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ESTIMATING THE FUZZY INEQUALITY ASSOCIATED WITH A FUZZY RANDOM VARIABLE IN RANDOM SAMPLINGS FROM FINITE POPULATIONS[1]

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In a recent paper we have introduced the fuzzy hyperbolic inequality index, to quantify the inequality associated with a fuzzy random variable in a finite population. In previous papers, we have also proven that the classical hyperbolic inequality index associated with real-valued random variables in finite populations can be unbiasedly estimated in random samplings.

The aim of this paper is to analyze the problem of estimating the population fuzzy hyperbolic index associated with a fuzzy random variable in random samplings from finite populations. This analysis will lead us to conclude that an unbiased (up to additive equivalences) estimator of the population fuzzy hyperbolic inequality index can be constructed on the basis of the sample index and the expected value of the values fuzzy hyperbolic inequality in the sample.

1. INTRODUCTION

The fuzzy hyperbolic inequality index has been introduced (Corral et al [4]) to measure the inequality associated with attributes which cannot be properly modeled by means of classical random variables, but rather they could be more suitably formalized in terms of fuzzy random variables (Puri and Ralescu [17]). This type of attributes arises often in Social Sciences and Psychology, where most of the involved variables are linguistic, but examples of them can be found also in Economics and Life Sciences (see, for instance, Gil et al [8], Cox [5], Gil and López–Díaz [9]).

On the other hand, in studies concerning the estimation from complete data of the inequality associated with a classical random variable in a large uncensused finite population, we have concluded that in random samplings with and without replacement (Gil et al [11]), stratified random sampling (Caso and Gil [2, 3], Gil and Martínez [10]) and single-stage cluster sampling (Gil and Gil [7]), the hyperbolic inequality index (which is an additively decomposable index, [1], and belonging to

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Gastwirth's family, [6]) behaves appropriately, so that unbiased estimates of the population index can be defined on the basis of the sample one.

In the present paper we are going to extend the last conclusion to the fuzzy case, by defining unbiased fuzzy estimates of the population fuzzy hyperbolic inequality index, where unbiasedness is intended in the sense of the fuzzy expected value (Puri and Ralescu [17]), and it holds up to additive equivalences (Mareš [14–16]).

2. PRELIMINARY CONCEPTS

Fuzzy random variables have been introduced by Puri and Ralescu [17] to formalize fuzzy-valued quantification processes associated with the outcomes of the performance of a random experiment.

Let $\mathbb{X}$ be a subset of the Euclidean space $\mathbb{R}^n$ ($n \geq 1$), and let $\mathcal{F}_c(\mathbb{X})$ be the class of fuzzy sets of $\mathbb{X}$, $\tilde{V} : \mathbb{X} \longrightarrow [0,1]$, such that: i) $\tilde{V}$ is upper semicontinuous, ii) the inverse image of the singleton $\{1\}$ is nonempty, iii) for each $\alpha \in (0,1]$ the α-cut set, $\tilde{V}_\alpha = \{x \in \mathbb{X}; \tilde{V}(x) \geq \alpha\}$, is convex, and iv) $\tilde{V}_0 = cl\,\{x \in \mathbb{X}; \tilde{V}(x) > 0\}$ is compact.

Given a measurable space $(\Omega, \mathcal{A})$, a *fuzzy random variable* associated with $(\Omega, \mathcal{A})$ is a $(\mathcal{A}, B_{d_\infty})$ measurable function $\mathcal{X} : \Omega \longrightarrow \mathcal{F}_c(\mathbb{R}^n)$, where B_{d_∞} is the σ-field generated by the open balls of $\mathcal{F}_c(\mathbb{R}^n)$ defined in terms of the d_∞ metric.

If $\mathcal{X}$ is a positive fuzzy random variable defined on a finite population of N individuals or units, $U_1, \ldots, U_N$, for which it takes on the values $\tilde{x}_1 = \mathcal{X}(U_1), \ldots, \tilde{x}_N = \mathcal{X}(U_N)$, with $\tilde{x}_1, \ldots, \tilde{x}_N \in \mathcal{F}_c\big((0,+\infty)\big)$, then the *fuzzy expected value* of $\mathcal{X}$ in the population is given (Puri and Ralescu [17]) by

$$\tilde{E}(\mathcal{X}) = \frac{1}{N} \odot (\tilde{x}_1 \oplus \ldots \oplus \tilde{x}_N),$$

where $\oplus$ and $\odot$ are the extension (see, for instance, Kaufmann and Gupta [13], Mareš [16]) of the addition and product of real numbers, respectively, based on Zadeh's extension principle [18].

On the other hand, the *population fuzzy hyperbolic index* associated with $\mathcal{X}$ is given (Corral et al [4]) by

$$\tilde{I}_H(\mathcal{X}) = \frac{1}{N} \odot \left[\left(\tilde{E}(\mathcal{X}) \oslash \tilde{x}_1\right) \oplus \ldots \oplus \left(\tilde{E}(\mathcal{X}) \oslash \tilde{x}_N\right)\right] \ominus 1,$$

where $\oslash$ and $\ominus$ mean the extension of the division and substraction of real numbers, respectively.

3. ESTIMATING THE POPULATION FUZZY HYPERBOLIC INEQUALITY IN RANDOM SAMPLING WITH REPLACEMENT

Let $\mathcal{X}$ be a positive fuzzy random variable which in a population of N individuals or sampling units, $U_1, \ldots, U_N$, for which it takes on the values $\tilde{x}_1 = \mathcal{X}(U_1), \ldots, \tilde{x}_N = \mathcal{X}(U_N)$, with $\tilde{x}_1, \ldots, \tilde{x}_N \in \mathcal{F}_c\big((0,+\infty)\big)$. Assume that a sample of size n is chosen at random and with replacement from the overall population. Let $[\tau]$ denote this generic sample and let $U_{\tau 1}, \ldots, U_{\tau n}$ be the individuals or units in it. Then,

Definition 3.1. The *sample fuzzy hyperbolic inequality index* associated with $\mathcal{X}$ in $[\tau]$ is given by

$$\tilde{I}_H(\mathcal{X}[\tau]) = \frac{1}{n} \odot \left[\left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_{\tau 1}\right) \oplus \ldots \oplus \left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_{\tau n}\right)\right] \ominus 1,$$

where $\tilde{x}_{\tau i} = \mathcal{X}(U_{\tau i})$ for all $i \in \{1, \ldots, n\}$, and

$$\tilde{E}(\mathcal{X}[\tau]) = \frac{1}{n} \odot (\tilde{x}_{\tau 1} \oplus \ldots \oplus \tilde{x}_{\tau n}).$$

The sample index $\tilde{I}_H(\mathcal{X}[\cdot])$ is defined on the space of the $CR_{N,n} = \binom{N+n-1}{n}$ distinct possible random samples with replacement of size n from the given population. On the basis of this sample index, we can construct an estimator whose fuzzy expected value on the space of the $CR_{N,n}$ samples can be related to the population index as follows:

Theorem 3.1. In random sampling with replacement of size n from the population of N individuals or sampling units, the fuzzy estimator $\frac{n}{n-1} \odot \tilde{I}_H(\mathcal{X}[\cdot])$ such that

$$\frac{n}{n-1} \odot \tilde{I}_H(\mathcal{X}[\tau]) = \frac{1}{n-1} \odot \left[\left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_{\tau 1}\right) \oplus \ldots \oplus \left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_{\tau n}\right)\right] \ominus \frac{n}{n-1},$$

for all $\tau \in \{1, \ldots, CR_{N,n}\}$, satisfies that

$$\tilde{E}\left(\frac{n}{n-1} \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) = \tilde{I}_H(\mathcal{X}) \oplus \left(\frac{1}{N(n-1)} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right),$$

(where the last fuzzy expected value, $\tilde{E}$, is intended to be applied on the space of the $CR_{N,n}$ random samples with replacement of size n).

Proof. Indeed,

$$\tilde{E}\left(\frac{n}{n-1} \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) = \left[p([1]) \odot \left(\frac{n}{n-1} \odot \tilde{I}_H(\mathcal{X}[1])\right)\right]$$
$$\oplus \ldots \oplus \left[p([CR_{N,n}]) \odot \left(\frac{n}{n-1} \odot \tilde{I}_H(\mathcal{X}[CR_{N,n}])\right)\right],$$

where $p([\tau])$ = probability of choosing the sample $[\tau]$, $\tau = 1, \ldots, CR_{N,n}$.

On the basis of the properties of the fuzzy operations $\oplus$ and $\odot$ (see, for instance, Kaufmann and Gupta [13], Mareš [16]), and because of the positiveness of $\mathcal{X}$, we have that

$$\tilde{I}_H(\mathcal{X}[\tau]) = \frac{1}{n} \odot \left[\left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_{\tau 1}\right) \oplus \ldots \oplus \left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_{\tau n}\right)\right] \ominus 1$$
$$= \frac{1}{n} \odot \left(\left[\left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_1\right) \odot t_1[\tau]\right] \oplus \ldots \oplus \left[\left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_N\right) \odot t_N[\tau]\right]\right) \ominus 1,$$

where $t_j[\tau]$ is the number of times that U_j appears in the sample $[\tau]$, $j = 1, \ldots, n$, and $\tau = 1, \ldots, CR_{N,n}$.

Since

$$\begin{aligned}
\tilde{E}(\mathcal{X}[\tau]) &= \frac{1}{n} \odot (\tilde{x}_{\tau 1} \oplus \ldots \oplus \tilde{x}_{\tau n}) \\
&= \frac{1}{n} \odot \big((\tilde{x}_1 \odot t_1[\tau]) \oplus \ldots \oplus (\tilde{x}_N \odot t_N[\tau]) \big),
\end{aligned}$$

then,

$$\begin{aligned}
\tilde{I}_H(\mathcal{X}[\tau]) = \frac{1}{n^2} \odot \{ [((\tilde{x}_1 \oslash \tilde{x}_1) \odot t_1[\tau] t_1[\tau]) \oplus \ldots \oplus ((\tilde{x}_N \oslash \tilde{x}_1) \odot t_N[\tau] t_1[\tau])] \\
\oplus \ldots \oplus [((\tilde{x}_1 \oslash \tilde{x}_N) \odot t_1[\tau] t_N[\tau]) \oplus \ldots \oplus ((\tilde{x}_N \oslash \tilde{x}_N) \odot t_N[\tau] t_N[\tau])] \} \ominus 1.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\tilde{E}\left(\frac{n}{n-1} \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) = \frac{1}{n(n-1)} \odot \{ [((\tilde{x}_1 \oslash \tilde{x}_1) \odot E(t_1[\cdot] t_1[\cdot])) \\
\oplus \ldots \oplus ((\tilde{x}_N \oslash \tilde{x}_1) \odot E(t_N[\cdot] t_1[\cdot]))] \oplus \ldots \oplus [((\tilde{x}_1 \oslash \tilde{x}_N) \odot E(t_1[\cdot] t_N[\cdot])) \\
\oplus \ldots \oplus ((\tilde{x}_N \oslash \tilde{x}_N) \odot E(t_N[\cdot] t_N[\cdot]))] \} \ominus \left(\frac{n}{n-1} \sum_{\tau=1}^{CR_{N,n}} p([\tau]) \right),
\end{aligned}$$

where the classical expected values $E(t_j[\cdot] t_{j'}[\cdot])$, for all $j, j' \in \{1, \ldots, N\}$, $j \neq j'$, are defined on the space of the $CR_{N,n}$ random samples with replacement of size n from the population.

The probability that the individual or unit U_j is drawn equals $1/N$ in each draw, so that the classical random variable $t_j[\cdot]$ is distributed on this space as a binomial $B(n, \frac{1}{N})$, and jointly the variables $t_1[\cdot], \ldots, t_N[\cdot]$ follow a multinomial distribution $M(n, \frac{1}{N}, \ldots, \frac{1}{N})$. Therefore,

$$\begin{aligned}
E(t_j[\cdot]) &= \frac{n}{N}, \quad E(t_j[\cdot] t_{j'}[\cdot]) = \frac{n(n-1)}{N^2}, \\
E(t_j[\cdot] t_j[\cdot]) &= \frac{n(n-1)}{N^2} + \frac{n}{N}, \quad j, j' \in \{1, \ldots, N\},\ j \neq j',
\end{aligned}$$

so that,

$$\begin{aligned}
&\tilde{E}\left(\frac{n}{n-1} \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) \\
&= \left(\frac{1}{N^2} \odot \{ [(\tilde{x}_1 \oslash \tilde{x}_1) \oplus \ldots \oplus (\tilde{x}_N \oslash \tilde{x}_1)] \oplus \ldots \oplus [(\tilde{x}_1 \oslash \tilde{x}_N) \oplus \ldots \oplus (\tilde{x}_N \oslash \tilde{x}_N)] \} \right) \\
&\oplus \left(\frac{1}{N(n-1)} \odot [(\tilde{x}_1 \oslash \tilde{x}_1) \oplus \ldots \oplus (\tilde{x}_N \oslash \tilde{x}_N)] \right) \ominus \frac{n}{n-1} \\
&= \left(\frac{1}{N} \odot \left[\left(\tilde{E}(\mathcal{X}) \oslash \tilde{x}_1 \right) \oplus \ldots \oplus \left(\tilde{E}(\mathcal{X}) \oslash \tilde{x}_N \right) \right] \ominus 1 \right)
\end{aligned}$$

$$\oplus \left(\frac{1}{N(n-1)} \odot \left[(\tilde{x}_1 \oslash \tilde{x}_1) \oplus \ldots \oplus (\tilde{x}_N \oslash \tilde{x}_N) \ominus N\right]\right)$$
$$= \tilde{I}_H(\mathcal{X}) \oplus \left(\frac{1}{N(n-1)} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right). \qquad \square$$

On the other hand, the remaining term

$$\frac{1}{N(n-1)} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]$$

(which could be viewed as a kind of "fuzzy bias") can be estimated as follows:

Theorem 3.2. In random sampling with replacement of size n from the population of N individuals or sampling units, the fuzzy estimator $\tilde{I}_H^{wv}(\mathcal{X}[\cdot])$ such that

$$\tilde{I}_H^{wv}(\mathcal{X}[\tau]) = \frac{1}{n} \odot \left[\tilde{I}_H(\{\tilde{x}_{\tau 1}\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_{\tau n}\})\right]$$

for all $\tau \in \{1, \ldots, CR_{N,n}\}$ (i.e., $\tilde{I}_H^{wv}(\mathcal{X}[\tau])$ = sample fuzzy expected value of the fuzzy *intra*values hyperbolic inequality index), satisfies that

$$\tilde{E}\left(\tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right) = \frac{1}{N} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right],$$

that is, its fuzzy expected value equals the population fuzzy expected value of the fuzzy hyperbolic inequality index within values.

Proof. Indeed,

$$\begin{aligned}\tilde{I}_H^{wv}(\mathcal{X}[\tau]) &= \frac{1}{n} \odot \left[\tilde{I}_H(\{\tilde{x}_{\tau 1}\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_{\tau n}\})\right] \\ &= \frac{1}{n} \odot \left[\left(\tilde{I}_H(\{\tilde{x}_1\}) \odot (t_1[\tau])\right) \oplus \ldots \oplus \left(\tilde{I}_H(\{\tilde{x}_N\}) \odot (t_N[\tau])\right)\right],\end{aligned}$$

so that if we consider its fuzzy expected value over all random samples with replacement of size n, $[\tau]$, with $\tau = 1, \ldots, CR_{N,n}$, we obtain that

$$\tilde{E}\left(\tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right) = \left[p([1]) \odot \left(\tilde{I}_H^{wv}(\mathcal{X}[1])\right)\right]$$
$$\oplus \ldots \oplus \left[p([CR_{N,n}]) \odot \left(\tilde{I}_H^{wv}(\mathcal{X}[CR_{N,n}])\right)\right],$$

whence

$$\begin{aligned}\tilde{E}\left(\tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right) &= \frac{1}{n} \odot \left[\left(\tilde{I}_H(\{\tilde{x}_1\}) \odot E(t_1[\cdot])\right) \oplus \ldots \oplus \left(\tilde{I}_H(\{\tilde{x}_N\}) \odot E(t_N[\cdot])\right)\right] \\ &= \frac{1}{N} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]. \qquad \square\end{aligned}$$

As a consequence of the preceding results, we now construct a fuzzy estimator of $\tilde{I}_H(\mathcal{X})$, which is unbiased up to additive equivalences (in Mareš' sense [14–16]).

Theorem 3.3. In random sampling with replacement of size n from the population of N individuals or sampling units, the fuzzy estimator

$$\hat{\tilde{I}_H(\mathcal{X})}([\cdot]) = \frac{1}{n-1} \odot \left[\left(n \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) \ominus \tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right],$$

satisfies that

$$\tilde{E}\left(\hat{\tilde{I}_H(\mathcal{X})}([\cdot])\right) \sim_{\oplus} \tilde{I}_H(\mathcal{X}),$$

($\sim_{\oplus}$ meaning the additive equivalence).

Proof. Indeed, on the basis of Theorems 3.1 and 3.2, and because of the properties of the fuzzy operations, we have that

$$\begin{aligned}
& \tilde{E}\left(\frac{1}{n-1} \odot \left[\left(n \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) \ominus \tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right]\right) \\
= \ & \tilde{E}\left(\frac{n}{n-1} \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) \ominus \tilde{E}\left(\frac{1}{n-1} \odot \tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right) \\
= \ & \tilde{I}_H(\mathcal{X}) \oplus \left(\frac{1}{N(n-1)} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right) \\
\ominus \ & \left(\frac{1}{N(n-1)} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right),
\end{aligned}$$

where

$$\begin{aligned}
& \left(\frac{1}{N(n-1)} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right) \\
\ominus \ & \left(\frac{1}{N(n-1)} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right),
\end{aligned}$$

is 0-symmetric, which guarantees the additive equivalence. □

4. ESTIMATING THE POPULATION FUZZY HYPERBOLIC INEQUALITY IN SIMPLE RANDOM SAMPLING

Let $\mathcal{X}$ be a positive fuzzy random variable which in a population of N individuals or sampling units, $U_1, \ldots, U_N$, for which it takes on the values $\tilde{x}_1 = \mathcal{X}(U_1), \ldots, \tilde{x}_N = \mathcal{X}(U_N)$, with $\tilde{x}_1, \ldots, \tilde{x}_N \in \mathcal{F}_c((0, +\infty))$. Assume that a sample of size n is chosen at random and without replacement from the overall population. Let $[\tau]$ denote this generic sample and let $U_{\tau 1}, \ldots, U_{\tau n}$ be the (distinct) individuals or units in it.

Then, the concept of the sample fuzzy hyperbolic inequality index associated with $\mathcal{X}$ in $[\tau]$ can be defined as for the random sampling with replacement (Definition 3.1).

The sample index $\tilde{I}_H(\mathcal{X}[\cdot])$ is defined on the space of the $C_{N,n} = \binom{N}{n}$ distinct possible random samples without replacement of size n from the given population. On the basis of this sample index, we can construct an estimator whose fuzzy expected value on the space of the $C_{N,n}$ samples can be related to the population index as follows:

Theorem 4.1. In random sampling without replacement of size n from the population of N individuals or sampling units, the fuzzy estimator $\frac{n(N-1)}{(n-1)N} \odot \tilde{I}_H(\mathcal{X}[\cdot])$ such that

$$\begin{aligned}
&\frac{n(N-1)}{(n-1)N} \odot \tilde{I}_H(\mathcal{X}[\tau]) \\
= \; &\frac{N-1}{(n-1)N} \odot \left[\left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_{\tau 1}\right) \oplus \ldots \oplus \left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_{\tau n}\right)\right] \ominus \frac{n(N-1)}{(n-1)N},
\end{aligned}$$

for all $\tau \in \{1, \ldots, C_{N,n}\}$, satisfies that

$$\begin{aligned}
&\tilde{E}\left(\frac{n(N-1)}{(n-1)N} \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) \\
= \; &\tilde{I}_H(\mathcal{X}) \oplus \left(\frac{N-n}{(n-1)N^2} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right),
\end{aligned}$$

(where the last fuzzy expected value, $\tilde{E}$, is intended to be applied on the space of the $C_{N,n}$ random samples without replacement of size n).

Proof. Indeed,

$$\begin{aligned}
&\tilde{E}\left(\frac{n(N-1)}{(n-1)N} \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) \\
= \; &\left[p([1]) \odot \left(\frac{n(N-1)}{(n-1)N} \odot \tilde{I}_H(\mathcal{X}[1])\right)\right] \\
&\oplus \ldots \oplus \left[p([C_{N,n}]) \odot \left(\frac{n(N-1)}{(n-1)N} \odot \tilde{I}_H(\mathcal{X}[C_{N,n}])\right)\right],
\end{aligned}$$

where $p([\tau]) = \frac{1}{C_{N,n}}$ = probability of choosing the sample $[\tau]$, $\tau = 1, \ldots, C_{N,n}$.

On the other hand,

$$\begin{aligned}
&\tilde{I}_H(\mathcal{X}[\tau]) = \frac{1}{n} \odot \left[\left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_{\tau 1}\right) \oplus \ldots \oplus \left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_{\tau n}\right)\right] \ominus 1 \\
= \; &\frac{1}{n} \odot \left(\left[\left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_1\right) \odot a_1[\tau]\right] \oplus \ldots \oplus \left[\left(\tilde{E}(\mathcal{X}[\tau]) \oslash \tilde{x}_N\right) \odot a_N[\tau]\right]\right) \ominus 1,
\end{aligned}$$

where $a_j[\tau] = 1$ if U_j appears in the sample $[\tau]$, $= 0$ otherwise, $j = 1, \ldots, n$, and $\tau = 1, \ldots, C_{N,n}$.

Since

$$\tilde{E}(\mathcal{X}[\tau]) = \frac{1}{n} \odot \left((\tilde{x}_1 \odot a_1[\tau]) \oplus \ldots \oplus (\tilde{x}_N \odot a_N[\tau])\right),$$

then,

$$\begin{aligned}
&\tilde{I}_H(\mathcal{X}[\tau]) = \frac{1}{n^2} \odot \{[((\tilde{x}_1 \oslash \tilde{x}_1) \odot a_1[\tau]a_1[\tau]) \oplus \ldots \oplus ((\tilde{x}_N \oslash \tilde{x}_1) \odot a_N[\tau]a_1[\tau])] \\
&\oplus \ldots \oplus [((\tilde{x}_1 \oslash \tilde{x}_N) \odot a_1[\tau]a_N[\tau]) \oplus \ldots \oplus ((\tilde{x}_N \oslash \tilde{x}_N) \odot a_N[\tau]a_N[\tau])]\} \ominus 1.
\end{aligned}$$

Consequently,

$$\tilde{E}\left(\frac{n(N-1)}{(n-1)N} \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) = \frac{N-1}{n(n-1)N} \odot \{[((\tilde{x}_1 \oslash \tilde{x}_1) \odot E(a_1[\cdot]a_1[\cdot]))$$
$$\oplus \ldots \oplus ((\tilde{x}_N \oslash \tilde{x}_1) \odot E(a_N[\cdot]a_1[\cdot]))] \oplus \ldots \oplus [((\tilde{x}_1 \oslash \tilde{x}_N) \odot E(a_1[\cdot]a_N[\cdot]))$$
$$\oplus \ldots \oplus ((\tilde{x}_N \oslash \tilde{x}_N) \odot E(a_N[\cdot]a_N[\cdot]))]\} \ominus \left(\frac{n(N-1)}{(n-1)N} \sum_{\tau=1}^{C_{N,n}} p([\tau])\right),$$

where the classical expected values $E(a_j[\cdot]a_{j'}[\cdot])$, for all $j, j' \in \{1, \ldots, N\}$, $j \neq j'$, are defined on the space of the $C_{N,n}$ random samples without replacement of size n from the population.

The probability that the individual or unit U_j belongs to a random sample equals n/N, so that the classical random variable $a_j[\cdot]$ behaves as a Bernoulli random variable $B\left(1, \frac{n}{N}\right)$, and the product variable $a_j[\cdot]a_{j'}[\cdot]$ behaves as a Bernouilli $B\left(1, \frac{n(n-1)}{N(N-1)}\right)$, for all $j, j' \in \{1, \ldots, N\}$, $j \neq j'$. Therefore,

$$E(a_j[\cdot]) = \frac{n}{N}, \quad E(a_j[\cdot]a_{j'}[\cdot]) = \frac{n(n-1)}{N(N-1)},$$
$$E(a_j[\cdot]a_j[\cdot]) = \frac{n}{N}, \quad j, j' \in \{1, \ldots, N\}, j \neq j',$$

so that

$$\tilde{E}\left(\frac{n(N-1)}{(n-1)N} \odot \tilde{I}_H(\mathcal{X}[\cdot])\right)$$
$$= \left(\frac{1}{N^2} \odot \{[(\tilde{x}_1 \oslash \tilde{x}_1) \oplus \ldots \oplus (\tilde{x}_N \oslash \tilde{x}_1)] \oplus \ldots \oplus [(\tilde{x}_1 \oslash \tilde{x}_N) \oplus \ldots \oplus (\tilde{x}_N \oslash \tilde{x}_N)]\}\right)$$
$$\oplus \left(\frac{N-n}{(n-1)N^2} \odot [(\tilde{x}_1 \oslash \tilde{x}_1) \oplus \ldots \oplus (\tilde{x}_N \oslash \tilde{x}_N)]\right) \ominus \frac{n(N-1)}{(n-1)N}$$
$$= \left(\frac{1}{N} \odot \left[\left(\tilde{E}(\mathcal{X}) \oslash \tilde{x}_1\right) \oplus \ldots \oplus \left(\tilde{E}(\mathcal{X}) \oslash \tilde{x}_N\right)\right] \ominus 1\right)$$
$$\oplus \left(\frac{N-n}{(n-1)N^2} \odot [(\tilde{x}_1 \oslash \tilde{x}_1) \oplus \ldots \oplus (\tilde{x}_N \oslash \tilde{x}_N) \ominus N]\right)$$
$$= \tilde{I}_H(\mathcal{X}) \oplus \left(\frac{N-n}{(n-1)N^2} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right). \qquad \square$$

On the other hand, the remaining term

$$\frac{N-n}{(n-1)N^2} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]$$

can be estimated as in the preceding sampling as follows:

Theorem 4.2. In random sampling without replacement of size n from the population of N individuals or sampling units, the fuzzy estimator $\tilde{I}_H^{wv}(\mathcal{X}[\cdot])$ such that

$$\tilde{I}_H^{wv}(\mathcal{X}[\tau]) = \frac{1}{n} \odot \left[\tilde{I}_H(\{\tilde{x}_{\tau 1}\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_{\tau n}\})\right]$$

for all $\tau \in \{1, \ldots, C_{N,n}\}$, satisfies that

$$\tilde{E}\left(\tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right) = \frac{1}{N} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right].$$

Proof. Indeed,

$$\begin{aligned}\tilde{I}_H^{wv}(\mathcal{X}[\tau]) &= \frac{1}{n} \odot \left[\tilde{I}_H(\{\tilde{x}_{\tau 1}\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_{\tau n}\})\right] \\ &= \frac{1}{n} \odot \left[\left(\tilde{I}_H(\{\tilde{x}_1\}) \odot (a_1[\tau])\right) \oplus \ldots \oplus \left(\tilde{I}_H(\{\tilde{x}_N\}) \odot (a_N[\tau])\right)\right],\end{aligned}$$

so that if we consider its fuzzy expected value over all random samples without replacement of size n, $[\tau]$, with $\tau = 1, \ldots, C_{N,n}$, we obtain that

$$\begin{aligned}\tilde{E}\left(\tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right) &= \left[p([1]) \odot \left(\tilde{I}_H^{wv}(\mathcal{X}[1])\right)\right] \\ &\oplus \ldots \oplus \left[p([C_{N,n}]) \odot \left(\tilde{I}_H^{wv}(\mathcal{X}[C_{N,n}])\right)\right],\end{aligned}$$

whence

$$\begin{aligned}\tilde{E}\left(\tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right) &= \frac{1}{n} \odot \left[\left(\tilde{I}_H(\{\tilde{x}_1\}) \odot E(a_1[\cdot])\right) \oplus . \oplus \left(\tilde{I}_H(\{\tilde{x}_N\}) \odot E(a_N[\cdot])\right)\right], \\ &= \frac{1}{N} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right].\end{aligned}$$ □

As a consequence of the preceding results, we now construct a fuzzy estimator of $\tilde{I}_H(\mathcal{X})$ in simple random sampling, which is unbiased up to additive equivalences.

Theorem 4.3. In random sampling without replacement of size n from the population of N individuals or sampling units, the fuzzy estimator

$$\left[\tilde{I}_H \hat{(\mathcal{X})}\right]^s([\cdot]) = \frac{1}{(n-1)N} \odot \left[\left(n(N-1) \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) \ominus \left((N-n) \odot \tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right)\right],$$

satisfies that

$$\tilde{E}\left(\left[\tilde{I}_H \hat{(\mathcal{X})}\right]^s([\cdot])\right) \sim_{\oplus} \tilde{I}_H(\mathcal{X}).$$

Proof. Indeed, on the basis of Theorems 4.1 and 4.2, and because of the properties of the fuzzy operations, we have that

$$\tilde{E}\left(\frac{1}{(n-1)N} \odot \left[\left(n(N-1) \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) \ominus \left((N-n) \odot \tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right)\right]\right)$$

$$
\begin{aligned}
&= \tilde{E}\left(\frac{n(N-1)}{(n-1)N} \odot \tilde{I}_H(\mathcal{X}[\cdot])\right) \ominus \tilde{E}\left(\frac{N-n}{(n-1)N} \odot \tilde{I}_H^{wv}(\mathcal{X}[\cdot])\right) \\
&= \tilde{I}_H(\mathcal{X}) \oplus \left(\frac{N-n}{(n-1)N^2} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right) \\
&\ominus \left(\frac{N-n}{(n-1)N^2} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&\left(\frac{N-n}{(n-1)N^2} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right) \\
\ominus &\left(\frac{N-n}{(n-1)N^2} \odot \left[\tilde{I}_H(\{\tilde{x}_1\}) \oplus \ldots \oplus \tilde{I}_H(\{\tilde{x}_N\})\right]\right),
\end{aligned}
$$

is 0-symmetric, which guarantees the additive equivalence. □

5. ILLUSTRATIVE EXAMPLE

To illustrate the application of the results in this paper we now consider the following example:

Example. The Government of a University is interested in estimating the inequality associated with the variable "mean time spent daily in the study of Mathematics" ($\mathcal{X}$), in the population of the 25,064 students taking courses in Mathematics in this university. For this purpose, a sample $[\tau]$ of 100 students is chosen at random and without replacement from this population, and the students in the sample are asked about their values for variable $\mathcal{X}$, the answers obtained being the following:

8 "not too much time,"

15 "less than 1 hour,"

9 "a bit less than 1 hour,"

12 "around 1 hour,"

11 "much more than 1 hour,"

17 "1 to 2 hours,"

10 "a bit more than 1 hour and a half,"

6 "approximately 2 hours,"

10 "not much more than 2 hours,"

2 "more than 2 hours."

If these answers are described by means of the triangular (Tri) and trapezoidal (Tra) fuzzy numbers (see, for instance, Cox [5], Mareš [16], Jang and Galley [12]) whose support is contained in $[.25, 4]$ (with $.25 = 15$ minutes) and given by (see Figure 1)

$\tilde{x}_1^* =$ "not too much time" $= Tri(.25, .25, 1)$,
$\tilde{x}_2^* =$ "less than 1 hour" $= Tra(.25, .25, 1, 1)$,
$\tilde{x}_3^* =$ "a bit less than 1 hour" $= Tra(.5, .75, 1, 1)$,
$\tilde{x}_4^* =$ "around 1 hour" $= Tri(.5, 1, 1.5)$,
$\tilde{x}_5^* =$ "much more than 1 hour" $= Tra(1, 1, 1.25, 2)$,
$\tilde{x}_6^* =$ "1 to 2 hours" $= Tra(1, 1, 2, 2)$,
$\tilde{x}_7^* =$ "a bit more than 1 hour and a half" $= Tra(1.5, 1.5, 1.75, 2)$,
$\tilde{x}_8^* =$ "approximately 2 hours" $= Tri(1.5, 2, 2.5)$,
$\tilde{x}_9^* =$ "not much more than 2 hours" $= Tri(2, 2, 2.25)$,
$\tilde{x}_{10}^* =$ "more than 2 hours" $= Tra(2, 2, 4, 4)$.

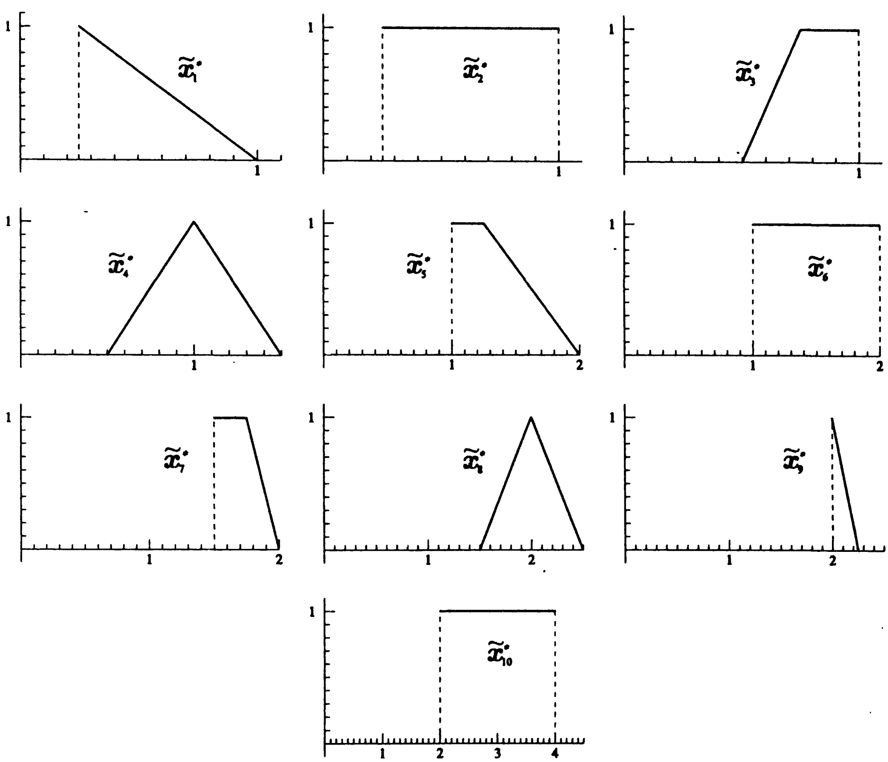

Fig. 1. Fuzzy numbers $\tilde{x}_1^* =$ "not too much time", $\tilde{x}_2^* =$ "less than 1 hour", $\tilde{x}_3^* =$ "a bit less than 1 hour", $\tilde{x}_4^* =$ "around 1 hour", $\tilde{x}_5^* =$ "much more than 1 hour", $\tilde{x}_6^* =$ "1 to 2 hours", $\tilde{x}_7^* =$ "a bit more than 1 hour and a half", $\tilde{x}_8^* =$ "approximately 2 hours", $\tilde{x}_9^* =$ "not much more than 2 hours," and $\tilde{x}_{10}^* =$ "more than 2 hours".

Then, the value of the sample index $\tilde{I}_H(\mathcal{X}[\tau])$ is given by the fuzzy number in $\mathcal{F}_c((-1, +\infty))$ represented in Figure 2.

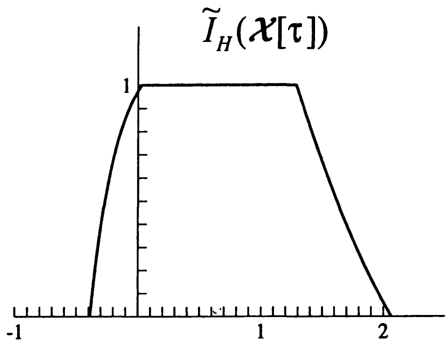

Fig. 2. Sample fuzzy hyperbolic inequality index associated with $\mathcal{X}$.

In virtue of Theorem 4.3, the population value $\tilde{I}_H(\mathcal{X})$ could be unbiasedly estimated (up to additive equivalences) by means of the estimate $\left[\widehat{\tilde{I}_H(\mathcal{X})}\right]^s([\tau])$ represented in Figure 3, which is a "correction" of the sample value.

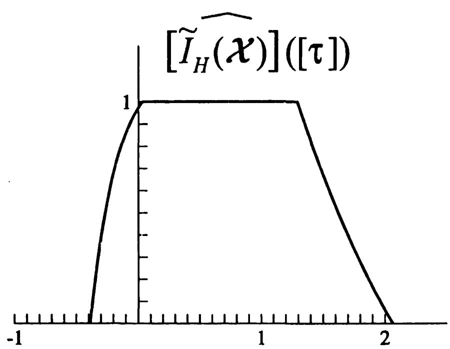

Fig. 3. Unbiased estimate of the population fuzzy hyperbolic inequality index associated with $\mathcal{X}$.

6. CONCLUDING REMARKS

The preceding results have allowed us to construct (up to additive equivalences) unbiased estimates of the fuzzy inequality index.

The study in this paper could be complemented by future studies concerning the quantification of the precision or the error associated with estimators in Sections 3 and 4, the estimation of this precision or error, and the estimation of adequate sample sizes. For this purpose, a counterpart of the standard error in the classical case must be considered in the fuzzy case, and possible measures for it could be stated on the basis of metrics defined on $\mathcal{F}_c(\mathbb{R})$.

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