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CONSIDERING UNCERTAINTY AND DEPENDENCE IN BOOLEAN, QUANTUM AND FUZZY LOGICS

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A degree of probabilistic dependence is introduced in the classical logic using the Frank family of t-norms known from fuzzy logics. In the quantum logic a degree of quantum dependence is added corresponding to the level of noncompatibility. Further, in the case of the fuzzy logic with P-states, (resp. T-states) the consideration turned out to be fully analogous to (resp. considerably different from) the classical situation.

1. INTRODUCTION

Statistical descriptions of the real-world systems give rise to three types of uncertainty. The first type is the *probabilistic uncertainty*. This uncertainty occurs in considerations affected by the real-world constraints. These constraints are however not always known with the required precision. The second type is the *quantum uncertainty*. This uncertainty occurs when the observations of the system cause irreversible changes of states. A typical example is a quantum experiment. And the third type is the *fuzzy uncertainty*. This uncertainty occurs when we study sets of events whose truth values are not necessarily only *true* or *false*. Sometimes the above types of uncertainty appear simultaneously as we shall discuss it later on.

Three principal mathematical structures have been pursued in the study of systems with these three types of uncertainties. These structures are Boolean logics for the probabilistic uncertainties, quantum logics for quantum uncertainties, and fuzzy logics for fuzzy uncertainties. The quantum logics and fuzzy logics are both generalizations of Boolean logics. A common generalization of the quantum and fuzzy logics is desirable but it does not seem to be easily available [14, 25]. We first want to contribute to this problem of a common generalization by indicating some requirements on such a generalization. Then we raise the question of dependence/independence of two events in a logic (in the setup of the respective types). The three abovementioned types of uncertainty correspond to three types of dependence in these structures. Technically, our problem can be restated as follows.

Let a, b be two elements of a (Boolean, quantum, fuzzy, resp.) logic L. Let m be a state on L, i.e., a mapping $m : L \to [0, 1]$ assigning to each "event" its "probability of occurrence". Assuming the values $m(a)$, $m(b)$

are known, what other "parameters" are necessary and sufficient to be known in order to determine the values of m on the entire sublogic $L_{a,b}$ of L generated by a, b?

Here we present some results towards the solution of the latter question.

In the next part, we present examples demonstrating the types of uncertainty. Without providing mathematical details, we indicate specific features of classical, quantum and fuzzy logics. Then we present systems with more than one type of uncertainty. The readers interested only in formal mathematical results may skip the first two parts and start in the third section. The third part recalls the Frank t-norms which play an important role in all logics in question. The fourth part introduces a common background for the definitions which follow. In the subsequent sections we deal with the notion of dependence in the specific logics and we investigate the degrees of dependence corresponding to the types of uncertainty.

2. THE ORIGIN OF THREE TYPES OF UNCERTAINTY

Probabilistic uncertainty is typically used to describe a system in which the initial conditions are not completely known. The conditions may have significant effect on the outputs of the system. Because of the lack of precision in calculating the results, one introduces a probability to obtain an "approximative" description of the system.

Example. When throwing a dice, we know exactly the physical laws which determine its motion. (Quantum phenomena cause some uncertainty but they are negligible in this case.) However, the exact prediction of the result requires a very precise measurement of the initial conditions. Such a precise measurement is impossible. Nevertheless, the results of the experiment are clearly visible and make a sharp distinction between a finite number of possible results. In fact, it is the very idea of such systems like a dice, roulette, lottery, etc., to amplify probabilistic uncertainty to give rise to outcomes which are no longer uncertain.

Quantum uncertainty is typical not only for quantum systems but also for sociology, psychology, medicine, artificial intelligence, etc. Its characteristic feature is the presence of noncompatibility. Two events a, b are called *noncompatible* if they can be observed separately but not simultaneously. The structure of the system excludes a simultaneous measurement and this obstacle cannot be overcome by a repeated measurement because we in principle cannot return to the same initial state. Thus "quantum phenomena" occur whenever the state of the system changes in the process of measuring.

Example. The position and the momentum of a particle cannot be measured independently because the joint error of these measurements is limited by the Plank's constant.

Example. Hypothetically, a patient who suffers the influenza receives two suggestions as regards a potential treatment: (1) take antibiotics, (2) drink some whisky. Both treatments, when applied separately, may have a measurable effect. However, they cannot be applied simultaneously. (Even if we try this, the effect certainly cannot be called simultaneous treatments.) The two methods of treatment cannot be compared because we cannot return the patient into the same initial state and repeat the experiment, applying another treatment.

Fuzzy uncertainty is encountered when we work with more than two truth values. This is the case in the representation of human knowledge and experience, because we usually think in unsharp (fuzzy) categories.

Example. When somebody says "a tall man went quickly through this street a few minutes ago", the only sharp (crisp) facts are "a man" and "this street". But "tall", "quickly" and "few minutes ago" are quantities which cannot be sufficiently described in the classical yes-no terms. There is no probabilistic uncertainty in this case – the experiment was already done and, after all, even a precise measurement will not give a precise answer to the question: "Is this man tall?"

Thus fuzzy uncertainty corresponds to the vagueness of data. Its description allows us to represent naturally the human reasoning. This is why fuzzy logics became a very successful tool in control and in expert systems.

Combined uncertainties. Until now, we have tried to present examples in which one type of uncertainty appears in its pure form (or at least prevails the other uncertainties).

Example. It is conjectured that the weather (in particular, the rain) can be influenced by putting some substance in the clouds. Experiments could be made to verify this conjecture. In order to evaluate their results properly, one must work with all three types of uncertainty, i.e., one must work with probability on a fuzzy quantum logic.

3. PRELIMINARIES – THE FRANK t-NORMS

Triangular norms were studied in the early sixties in the area of probabilistic metric spaces (see [27]) and even in earlier works (see [2] for their overview). They are often used in fuzzy logic in order to obtain a fuzzy conjunction.

A *triangular norm* (t-norm) is an operation $T : [0,1]^2 \to [0,1]$ which is commutative, associative, monotone in each component, and which satisfies the boundary condition $T(1,\alpha) = \alpha$ (see e.g. [2, 27]). The *Frank family of t-norms* T_s, $s \in [0,\infty]$, was defined in [6]. For $s \in (0,\infty)\setminus\{1\}$, the Frank t-norms are defined by the formula

$$T_s : (\alpha,\beta) \mapsto \log_s \left(1 + \frac{(s^\alpha - 1)(s^\beta - 1)}{s-1}\right).$$

The limit cases coincide with the most frequently used t-norms:

$$\begin{array}{lll} T_0 : (\alpha, \beta) \mapsto \min(\alpha, \beta) & & (\text{minimum } t\text{-norm }), \\ T_\infty : (\alpha, \beta) \mapsto \max(\alpha + \beta - 1, 0) & & (\text{Lukasiewicz } t\text{-norm }), \\ T_1 : (\alpha, \beta) \mapsto \alpha \cdot \beta & & (\text{product } t\text{-norm }). \end{array}$$

Depending on the index s, the Frank t-norms T_s span the whole range between T_∞ and T_0 (see [6] and an unpublished result by Takacz which can be found in [2]):

Theorem 1. For fixed $\alpha, \beta \in (0, 1)$, the function

$$F_{\alpha,\beta} : s \mapsto T_s(\alpha, \beta)$$

is continuous and strictly decreasing. It is a bijection of $[0, \infty]$ onto $[T_\infty(\alpha, \beta), T_0(\alpha, \beta)]$.

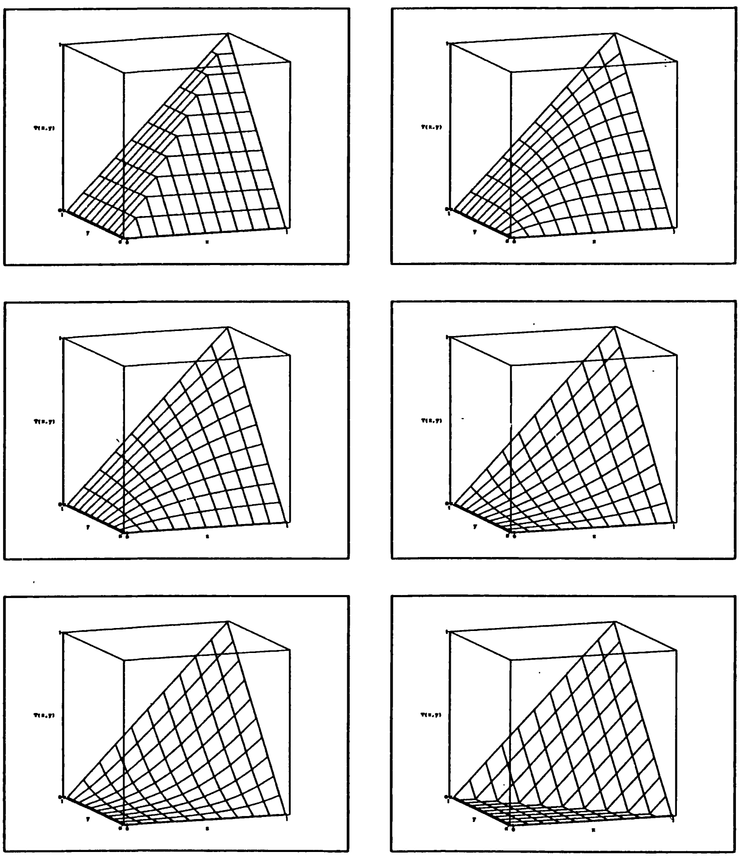

Fig. 1. The Frank t-norms T_0, $T_{0.01}$, T_0, T_{10}, T_{100}, T_∞ (drawn in Maple V.4).

In the sequel, the Frank t-norms will play an essential role not only in the study of fuzzy logics, but, surprisingly, also in the classical logic and the quantum logic. It should be noted that the Frank t-norms have been used in similar context in [4].

4. GENERAL PROBLEM OF DEPENDENCE

In the rest of this paper, we shall deal with the mathematical questions concerning the dependence of two events. The respective mathematical models sometimes allow to distinguish various types of dependence corresponding to specific types of uncertainty. Let us first formulate a general question; it will be precised in the specific logics.

Let L be a *logic* (= the collection of observable events of a system). We assume that the logical operations $\wedge, \vee, ', 0, 1$ are defined on L. They are subject to axioms specific for the logic in question. Moreover, because of dealing with measure-theoretic properties, we assume that the operations $\wedge, \vee$ are defined also for (countable) sequences of elements of L.

A *state* on L is usually a σ-additive mapping $m : L \longrightarrow [0,1]$ such that $m(1) = 1$. (The expression of σ-additivity may not be identical in the respective logics.) For $x \in L$, the value $m(x)$ represents the degree to which x is satisfied.

Two elements $a, b \in L$ generate a sublogic, $L_{a,b}$, of L. Let m be a state on L. The restriction $m|L_{a,b}$ is determined by $m(a)$, $m(b)$, and some other parameters, corresponding to the degrees of freedom of $m|L_{a,b}$ in the state space of $L_{a,b}$. We shall show that these parameters can be sometimes interpreted as "degrees of dependence" of the events a, b.

5. CLASSICAL LOGIC

In the classical logic, the event structure L is assumed to be a Boolean σ-algebra. A state is a σ-additive probability measure. This is equivalent to the following definition: A mapping $m : L \longrightarrow [0,1]$ is a state if it satisfies the following conditions:

(s1) $m(0) = 0, \ m(1) = 1$,

(s2) $m(a \vee b) + m(a \wedge b) = m(a) + m(b)$ for all $a, b \in L$,

(s3) $m(\lim_{n \in N} a_n) = \lim_{n \in N} m(a_n)$ for each increasing sequence $(a_n)_{n \in N}$ in L.

The sublogic (Boolean sub-σ-algebra) $L_{a,b}$ of L generated by $a, b \in L$ is an epimorphic image of the free Boolean algebra $F_{\mathrm{BA}} \cong 2^4$ with two free generators. In order to study our question of dependence of two events a, b, it is sufficient to study the case when $L = L_{a,b} \cong F_{\mathrm{BA}}$ and a, b are the free generators of L.

An easy observation shows that the state space of L $(= F_{\mathrm{BA}})$ is a tetrahedron. More exactly, a state m is uniquely determined by the values $m(a)$, $m(b)$, $m(a \wedge b)$, where $m(a), m(b) \in [0,1]$ are arbitrary, and $m(a \wedge b)$ satisfies the inequality

$$T_\infty(m(a), m(b)) \le m(a \wedge b) \le T_0(m(a), m(b)).$$

Instead of $m(a \wedge b)$, we may determine m using another parameter which plays a more symmetric role with respect to complements. Following [21], we use a *degree of probabilistic dependence*, $p_m(a, b)$. It is a parameter from $[-1, 1]$ defined for all $a, b \in L$ with $m(a), m(b) \in (0, 1)$ and which is subject to the following axioms:

(p1) $p_m(a, b) = p_m(b, a)$,

(p2) $p_m(a', b) = -p_m(a, b)$,

(p3) if $m_1(a_1) = m_2(a_2)$, $m_1(b_1) = m_2(b_2)$ and $m_1(a_1 \wedge b_1) < m_2(a_2 \wedge b_2)$, then $p_{m_1}(a_1, b_1) < p_{m_2}(a_2, b_2)$,

(p4) $p_m(a, b) = 0$ iff $m(a \wedge b) = T_1(m(a), m(b))$,

(p5) $p_m(a, b) = 1$ iff $m(a \wedge b) = T_0(m(a), m(b))$,

(p6) $p_m(a, b) = -1$ iff $m(a \wedge b) = T_\infty(m(a), m(b))$.

The motivation is that the value $p_m(a, b) = 0$ corresponds to the independence of a, b, while the extreme cases $p_m(a, b) = 1$, resp. $p_m(a, b) = -1$, correspond to the maximal positive, resp. negative, dependence of a, b. A degree of probabilistic dependence with the above properties may be obtained as follows:

$$p_m(a, b) = g(s_{a,b}),$$

where $s_{a,b} \in [0, \infty]$ such that

$$m(a \wedge b) = T_{s_{a,b}}(m(a), m(b))$$

and

$$g(s_{a,b}) = 1 - \frac{4}{\pi} \arctan s_{a,b}.$$

Remark. The existence of $s_{a,b}$ with the above properties is guaranteed by Theorem 1. In order to satisfy (p1)–(p6), g may be any decreasing bijection $g : [0, \infty] \to [-1, 1]$ satisfying $g(1/s) = -g(s)$ for all $s \in [0, \infty]$. On the other hand, the Frank family cannot be replaced by any other family of t-norms.

For $x \in L$ and $i \in \{1, -1\}$, we denote

$$x^i = \begin{cases} x & \text{if } i = 1, \\ x' & \text{if } i = -1. \end{cases}$$

With this notation, there exists a function $P : [0, 1]^2 \times [-1, 1] \to [0, 1]$ such that

$$m(a^i \wedge b^j) = P(m(a^i), m(b^j), p_m(a^i, b^j)) \tag{P}$$

for all i, $j \in \{1, -1\}$. The explicit expression for P is

$$P(u, v, p) = T_{g^{-1}(p)}(u, v).$$

To complete the rules, we recall the following formulas:

$$m(a^i) = m(a)^i, \quad m(b^i) = m(b)^i,$$

$$p_m(a^i, b^j) = i \cdot j \cdot p_m(a, b).$$

The exponent -1 on the right-hand side of the latter equalities should be interpreted as the standard fuzzy negation, i.e.,

$$m(a)^i = \begin{cases} m(a) & \text{if } i = 1, \\ 1 - m(a) & \text{if } i = -1, \end{cases}$$

and analogously for $m(b)^i$. The same convention is used in the sequel for negations of probability values. Thus, (P) can be reformulated to the form

$$m(a^i \wedge b^j) = P(m(a)^i, m(b)^j, i \cdot j \cdot p_m(a, b)).$$

Since each element of L can be expressed as an orthogonal join of elements of the form $a^i \wedge b^j$, the formula (P) shows how all values of m on L can be computed from three parameters, $m(a)$, $m(b)$, and the degree of probabilistic dependence, $p_m(a, b)$.

6. QUANTUM LOGIC

In the quantum logic, the event structure is usually supposed to be a *σ-orthomodular lattice* (σ-OML), i.e., a lattice L with bounds $0, 1$ and a unary operation (orthocomplementation) $' : L \to L$ such that

1. $x' \leq y' \iff y \leq x$,
2. $x'' = x$,
3. $x \wedge x' = 0$,
4. if $(x_n)_{n \in N}$ is an *orthogonal* sequence in L (i.e., $x_k \leq x_n$ whenever $k \neq n$), then $\bigvee_{n \in N} x_n$ exists in L,
5. $x \leq y \Longrightarrow y = x \vee (x' \wedge y)$ (the *orthomodular law*).

If we consider only finite sequences in the condition 4, we obtain the definition of an *orthomodular lattice* (OML). In comparison to Boolean σ-algebras, the absorption laws

$$a = a \wedge (a \vee b), \quad a = a \vee (a \wedge b)$$

are relaxed, and the distributivity is replaced by the orthomodular law (the orthomodular law is obviously a weaker condition). A Boolean σ-algebra is a special case of a σ-OML. Another typical example of a σ-OML is the lattice of all closed subspaces in a Hilbert space. A σ-OML is a Boolean σ-algebra if and only if the *commutator*,

$$\mathrm{com}(x, y) = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y'),$$

is zero for each two elements x, y (see, e. g. [23]).

A state on a σ-OML L can be defined just as in Boolean σ-algebras, i.e., as a mapping $m : L \longrightarrow [0,1]$ satisfying the conditions (s1), (s2), (s3) from the previous section. The sublogic (sub-σ-OML) $L_{a,b}$ of L generated by $a, b \in L$ is an epimorphic image of the free OML F_{OML} with two free generators. In order to study our question of dependence of two events a, b, it is sufficient to study the case when $L = L_{a,b} \cong F_{\mathrm{OML}}$ and a, b are the free generators of L. The structure of F_{OML} is described, e. g., in [1, 7].

For $x \in L$, the interval $[0, x]_L = \{y \in L \ : \ y \leq x\}$, equipped with the operations inherited from L (in particular, with an orthocomplementation $'^x : y \mapsto y' \wedge x$), is again an OML.

The free OML $L \cong F_{\mathrm{OML}}$ can be expressed as a direct product of two intervals:

$$L = [0, c]_L \times [0, c']_L,$$

where $c = \mathrm{com}(a, b)$, [1, 7]. Each $x \in L$ admits an orthogonal decomposition $x = (x \wedge c) \vee (x \wedge c')$. The interval $[0, c']_L$ is isomorphic to 2^4 ($\cong F_{\mathrm{BA}}$), where 2^4 is the free Boolean algebra with two free generators, $a \wedge c'$, $b \wedge c'$. Thus each state m_B on $[0, c']_L$ can be described the same way as in the previous section. The interval $[0, c]_L = \{0, c, a \wedge c, a' \wedge c, b \wedge c, b' \wedge c\}$ is isomorphic to the modular ortholattice known as MO2 (see [1, 7]). The description of the state spaces of MO2 and F_{OML} can be found, e. g., in [3, 11, 19, 26]. There is only one state, m_Q, on $[0, c]_L \cong \mathrm{MO2}$. It attains the value $1/2$ at all elements $a \wedge c, a' \wedge c, b \wedge c, b' \wedge c$. Each state m on L is a convex combination of a "classical" state $x \mapsto m_B(x \wedge c')$ and a "purely quantum" state $x \mapsto m_Q(x \wedge c)$. Thus,

$$m(x) = q_m(a, b) \cdot m_Q(x \wedge c) + (1 - q_m(a, b)) \cdot m_B(x \wedge c'),$$

for some $q_m(a, b) \in [0, 1]$. As m_Q contributes by $q_m(a, b)/2$ to both $m(a)$ and $m(b)$,

$$m(a) = q_m(a, b)/2 + (1 - q_m(a, b)) \cdot m_B(a \wedge c'),$$

$$m(b) = q_m(a, b)/2 + (1 - q_m(a, b)) \cdot m_B(b \wedge c'),$$

it gives rise to a new form of dependence between a and b. The coefficient $q_m(a, b) \in [0, 1]$ of the state m_Q can be considered as a *degree of quantum dependence*. It can be computed from the formula

$$q_m(a, b) = m(\mathrm{com}(a, b)).$$

Also, $q_m(a, b)$ is a new degree of freedom in the state space of L. Since m_B belongs to a tetrahedron (= a three-dimensional simplex) and m_Q is unique, m belongs to their convex hull which is the four-dimensional simplex (= a convex hull of 5 points which do not belong to a subspace of dimension less than 4). Using the same idea as before, a state m is seen to be uniquely determined by the values $m(a)$, $m(b)$, the degree of probabilistic dependence $p_m(a, b)$, and the degree of quantum dependence $q_m(a, b)$. Following the previous analysis, all elements of L can be expressed as orthogonal joins of elements of one of the following forms:

$$a^i \wedge b^j, \quad i, j \in \{1, -1\}, \quad a^i \wedge c, \quad i \in \{1, -1\}, \quad b^i \wedge c, \quad i \in \{1, -1\}.$$

Thus all values of a state m on L can be easily computed provided we know the values $m(a)$, $m(b)$ and $m(a^i \wedge b^j)$ for $i, j \in \{1, -1\}$. There is a function $Q : [0,1]^2 \times [-1,1] \times [0,1] \to [0,1]$ such that

$$m(a^i \wedge b^j) = Q(m(a^i), m(b^j), p_m(a^i, b^j), q_m(a^i, b^j)) \tag{Q}$$

for all $i, j \in \{1, -1\}$. In comparison to (P), the degree of probabilistic dependence $p_m(a,b)$ must be generalized to the quantum logic case, and the function Q is modified accordingwise. Let us define $p_m(a,b)$ as the degree of probabilistic dependence in the Boolean algebra $[0, c']_L$, namely

$$p_m(a,b) = p_{m_B}(a \wedge c', b \wedge c').$$

The explicit formula is

$$p_m(a,b) = g(s_{a,b}),$$

where $s_{a,b} \in [0, \infty]$ is such that

$$m_B(a \wedge b) = T_{s_{a,b}}(m_B(a \wedge c')), m_B(b \wedge c')))$$

(g is the same as in the previous section). The left-hand side of the latter equality is written in a simplified form because $a \wedge b \wedge c' = a \wedge b$. In terms of m and $q_m(a,b)$, we can compute $s_{a,b}$ from the equation

$$\frac{m(a \wedge b)}{1 - q_m(a,b)} = T_{s_{a,b}} \left(\frac{m(a) - q_m(a,b)/2}{1 - q_m(a,b)}, \frac{m(b) - q_m(a,b)/2}{1 - q_m(a,b)} \right).$$

We obtain Q in the form

$$Q(u, v, p, q) = (1 - q) \cdot T_{g^{-1}(p)} \left(\frac{u - q/2}{1 - q}, \frac{v - q/2}{1 - q} \right).$$

The values of m on the elements of the form $x \wedge c$, where $x \in \{a, a', b, b'\}$, are equal:

$$m(x \wedge c) = q_m(a,b)/2.$$

The relation to orthocomplements in (Q) is the following:

$$m(a^i) = m(a)^i, \quad m(b^i) = m(b)^i,$$
$$p_m(a^i, b^j) = i \cdot j \cdot p_m(a,b),$$
$$q_m(a^i, b^j) = q_m(a,b).$$

Remark. There is no reason to define a sign of the degree of quantum dependence. Its role with respect to orthocomplements is entirely symmetric. It causes a zero contribution to all $m(a^i \wedge b^j)$, $i, j \in \{1, -1\}$, and the same positive contribution, $q_m(a,b)/2$, to all $m(x)$, $x \in \{a, a', b, b'\}$.

We conclude that the quantum logic possesses the probability uncertainty and the quantum uncertainty. In the state space, they correspond to two degrees of freedom, the degree of probabilistic dependence and the degree of quantum dependence. The classical logic is included as a special case of a quantum logic with a zero degree of quantum dependence.

7. FUZZY LOGIC

After numerous attempts in recent years, there is still no unique way of defining a fuzzy logic. There are different fuzzy generalizations of the classical logical connectives and they correspond to various mathematical structures. We shall concentrate on tribes of fuzzy sets.

Recall that, for a t-norm, T, the standard fuzzy negation $\eta(\alpha) = 1 - \alpha$ gives rise to the dual t-conorm S by the de Morgan formula $S(\alpha, \beta) = \eta(T(\eta(\alpha), \eta(\beta)))$. Let us extend the fuzzy negation η, the t-norm T, and the dual t-conorm S (which are operations on $[0,1]$) to the fuzzy complement $'$, the fuzzy intersection $\wedge$, and the fuzzy union $\vee$ on $[0,1]^X$ in the manner defined below (the pointwise extension):

$$(a')(x) = \eta(a(x)) = 1 - a(x),$$

$$(a \wedge b)(x) = T(a(x), b(x)),$$

$$(a \vee b)(x) = S(a(x), b(x)).$$

A collection $L \subset [0,1]^X$ is said to be a T-*tribe* on X if

1. the constant function $1 \in L$,
2. $a \in L \Longrightarrow a' \in L$,
3. $(a_n)_{n \in N} \subset L \Longrightarrow \bigvee_{n \in N} a_n \in L$.

A T-tribe on X is a fuzzy generalization of a σ-algebra of subsets of X. In this paper, we shall restrict our attention to T_s-tribes with respect to Frank t-norms T_s.

Theorem 2. [2] Let $s \in (0, \infty)$. Each T_s-tribe is a T_∞-tribe, and each T_∞-tribe is a T_0-tribe.

States on fuzzy logics allow for different generalizations. We shall deal with P-states (introduced in [22]) and T-states (studied in [2]). To distinguish the two cases, we shall speak of P-states and T-states. In the special case of Boolean σ-algebras, both definitions coincide with the ordinary states (i. e., with the probability measures).

8. P-STATES

The P-states were originally introduced on T_0-tribes (see [5, 22]). As was shown in [18], the results can be generalized to d^3-lattices. The d^3-lattices are special Kleene algebras which possess a common generalization of T_0-tribes and Boolean σ-algebras. (In contrast to T_0-tribes, d^3-lattices include all Boolean σ-algebras.)

A *Kleene algebra* is a bounded distributive lattice with a unary operation $'$ such that

$$a'' = a, \quad a \leq b \Longrightarrow b' \leq a', \quad a \wedge a' \leq b \vee b'$$

for all a, b. A Kleene algebra L is a *d^3-lattice* if it is a σ-complete lattice and if it satisfies the distributivity condition

$$b \wedge \bigvee_{n\in N} a_n = \bigvee_{n\in N} (b \wedge a_n).$$

A T_0-tribe is a typical example of a d^3-lattice. In this case, $'$ coincides with the fuzzy complement and (countable) lattice-theoretical meets and joins coincide with fuzzy intersections and unions. A d^3-lattice L is a Boolean σ-algebra iff $a \wedge a' = 0$ for all $a \in L$; in this Boolean case, P-states coincide with states on a Boolean σ-algebra. Throughout this section, we assume that L is a d^3-lattice.

A *P-state* on L ('P' for probability) is a mapping $m : L \longrightarrow [0,1]$ such that

(s4) $m(a \vee a') = 1$ for all $a \in L$,

(s5) $m\left(\bigvee_{n\in N} a_n\right) = \sum_{n\in N} m(a_n)$ if $(a_n)_{n\in N}$ is sequence in L which is orthogonal, i.e., if $a_m \leq a'_n$ whenever $m \neq n$.

Each P-state satisfies also the conditions (s1), (s2), (s3).

Let L, M be d^3-lattices. A mapping $h : L \longrightarrow M$ is called a *σ-homomorphism* if

$$h(a') = h(a)', \qquad h\left(\bigvee_{n\in N} a_n\right) = \bigvee_{n\in N} h(a_n).$$

The following theorem characterizes P-states.

Theorem 3. [18, 20] Let L be a d^3-lattice. There is a Boolean σ-algebra B and a σ-homomorphism $h : L \longrightarrow B$ such that each mapping $m : L \longrightarrow [0,1]$ is a P-state iff it is of the form $h \circ \mu$ for some state μ on B. The Boolean σ-algebra B is called a *Boolean representation* of L.

Let $a, b \in L$ and let $L_{a,b}$ be the sub-d^3-lattice of L generated by a, b. Again, $L_{a,b}$ is a homomorphic image of the free d^3-lattice with two free generators. This free d^3-lattice coincides with the free Kleene algebra, F_{KA}, with two free generators. In order to describe the dependence of two elements of a d^3-lattice, we shall restrict to the case when $L = L_{a,b} \cong F_{\mathrm{KA}}$. The free Kleene algebra F_{KA} is finite. Its Boolean representation is $F_{\mathrm{BA}} \cong 2^4$. The P-states on $L \cong F_{\mathrm{KA}}$ are in a one-to-one correspondence with the states on F_{BA}. The values of a P-state m on L are uniquely determined by $m(a)$, $m(b)$ and the degree of probabilistic dependence, $p_m(a,b)$. We have

$$m(a^i \wedge b^j) = P(m(a^i), m(b^j), p_m(a^i, b^j))$$

for all $i, j \in \{1, -1\}$, where P is the same function as in the case of Boolean σ-algebras. The elements $a \wedge a'$, $b \wedge b'$ may be nonzero, but m vanishes on them. As concerns the dependence of two events, P-states do not bring anything new in comparison to states on Boolean σ-algebras.

9. T-STATES

The T-states were studied in numerous papers, particularly in [2], where they were applied to the game theory. Fairly deep mathematical results were derived there, e. g., a generalization of Ljapunov's theorem was established.

Let T be a t-norm and S its dual t-conorm. Let us denote by $\wedge$, resp. $\vee$, the fuzzy intersection, resp. union, induced by T, resp. S. As usual, $'$ stands for the standard fuzzy complement.

Let L be a T-tribe. A mapping $m : L \to [0,1]$ is a *T-state* if it satisfies the conditions (s1), (s2), (s3) (see the section "Classical logic"). In this section, we shall study the Frank t-norms T_s, $s \in (0, \infty]$. We assume that L is a T_s-tribe on X and m is a T_s-state on L. (Sometimes the case $s = \infty$ will be considered separately.)

The structure of T_s-tribes was clarified in a series of papers [2, 9, 10, 12, 13, 17]. Obviously, a T_s-tribe L on X is a σ-algebra iff $L \subset \{0,1\}^X$. The set $C(L) = L \cap \{0,1\}^X$ is a σ-algebra of subsets of X.

Theorem 4. [2] All elements of L are $C(L)$-measurable.

Recall that a *support* of a fuzzy subset a of X is the crisp set $\operatorname{Supp} a = \{x \in X : a(x) > 0\}$. We say that L is a *weakly generated tribe* [17] if there is a σ-ideal Δ in $C(L)$ such that $L = \{a \in [0,1]^X : a \text{ is } C(L)\text{-measurable}, \operatorname{Supp} a \cap \operatorname{Supp} a' \in \Delta\}$. If the ideal Δ is principal, $\Delta = [0, d]_L$ for some $d \in L$, then L is called a *semigenerated tribe* [12].

Theorem 5. [10, 13] If $s \in (0, \infty)$, then L is a weakly generated tribe. If, moreover, L is generated by a countable set of elements of L, it is semigenerated.

Remark. If $s = \infty$, then the structure of L is more complex. A generalization of the latter theorem for T_∞-tribes is given in [10].

The following theorem characterizes T_s-states.

Theorem 6. [2, 15] Let $s \in (0, \infty]$. Each T_s-state m on L is a convex combination of two T_s-states m_1, m_2 of the following forms:

(m1) $m_1(a) = \int a \, d\mu_1$,

(m2) $m_2(a) = \mu_2(\operatorname{Supp} a)$,

where μ_1, μ_2 are states on the σ-algebra $C(L)$. If $s = \infty$, then m is of the form (m1) (i.e., $m = m_1$) because m_2 is not a T_∞-state.

In a certain contrast to the previous sections, $m(c \wedge c')$ may be nonzero for $s \in (0, \infty)$ (not for $s = \infty$).

Let $a, b \in L$ and let $L_{a,b}$ be the sublogic (= sub-T_s-tribe) of L generated by a, b. As T_s-tribes do not form an equational class, we cannot speak of a "free T_s-tribe with two free generators". Only in the case of T_∞-tribes, MV-algebras [16] could be the corresponding algebraic structures. The difficulties arise when we try to introduce countable operations. According to Theorem 4, all elements of $L_{a,b}$ (in particular, the elements a and b) are $C(L_{a,b})$-measurable. This means that $C(L_{a,b})$ contains preimages of all Borel subsets of $[0,1]$ under a and b. Suppose that at least one of a, b has an infinite range. Then also $C(L_{a,b})$ is infinite. According to Theorem 5, $L_{a,b}$ is a semigenerated tribe which is not a σ-algebra. Theorem 6 gives infinitely many T_s-states of the form (m1) on $L_{a,b}$. (Though the structure of T_∞-tribes is more complex, the principal conclusions remain valid for them, too.) In contrast to the previous sections, there is no chance to determine a T_s-state on $L_{a,b}$ by finitely many parameters. Thus, this type of fuzzy logic admits infinitely many new degrees of freedom. Therefore a much more rich structure of the T_s-state space comes into existence than in the logics studied in previous sections. We have a new area of problems typical for fuzzy logics.

10. CONCLUSIONS

In the classical logic (i.e., in a Boolean σ-algebra), we have introduced a degree of probabilistic dependence. This has been defined using the Frank family of t-norms known from fuzzy logics. The result presents a rather nonstandard use of fuzzy set techniques in Boolean σ-algebras. In the quantum logic (i.e. in a σ-OML), we have added a degree of quantum dependence corresponding to the level of noncompatibility. Further, in the case of the fuzzy logic with P-states, the consideration turned out to be fully analogous to the classical situation. Finally, in the case of the fuzzy logic with T-states, the investigation has proved to be considerably different from the previous ones – we have infinitely many degrees of freedom corresponding to infinitely many types of dependence.

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