jA \times R - \frac{1}{2}\left[\sum_{h=1}^{kp}({}^tR_h + R_h)\right]{}^t\widetilde{\phi}J^j\widetilde{\phi}, \\ & \qquad j \in \{0,1,\ldots,kp\}. \end{aligned}$$

Now, let us consider the situation where the probability generating function $\widehat{L}$ is not injective. In this case, the randomly sampled model may be reduced. Denote by $\widehat{L}_1, \widehat{L}_2, \ldots, \widehat{L}_b$ the distinct values in the sequence $\left(\widehat{L}(r_1), \widehat{L}(r_2), \ldots, \widehat{L}(r_p)\right)$. Let us divide the set $\{1,2,\ldots,p\}$ into b distinct and non-empty classes $E_1, E_2, \ldots, E_b$ such that $E_j = \{h/\widehat{L}(r_h) = \widehat{L}_j\}$. Then clearly $\sum_{j=1}^{kp} \frac{R_j}{1-z^{-1}\widehat{L}(r_j)} = \sum_{j=1}^{b} \frac{R'_j}{1-z^{-1}\widehat{L}(r_j)}$ where $R'_j = \sum_{h\in E_j} R_h$.

This leads to Corollary 1 where the matrix A is of dimension $(b+1) \times b$ and $R' = {}^t({}^tR'_1, {}^tR'_2, \ldots, {}^tR'_b)$.

Corollary 1. Assume that the poles of X are simple and that $(q-p) \geq 0$. If b denotes the number of distinct and non zero values of $\widehat{L}$, then the spectrum of the process $\widetilde{X}$ is

$$\widehat{C}_{\widetilde{X}}(z) = \frac{\sum_{j=-(b+n)}^{b+n} \widetilde{V}_j(\Phi,\Theta,L) z^j}{\prod_{j=1}^{b}(1-z\widehat{L}_j)(1-z^{-1}\widehat{L}_j)},$$

n is the highest integer such that $n\ell \leq q-p$, and

$$\widetilde{V}_j(\Phi,\Theta,L) = \begin{cases} \widetilde{V}_j^{(AR)}(\Phi,\Theta,L) + \widetilde{\Gamma}_{j+n+b}(\Phi,\Theta,L), & \forall j \in \{0,1,\ldots,b\} \\ \widetilde{\Gamma}_{j+n+b}(\Phi,\Theta,L), & \forall j \in \{b+1,\ldots,b+n\} \end{cases}$$

where:

$$\widetilde{V}_j^{(AR)} = {}^t R^t A J^j \widetilde{\phi} + {}^t \widetilde{\phi}^t J^j A \times R - \frac{1}{2}\left[\sum_{h=1}^{kp}({}^t R_h + R_h)\right] {}^t \widetilde{\phi} J^j \widetilde{\phi}, \quad \forall j \in \{0,1,\ldots,b\},$$

and

$$\widetilde{\Gamma}_j(\Phi,\Theta,L) = \sum_{h=0}^{j} \left({}^t M \Delta_L^{h-n} \Psi\right) \widetilde{\varphi}_{b-(i-h)}, \quad \forall j \in \{0,1,\ldots,b+n\}.$$

The numerator spectrum coefficients satisfy

$${}^t\widetilde{V}_{-j} = \widetilde{V}_j, \quad \forall j \in \{1,2,\ldots,n+b\}.$$

Let us consider the $AR(p)$ models.

Corollary 2. Assume that $q=0$ and that the poles of X are simple. Then the spectrum of the process $\widetilde{X}$ is

$$\widehat{C}_{\widetilde{X}}(z) = \frac{\sum_{j=-kp}^{kp} \widetilde{V}_j(\Phi,0,L) z^j}{\prod_{j=1}^{kp}(1-z\widehat{L}(r_j))(1-z^{-1}\widehat{L}(r_j))},$$

where

$$\widetilde{V}_j(\Phi,0,L) = {}^t R^t A J^j \widetilde{\phi} + {}^t \widetilde{\phi}\, {}^t J^j A \times R - \frac{1}{2}\left[\sum_{h=1}^{kp}({}^t R_h + R_h)\right] {}^t \widetilde{\phi} J^j \widetilde{\phi}, \quad \forall j \in \{0,1,\ldots,kp\}.$$

The numerator spectrum coefficients satisfy

$${}^t\widetilde{V}_{-j} = \widetilde{V}_j, \quad \forall j \in \{1,2,\ldots,kp\}.$$

P r o o f. In this case, we have no pole at 0 and the spectrum of $\widetilde{X}$ is as in Remark 3. □

Let us now consider the $MA(q)$ models. Denote $\Omega = {}^t\left(\Sigma_\epsilon^{\frac{1}{2}}, \Theta_1\Sigma_\epsilon^{\frac{1}{2}}, \ldots, \Theta_q\Sigma_\epsilon^{\frac{1}{2}}\right)$.

Corollary 3. Assume that $p = 0$. Then the spectrum of the process $\widetilde{X}$ is

$$\widehat{C}_{\widetilde{X}}(z) = \sum_{j=-n}^{n} \widetilde{V}_j(0, \Theta, L) z^j$$

where n is the highest integer such that $n\ell \leq q$ and the coefficients $\widetilde{V}_j$ are quadratic in the parameter matrices Θ_j,

$$\widetilde{V}_j(0, \Theta, L) = {}^t\Omega \Delta_L^j \Omega, \quad \forall j \in \{0, 1, 2, \ldots, n\}.$$

These coefficients satisfy

$${}^t\widetilde{V}_{-j}(0, \Theta, L) = \widetilde{V}_j(0, \Theta, L), \quad \forall j \in \{1, 2, \ldots, n\}.$$

Proof. For $p = 0$, we obtain

$$\begin{aligned} \widetilde{V}_j(0, \Theta, L) &= \widetilde{\Gamma}_{j+n}(0, \Theta, L), \quad \forall j \in \{1, \ldots, n\} \\ &= \sum_{h=0}^{j+n} \widetilde{V}_{h-n}^{(0)} \widetilde{\varphi}_{-(j+n-h)}. \end{aligned}$$

All the terms $\widetilde{\varphi}_{-(j+n-h)}$ vanish except when $h = j + n$. So

$$\widetilde{V}_j(0, \Theta, L) = \widetilde{V}_j^{(0)}(0, \Theta, L) = {}^tM \Delta_L^j \Psi.$$

Given that the matrix Δ_L^j has the form

$$\begin{pmatrix} 0_{j\ell k} & 0_{j\ell k} & \ldots & 0_k & 0_k \\ L_{j\ell}^{*j} I_k & 0_k & \ldots & 0_k & 0_k \\ L_{j\ell+1}^{*j} I_k & L_{j\ell}^{*j} I_k & 0_k & \ldots & 0_k \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ L_q^{*j} I_k & L_{q-1}^{*j} I_k & L_{j\ell+1}^{*j} I_k & L_{j\ell}^{*j} I_k & {}^t0_{j\ell k} \end{pmatrix},$$

where the matrix $0_{j\ell k}$ is as $0_{j\ell k} = {}^t(\underbrace{0_k, \ldots, 0_k}_{j\ell \text{ times}})$; we obtain,

$$\widetilde{V}_j(0, \Theta, L) = \sum_{h=j\ell}^{q} L_h^{*j} \, {}^tM_{q-h}.$$

The initial model is a $MA(q)$, so the matrix $M(x)$ is as follows

$$M(x) = \Theta(x) \Sigma_\epsilon {}^t\Theta_1(x) = x^q \Theta(x) \Sigma_\epsilon {}^t\Theta(x^{-1}) = x^q \sum_{j=-q}^{q} C(j) x^j.$$

Therefore the matrix coefficients $M_0, M_1, \ldots, M_q$ are respectively equal to $C(-q)$, $C(-q+1), \ldots, C(0)$ and the matrix covariances may be written by Lemma 1

$$C(j) = \begin{cases} {}^t\Omega J^j \Omega, & \text{if } j \geq 0 \\ {}^t\Omega \, {}^tJ^j \Omega, & \text{if } j < 0. \end{cases}$$

Hence

$$\widetilde{V}_j(0, \Theta, L) = \sum_{h=j\ell}^{q} L_h^{*j} \, {}^tC(-h) = {}^t\Omega \left(\sum_{h=j\ell}^{q} L_h^{*j} J^h \right) \Omega = {}^t\Omega \Delta_L^j \Omega. \qquad \square$$

4. NUMERICAL EXAMPLES

In this section, let us examine some simple cases. The process considered is two-dimensional and the sampling law is such that $L_1 \neq 0$.

- Let X be a first order moving average process: $X_t = \epsilon_t + \Theta \epsilon_{t-1}$, where $\Theta = \begin{pmatrix} 0.5 & -1 \\ 0 & 0.5 \end{pmatrix}$ and $\Sigma_\epsilon = I_2$.

  The spectrum of the process X is given by:

$$\widehat{C}_X(z) = C_X(-1)z^{-1} + C_X(0) + C_X(1)z = \Sigma_\epsilon {}^t\Theta z^{-1} + \left(\Sigma_\epsilon + \Theta \Sigma_\epsilon {}^t\Theta\right) + \Theta \Sigma_\epsilon z.$$

So

$$z\widehat{C}_X(z) = \begin{pmatrix} \frac{1}{2}z^2 + \frac{9}{4}z + \frac{1}{2} & -z^2 - \frac{1}{2}z \\ -\frac{1}{2}z - 1 & \frac{1}{2}z^2 + \frac{5}{4}z + \frac{1}{2} \end{pmatrix}.$$

The spectrum of the process $\widetilde{X}$ is obtained by applying Corollary 3:

$$\begin{aligned} \widehat{C}_{\widetilde{X}}(z) &= \widetilde{V}_{-1}(0, \Theta, L)z^{-1} + \widetilde{V}_0(0, \Theta, L) + \widetilde{V}_1(0, \Theta, L)z \\ &= L_1 \Sigma_\epsilon {}^t\Theta z^{-1} + \left(\Sigma_\epsilon + \Theta \Sigma_\epsilon {}^t\Theta\right) + L_1 \Theta \Sigma_\epsilon z. \end{aligned}$$

So

$$z\widehat{C}_{\widetilde{X}}(z) = \begin{pmatrix} \frac{1}{2}L_1 z^2 + \frac{9}{4}z + \frac{1}{2}L_1 & -L_1 z^2 - \frac{1}{2}z \\ -\frac{1}{2}z - L_1 & \frac{1}{2}L_1 z^2 + \frac{5}{4}z + \frac{1}{2}L_1 \end{pmatrix}.$$

In Tables 1 and 2, we compute the zeros of the process $\widetilde{X}$ for different values of L_1.

a) When the sampling distribution is a Bernoulli law, we have
$L_j = p^{j-1}(1-p)^{2-j}, \quad j \in \{1, 2\}$.

a) When the sampling distribution is a Poisson law, we have
$L_j = e^{-\lambda} \frac{\lambda^{j-1}}{(j-1)!}, \quad j \geq 1$.

Remark. In all the tables, we only report the models with zeros inside the unit disk.

Table 1. Bernoulli law.

L_1	zeros	modules of zeros
0.1	$-0.0245 \pm 0.0195i$	0.0313
0.2	$-0.0494 \pm 0.0391i$	0.0630
0.3	$-0.0755 \pm 0.0587i$	0.0956
0.4	$-0.1033 \pm 0.0783i$	0.1296
0.5	$-0.1338 \pm 0.0979i$	0.1658
0.6	$-0.1684 \pm 0.1173i$	0.2053
0.7	$-0.2096 \pm 0.1357i$	0.2497
0.8	$-0.2621 \pm 0.1512i$	0.3026
0.9	$-0.3381 \pm 0.1558i$	0.3722

We notice, from Table 1, that the zeros of the sampled process $\widetilde{X}$, are more stable than those of the process X. We also see, that the more L_1 is small, the more these zeros are stable. So the more the sampling process has increments of longer 2, the more the zeros of $\widetilde{X}$ are stable. In Kadi [10], we study by means of numerical examples the behaviour of the zeros of $\widetilde{X}$ in relation with the zeros and the poles of X in the univariate case. The same properties are reported.

Table 2. Poisson law.

λ	L_1	zeros	modules of zeros
0.1	0.9048	$-0.3427 \pm 0.1553i$	0.3764
0.3	0.7408	$-0.2293 \pm 0.1426i$	0.2700
0.4	0.6703	$-0.1965 \pm 0.1304i$	0.2358
0.6	0.5488	$-0.1501 \pm 0.1074i$	0.1846
1	0.3679	$-0.0941 \pm 0.0720i$	0.1185
3	0.0498	$-0.0122 \pm 0.0097i$	0.0156
6	0.0025	$(-06046 \pm 0.4837i) \times 10^{-3}$	0.7742×10^{-3}
9	1.2341×10^{-4}	$(-0.3010 \pm 0.2408i) \times 10^{-4}$	0.3855×10^{-4}

We notice the same behaviour as in the case of the Bernoulli law.

- Let X be a first order autoregressive process: $X_t + \Phi X_{t-1} = \epsilon_t$, where $\Phi = \begin{pmatrix} 0.25 & 1 \\ 0 & -0.5 \end{pmatrix}$ and $\Sigma_\epsilon = I_2$.

The spectrum of the process $\widetilde{X}$ is obtained by applying Corollary 2:

$$\prod_{j=1}^{2}\left(1-z\widehat{L}(r_j)\right)\left(1-z^{-1}\widehat{L}(r_j)\right)\widehat{C}_{\widetilde{X}}(z)=\sum_{j=-2}^{2}\widetilde{V}_j(\Phi,0,L)z^j.$$

To compute the coefficients $\widetilde{V}_j$, we need:

$$\begin{aligned}
M(x) &= \begin{pmatrix} -\frac{1}{2}x^2+\frac{9}{4}x-\frac{1}{2} & -1-\frac{1}{4}x \\ -x^2-\frac{1}{4}x & \frac{1}{4}x^2+\frac{17}{16}x-\frac{1}{4}\end{pmatrix} \\
|\widetilde{\Phi}(z)| &= \widetilde{\phi}_0 z^2+\widetilde{\phi}_1 z+\widetilde{\phi}_2 \\
&= z^2-\left(\widehat{L}(r_1)+\widehat{L}(r_2)\right)z+\widehat{L}(r_1)\widehat{L}(r_2) \\
A &= \begin{pmatrix} I_2 & I_2 \\ -\widehat{L}(r_2)I_2 & -\widehat{L}(r_1)I_2 \\ 0_2 & 0_2 \end{pmatrix}
\end{aligned}$$

$$R_1=\frac{M(r_1)}{(1-r_1^2)(1-r_1r_2)(r_1-r_2)} \quad\text{and}\quad R_2=\frac{M(r_2)}{(1-r_2^2)(1-r_1r_2)(r_2-r_1)}.$$

Then we obtain:

$$\begin{aligned}
\widetilde{V}_0(\Phi,0,L) &= \frac{1}{2}(R_1+R_2+{}^tR_1+{}^tR_2)-(\widetilde{\phi}_1\widehat{L}(r_2)+\frac{1}{2}\widetilde{\phi}_1^2+\frac{1}{2}\widetilde{\phi}_2^2)(R_1+{}^tR_1) \\
&\quad -(\widetilde{\phi}_1\widehat{L}(r_1)+\frac{1}{2}\widetilde{\phi}_1^2+\frac{1}{2}\widetilde{\phi}_2^2)(R_2+{}^tR_2) \\
\widetilde{V}_1(\Phi,0,L) &= -(\widehat{L}(r_2)+\frac{1}{2}\widetilde{\phi}_1+\frac{1}{2}\widetilde{\phi}_1\widetilde{\phi}_2)(R_1+{}^tR_1)-(\widehat{L}(r_1)+\frac{1}{2}\widetilde{\phi}_1 \\
&\quad +\frac{1}{2}\widetilde{\phi}_1\widetilde{\phi}_2)(R_2+{}^tR_2) \\
\widetilde{V}_2(\Phi,0,L) &= -\frac{1}{2}\widetilde{\phi}_2\left(R_1+R_2+{}^tR_1+{}^tR_2\right).
\end{aligned}$$

In Tables 3 and 4, we compute the poles and the zeros of the process $\widetilde{X}$ for different values of the Bernoulli parameter p and the Poisson parameter λ.

a) When the sampling distribution is a Bernoulli law, we have $\widehat{L}(z)=(1-p)z+pz^2$.

b) When the sampling distribution is a Poisson law, we have $\widehat{L}(z)=z\exp(\lambda(z-1))$.

Table 3. Bernoulli law.

p	poles	zeros
0.1	−0.2188; 0.4750	−0.1392; 0.1409
0.2	−0.1875; 0.4500	−0.1228; 0.1242
0.3	−0.1563; 0.4250	−0.1055; 0.1065
0.4	−0.1250; 0.4000	−0.0871; 0.0878
0.5	−0.093; 0.3750	−0.676; 0.0679; 0.7280
0.6	−0.0625; 0.3500	−0.466; 0.0468; 0.5559
0.7	−0.0313; 0.3250	−0.8662; −0.0242; 0.0242; 0.4575
0.8	1.387×10^{-17}; 0.3000	−0.5467; 0.0000; 0.0000; 0.3862
0.9	0.0313; 0.2750	−0.4207; −0.0261; 0.0260; 0.3295

Table 4. Poisson law.

λ	poles	zeros
0.1	−0.2206; 0.4756	−0.1401; 0.1418
0.3	−0.1718; 0.4304	−0.1139; 0.1151
0.4	−0.1516; 0.4094	−0.1224; 0.1033
0.6	−0.1181; 0.3704	−0.0822; 0.0828; 0.9973
0.9	−0.0812; 0.3188	−0.0585; 0.0587; 0.5320
1	−0.0716; 0.3033	−0.8435; −0.0521; 0.0523; 0.4825
3	−0.0059; 0.1116	−0.1472; −0.0046; 0.0046; 0.1425
6	-1.3827×10^{-4}; 0.0249	−0.0314; 0.0313
9	-3.2518×10^{-6}; 0.0056	−0.0063; 0.0069

In the univariate case, we observe on many examples that the zeros of $\widetilde{X}$ are more stable than those of X. This property needs to be more studied to determine the conditions under which this property holds.

(Received September 19, 1996.)

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