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GENERALIZED MÖBIUS TRANSFORMATION OF KNOWLEDGE BASES

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Möbius transformation is an important tool for establishing weights of compositional expert systems rules from conditional weights.

In this paper, an applicability of Möbius transformation of rule bases is also extended to knowledge bases with elementary disjunctions in antecedents of rules. This paper contains an existence theorem, an algorithm of the transformation and some open problems which tend to maximal generality as well.

1. INTRODUCTION

The first ideas of how to establish weights of compositional expert systems rules from conditional weights related to real evidence, data, or experience were published in 1984 [5]. The solution of the problem is based on Möbius transformation. For a full description of Möbius transformation see e.g. [6, 7].

Suppose a knowledge base with two rules $A \Rightarrow H$ and $A\&B \Rightarrow H$ with conditional weights (a belief in favour of the hypothesis if the antecedent holds). When $A\&B$ hold we want to infer H with the same belief as in the knowledge base with the rule $A\&B \Rightarrow H$ only, i.e. avoiding the use of the first one. So, if we know the belief of H obtained from each rule with a conditional belief as its weight when the rule is considered individually, we have to find new weights. We have to find new weights so that when both rules are present we can still infer the same resulting weight of H as if only the second one was present. Thus, it is not possible to use conditional beliefs as weights of rules, resp. weights must be adequately redefined. In this paper, we show how this can be achieved.

The possibility of using Möbius transformation is not restricted to MYCIN-like systems, it is also important for a common generalization of MYCIN-like systems and fuzzy expert systems which use a composition of fuzzy relations like Conorm-CADIAG-2 extended by the handling of negative knowledge, see [4]. The system is derived from the fuzzy expert system CADIAG-2 [1].

The original Möbius transformation is formulated and used only for rules of a special form. The present work generalizes it for a wider class of rules.

Necessary preliminaries are introduced and the original Möbius transformation theorem for MYCIN-like systems is stated in the second section.

Section 3 describes ideas of how to also extend the field of Möbius transformation applicability to rules with elementary disjunctions in antecedents. Some principal problems are shown. It is suggested how they can be, more or less, overcome. The existence theorem is stated at the end of the section. In Section 4, the possibilities of the Möbius transformation simplification obtained by a simple minded algorithm and tools for improvement of the algorithm are introduced. Section 5 presents the improved algorithm for founded knowledge bases.

In Section 6 is a comparison of Möbius transformation for MYCIN-like systems with an introduced generalization. After that conclusions and ideas for future work follow.

The text is limited to some instructive simplifications; for a complete text with more details, proofs and an algorithm to compute Möbius transformation for more general knowledge bases, see [3].

2. MÖBIUS TRANSFORMATION

Preliminaries

In this paper, we shall consider *low knowledge bases*, i.e. knowledge bases without intermediate propositions, there are only questions (symptoms) and goals (hypotheses, diagnoses). In this section, let us suppose rules $A \Rightarrow S(w)$, where the antecedent A is an elementary conjunction of questions, the succedent S is a goal, and w is a weight. An *elementary conjunction* (of questions) is a conjunction of *literals* (of questions), i.e. questions or their negations, where every question has at most one occurrence in the elementary conjunction. Let the weights be in the interval $[-1, 1]$. A global weight of a hypothesis H is computed using a group operation $\oplus$ on $[-1, 1]$ as the $\oplus$-sum of contributions (effects) of all rules whose succedent is H (rules leading to H). A *three-valued questionnaire* q is a mapping of questions into the set $\{-1, 0, 1\}$, i.e. there are only answers $\{-1, 0, 1\}$ (i.e. No, I don't know, Yes). Each questionnaire of this kind can be represented by an elementary conjunction (positive literal for 1, negative one for -1, and no literal for 0). All of the above terminology corresponds to that used in monography [6].

An *elementary disjunction* is a disjunction of literals. An *ecd knowledge base* (elementary-conjunction-disjunction) is a knowledge base such that antecedents of rules are either elementary conjunctions or elementary disjunctions.

Further, we shall use the following terminology. A rule $R: A \Rightarrow S(w)$ is a *simple rule* if its antecedent A is a literal; R is a *conjunctive/disjunctive rule* if A is a conjunction/disjunction; R is a *maximal conjunctive/disjunctive rule* if there is no rule $B \Rightarrow S(w_B)$ in the knowledge base, so that A is a subconjunction/subdisjunction[1] of B. A conjunction $Conj = A\&B\&\cdots\&K$ is a conjunctive translation of a disjunction $Disj = A \vee B \vee \cdots \vee K$, a rule $Conj \Rightarrow H$ is a conjunctive translation of the disjunctive rule $Disj \Rightarrow H$.

[1] Formula F is a subconjunction/subdisjunction of formula G if it is its subformula, i.e. $a\&b$ is the subconjunction of $a\&b\&c$, $a \vee b$ is the subdisjunction of $a \vee b \vee c$, etc.

An *ecd* knowledge base Θ is *founded* if, for every H that appears as a succedent, it contains rules $A \Rightarrow H$ for every literal A which is involved in a disjunctive rule $A \vee Disj \Rightarrow H$ for any elementary disjunction $Disj$. An *ecd* knowledge base Θ is *weakly founded* if for every literal A from any disjunctive rule $A \vee Disj \Rightarrow H$ there is a simple rule $A \Rightarrow H$ or a conjunctive rule $A \& Conj \Rightarrow H$ (for some $Conj$) included in Θ.
Notice, that every knowledge base without a disjunction in the antecedents is founded.

Möbius transformation theorem

Let us denote by $\beta(H|E)$ a conditional expert belief[2] that hypothesis H is valid/invalid if only the evidence E is known. For Möbius transformation, we shall suppose that a given (original) weight w of the rule $R : A \Rightarrow S(w)$ represents the expert's belief that the weight of S is w provided that only A holds ($w = \beta(S|A)$).

Weights of rules are transformed by Möbius transformation; to distinguish them, we shall denote the original (given, source, conditional) weight of the rule R as w_R^0 (or $w_{S,A}^0$), while Möbius weight (i.e. weight after transformation) is represented as w_R (or $w_{S,A}$). We will also use w_A^0 and w_A if the succedent of a rule is clear from the context.

We say, that a set of rules is *weakly sound* if for every two rules sharing the same succedent such that $Ant_1 \subset Ant_2$ (Ant_1 is a subconjunction of Ant_2, or Ant_2 implies Ant_1) holds: if $w_{Ant_1}^0 = 1$ then $w_{Ant_2}^0 = 1$ also holds.

We say, that a low knowledge base is weakly sound if its set of rules is weakly sound.

Theorem 1. Let Θ_0 be a weakly sound set of rules such that $w_{H,E}^0 = \beta(H|E)$, i.e. $\Theta_0 = \{E \Rightarrow H(\beta(H|E)\}$. Then there exists a weighting of rules which forms a knowledge base $\Theta = \{E \Rightarrow H(w_{H,E})\}$ of MYCIN-like expert system, such that for any three-valued questionnaire E_q and hypothesis H for which $\beta(H|E_q)$ is defined, it holds that

$$W_\Theta(H|E_q) = \beta(H|E_q),$$

where $W_\Theta(H|E_q)$ is a global weight of the hypothesis H given by E_q,
$W_\Theta(H|E_q) = \oplus\{w_{H,E'}|E' \subseteq E_q\}$.

The new knowledge base Θ is called *Möbius transform* of the source rule base Θ_0. For details see [6, 7].

Note: There is no limitation to questionnaire values (to possible answers of a user) for Möbius transform of a rule base existence. But, the equation $W_\Theta(H|E_q) = \beta(H|E_q)$ only makes sense for three-valued questionnaires.

[2] $\beta(H|E)$ is positive if expert beliefs that H is more likely valid than its negation $\neg H$, $\beta(H|E)$ is negative if the expert beliefs that H is rather invalid, i.e. that $\neg H$ is more likely valid than H. Of course, it is possible to use another set of weights e.g. $W = [0,1]$, but in such a case $\oplus$ must be a group operation on $W = [0,1]$.

3. INCLUDING A DISJUNCTION INTO MÖBIUS TRANSFORMATION

First have a look at the principal idea of Möbius transformation for MYCIN-like systems. We suppose conditional rules, where their weights rely on the expert's belief that the succedent holds if the antecedent of the rule is true. Let us have the following rules

$$A \Rightarrow H(w_A^0),$$
$$B \Rightarrow H(w_B^0),$$
$$A\&B \Rightarrow H(w_\&^0).$$

Thus, if both of A and B are true we want to infer the conditional weight $w_\&^0$ (a belief that H holds if $A\&B$ is given) as a result, while MYCIN-like system infers $w_A^0 \oplus w_B^0 \oplus w_\&^0$. It is trivial that $w_A^0 \oplus w_B^0 \oplus w_\&^0 \neq w_\&^0$ in general, $w_\&^0$ may be greater, equal or less than $w_A^0 \oplus w_B^0$. In the case $w_\&^0 = w_A^0 \oplus w_B^0$, the third rule is redundant and so we remove it from the knowledge base. Otherwise, we make a transformation of the weight $w_\&^0$ of rule $A\&B \Rightarrow H$ to $w_\& = w_\&^0 \ominus (w_A^0 \oplus w_B^0)$, where $a \ominus b$ is an abbreviation of $a \oplus (-b)$. The resulting Möbius weight $w_\&$ is positive if $w_\&^0 > w_A^0 \oplus w_B^0$, w is negative if $w_\&^0 < (w_A^0 \oplus w_B^0)$ (a positive or negative effect of the rule – support/unsupport of hypothesis), and $w_\& = 0$ if $w_\&^0 = w_A^0 \oplus w_B^0$ (the rule is redundant as before).

We can easily verify that we obtain expected results: for A we get w_A^0, for B we get w_B^0, and finally, for A, B we get $w_A^0 \oplus w_B^0 \oplus (w_\&^0 \ominus (w_A^0 \oplus w_B^0)) = w_\&^0$.

Now, we shall try to apply this simple idea to *ecd* knowledge bases, i.e. knowledge bases in which antecedents can be a conjunction or a disjunction of literals (propositions or their negations), i.e. an *elementary conjunction* or an *elementary disjunction*. A very simple example follows:

$$\begin{aligned} &A \Rightarrow H(w_A^0), \\ &B \Rightarrow H(w_B^0), \\ &A \vee B \Rightarrow H(w_\vee^0). \end{aligned} \qquad (*)$$

For A we want a resulting weight w_A^0 instead of $w_A^0 \oplus w_\vee^0$, thus, we change w_A^0 with $w_A = w_A^0 \ominus w_\vee^0$, and analogically, $w_B = w_B^0 \ominus w_\vee^0$. If we know that $A \vee B$ is true and we are not able to specify whether A or B or $A\&B$, then the third rule is the only one which fires, and we keep its weight $w_\vee^0$.

After this transformation, we really get w_A^0 for A, we get w_B^0 for B, but for $A\&B$, we obtain $w_A^0 \ominus w_\vee^0 \oplus w_B^0 \ominus w_\vee^0 \oplus w_\vee^0 = w_A^0 \oplus w_B^0 \ominus w_\vee^0$ which is not equal to the value $w_A^0 \oplus w_B^0$ assumed because there is no rule $A\&B \Rightarrow H$. (We shall discuss this assumption later on.)

Let us formulate our problem more precisely, we get a set of equations, where

w_A, w_B, and w are modifications of weights w_A^0, w_B^0, and $w_\vee^0$, respectively:

$$w_\vee = w_\vee^0,$$
$$w_A \oplus w_\vee = w_A^0,$$
$$w_B \oplus w_\vee = w_0^B,$$
$$w_A \oplus w_B \oplus w_\vee = w_A^0 \oplus w_B^0.$$

Thus, we get $(w_A^0 \ominus w_\vee) \oplus (w_B^0 \ominus w_\vee) \oplus w_\vee = w_A^0 \oplus w_B^0$, hence $w_\vee = 0$. The system of equations has the only solution $w_\vee^0 = w_\vee = 0$, which only describes a situation without the rule with disjunction, contrary to our initial assumption (*).

There are two possibilities of how to overcome the problem. First, to express the rule $A \vee B \Rightarrow H(w_\vee^0)$ in another way without a disjunction in the antecedent, i.e. preliminary modification of the knowledge base before the application of Möbius transformation, or second, to modify our approach of understanding disjunctive rules.

Three possibilities of rewriting a disjunctive rule $A \vee B \Rightarrow H(w)$ were studied in [3]:

(1) $A \vee B = \neg(\neg A \& \neg B)$,

(2) a substitution of two rules $\neg A \& \neg B \Rightarrow C(1)$ and $\neg C \Rightarrow H(w)$, and

(3) a substitution of three rules $A \Rightarrow D(1)$, $B \Rightarrow D(1)$, $D \Rightarrow H(w)$.

The first two cases are unacceptable to our purpose, the third one turns us to a further complicated modification of the knowledge base before Möbius transformation.

So, we shall turn our attention to a better understanding of the rule $A \vee B \Rightarrow H(w)$. What does the rule mean? How do we understand $A \vee B$?

If $A \vee B$ holds, it means that either we want and we can distinguish one of the following possibilities: only A holds, only B holds, both A and B hold or we cannot distinguish or we don't like to distinguish them. Thus, we can rewrite (*) as

$$A \Rightarrow H(w_A^0),$$
$$B \Rightarrow H(w_B^0),$$
$$(A \vee B) \,\&\, A \Rightarrow H(w_4),$$
$$(A \vee B) \,\&\, B \Rightarrow H(w_5),$$
$$(A \vee B) \,\&\, (A \,\&\, B) \Rightarrow H(w_6),$$
$$(A \vee B) \,\&\, (A \vee B) \Rightarrow H(w_3).$$

The 3rd and 4th rule is a copy of the first and second one, hence we can remove them. We can simplify the antecedent of the last two rules, thus, all of the original rules remain in the knowledge base and there is only one new rule $A \& B \Rightarrow H(w_6)$. To derive an expected Möbius transform of the knowledge base we put $w_6 = w_A^0 \oplus w_B^0$, because there is no other precise specification for $A\&B$ given by an expert. So we can rewrite our knowledge base in the following way:

$$A \Rightarrow H(w_A^0),$$
$$B \Rightarrow H(w_B^0),$$
$$A \,\&\, B \Rightarrow H(w_A^0 \oplus w_B^0),$$
$$A \vee B \Rightarrow H(w_\vee^0).$$

Antecedents of rules are elementary conjunctions or elementary disjunctions again.

Now, we can apply the idea of Möbius transformation to our modified knowledge base, hence, we get

$$A \Rightarrow H(w_A^0 \ominus w_\vee^0),$$
$$B \Rightarrow H(w_B^0 \ominus w_\vee^0),$$
$$A \,\&\, B \Rightarrow H(w_\vee^0),$$
$$A \vee B \Rightarrow H(w_\vee^0),$$

($\vee$, $\&$ are used in indices as abbreviations of $A \vee B$ and $A\&B$; $w_\&$ was derived as follows: $w_\& = w_\&^0 \ominus (w_A \oplus w_B \oplus w_\vee) = w_A^0 \oplus w_B^0 \ominus (w_A^0 \ominus w_\vee^0 \oplus w_B \ominus w_\vee^0 \oplus w_\vee^0) = w_\vee^0$). We can easily verify that if only A holds, then we get w_A^0. Similarly, if only B holds, then we get w_B^0. If $A\&B$ holds, we get $w_\vee^0 \oplus (w_A^0 \ominus w_\vee^0) \oplus (w_B^0 \ominus w_\vee^0) \oplus w_\vee^0 = w_A^0 \oplus w_B^0$ as it should be. And finally, if we only know that $A \vee B$ holds, then we get $w_\vee^0$.

We have succeeded in the first trivial example of Möbius transformation for rules with a disjunction in the antecedent. Now, let us consider the following, more complicated, yet still a simple example of a knowledge base:

$$\Rightarrow H(w_0)^3,$$
$$A \Rightarrow H(w_A^0),$$
$$B \Rightarrow H(w_B^0),$$
$$C \Rightarrow H(w_C^0),$$
$$A \vee B \Rightarrow H(w_{A\vee B}^0),$$
$$A \vee C \Rightarrow H(w_{A\vee C}^0),$$
$$A \vee B \vee C \Rightarrow H(w_\vee^0).$$

From now on, $\vee$ in indices means an abbreviation of disjunction of all the literals used, here $A \vee B \vee C$.

[3] w_0 is an 'apriori' weight of the hypothesis H, to be precise it should be represented as w_0^0 or $w_\top^0$, where $\top$ is tautology. Apriori weight of H, i.e. weight of rule $\Rightarrow H(w_0)$ is not changed by Möbius transformation, there is always $w_0 = w_0^0$ ($w_\top = w_\top^0$), thus, we can use w_0 as an abbreviation.

In the present example we have disjunctions $A \vee B$, $A \vee C$, and $A \vee B \vee C$. According to our interpretation of them we obtain five new rules with antecedents $A\&B\&C$, $A\&B$, $A\&C$, $B\&C$, $B \vee C$. We are looking for a "Möbiable" knowledge base as close to the original one as possible. Thus, we try, at first, to only add the rule $A\&B\&C \Rightarrow H$ (the rule with a maximal conjunction in the antecedent), and second, also the rules $A\&B \Rightarrow H$ and $A\&C \Rightarrow H$ (conjunctive translations of the original ones), but both of these attempts are unsuccessful, see [3]. In this case it is necessary and sufficient to add four rules $A\&B\&C \Rightarrow H$, $A\&B \Rightarrow H$, $A\&C \Rightarrow H$, and $B\&C \Rightarrow H$, see [3] again. It is not necessary to add a rule with the antecedent $B \vee C$. Until now, there is no explanation of which rules should be added to the new knowledge base and which ones should not, and so we shall look for it later.

We have illustrated in the above examples that Möbius transforms of the analyzed knowledge bases exist and so, it makes sense to speak of Möbius transformation of rule bases with an elementary disjunction in the antecedents. Moreover, we can say the following:

Lemma 2. Let Θ_0 be a weakly sound low *ecd* knowledge base. If we can also explicitly set or estimate implicit weights for nonincluded combinations of literals, then Möbius transform of the knowledge base Θ_0 exists.

Note: weak soundness conditions in the present situation are as follows:
for every two rules sharing the same succedent such that $Ant_1 \subset Ant_2$ holds: if $w^0_{Ant_1} = 1$, then $w^0_{Ant_2} = 1$, where $Ant_1 \subset Ant_2$ means Ant_2 implies Ant_1 ($\subset$ indicates implication not a subset nor a subformula, i. e. $Conj_1$ is a subconjunction of $Conj_2$ or $Disj_2$ is a subdisjunction of $Disj_1$ or a subdisjunction of $Disj_1$ exists which is a subconjunction of $Conj_2$).

Proof. Let us show a simple idea of a construction of this Möbius transform. We have seen that maybe it is necessary to add some rules during transformation. So we have computed transformed (Möbius) weights of all possible rules.

Let us take all the elementary disjunctions from the longest to one-element ones and for every disjunction $Disj$ compute $\oplus$-combination c of all applicable rules provided only $Disj$ holds. Put $w_{Disj} = w^0_{Disj} \ominus c$. If rule $Disj \Rightarrow H$ exists and $w_{Disj} \neq w^0_{Disj}$, then rewrite the weight of the rule, if $w_{Disj} \neq 0$ and the rule does not exist, then add the rule $Disj \Rightarrow H(w_{Disj})$ into the knowledge base.

Let us go through all elementary conjunctions from one-element to the longest possible one analogically.

This construction is very simple, nevertheless, it is possible to show that the resulting transformed knowledge base is Möbius transform of the source knowledge base Θ_0. □

At this moment, we know how to construct, in a simple yet noneffective way, the requested Möbius transform. So, it is logical to look for its improvement. A decision-making of whether a new possible rule will be added or not depends on the Möbius weight of the possible rule. Therefore, we need a more sophisticated way of computing these weights. In the next subsection it is shown how to compute Möbius weights of rules in knowledge bases in which all possible elementary conjunctions and elementary disjunctions form antecedents of rules with the same succedent.

Formulas for computing Möbius weights

Generally, for three questions/symptoms, we have the following knowledge base[4]:

$$
\begin{aligned}
&\Rightarrow H(w_0),\\
&A \Rightarrow H(w_A^0),\\
&B \Rightarrow H(w_B^0),\\
&C \Rightarrow H(w_C^0),\\
&A \,\&\, B \Rightarrow H(w_{A\&B}^0),\\
&A \,\&\, C \Rightarrow H(w_{A\&C}^0),\\
&B \,\&\, C \Rightarrow H(w_{B\&C}^0),\\
&A \,\&\, B \,\&\, C \Rightarrow H(w_{\&}^0),\\
&A \vee B \Rightarrow H(w_{A\vee B}^0),\\
&A \vee C \Rightarrow H(w_{A\vee C}^0),\\
&B \vee C \Rightarrow H(w_{B\vee C}^0),\\
&A \vee B \vee C \Rightarrow H(w_{\vee}^0).
\end{aligned}
$$

By the recomputation of weights keeping the original principal idea of Möbius transformation, we obtain the following Möbius weights of rules:

$$
\begin{aligned}
w_0 &= w_0^0\\
w_\vee &= w_\vee^0 \ominus w_0\\
w_{A\vee B} &= w_{A\vee B}^0 \ominus w_\vee \ominus w_0 = w_{A\vee B}^0 \ominus w_\vee^0\\
w_A &= w_A^0 \ominus w_{A\vee B} \ominus w_{A\vee C} \ominus w_\vee \ominus w_0 = w_A^0 \ominus w_{A\vee B}^0 \ominus w_{A\vee C}^0 \oplus w_\vee^0\\
w_{A\&B} &= w_{A\&B}^0 \ominus w_A \ominus w_B \ominus w_{A\vee B} \ominus w_{A\vee C} \ominus w_{B\vee C} \ominus w_\vee \ominus w_0\\
&= w_{A\&B}^0 \ominus w_A^0 \ominus w_B^0 \oplus w_{A\vee B}^0\\
w_\& &= w_\&^0 \ominus w_{A\&B} \ominus w_{A\&C} \ominus w_{B\&C} \ominus w_A \ominus w_B \ominus w_C\\
&\qquad \ominus w_{A\vee B} \ominus w_{A\vee C} \ominus w_{B\vee C} \ominus w_\vee \ominus w_0\\
&= w_\&^0 \ominus w_{A\&B}^0 \ominus w_{A\&C}^0 \ominus w_{B\&C}^0 \oplus w_A^0 \oplus w_B^0 \oplus w_C^0 \ominus w_\vee^0
\end{aligned}
$$

Möbius weights $w_{A\vee C}$, $w_{B\vee C}$, w_B, w_C, $w_{A\&C}$, and $w_{B\&C}$ are computed analogically.

By using them, we can perform Möbius transformation of the source knowledge

[4] To be precise for three questions it should be a more complicated knowledge base, the presented one corresponds to three literals of three different questions, for a general knowledge base with three questions see [3].

base, and we shall obtain the following Möbius transform:

$$\begin{aligned}
&\Rightarrow H(w_0^0),\\
&A \Rightarrow H(w_A^0 \ominus w_{A\vee B}^0 \ominus w_{A\vee C}^0 \oplus w_\vee^0),\\
&B \Rightarrow H(w_B^0 \ominus w_{A\vee B}^0 \ominus w_{B\vee C}^0 \oplus w_\vee^0),\\
&C \Rightarrow H(w_C^0 \ominus w_{B\vee C}^0 \ominus w_{A\vee C}^0 \oplus w_\vee^0),\\
&A\,\&\,B \Rightarrow H(w_{A\&B}^0 \ominus w_A^0 \ominus w_B^0 \oplus w_{A\vee B}^0),\\
&A\,\&\,C \Rightarrow H(w_{A\&C}^0 \ominus w_A^0 \ominus w_C^0 \oplus w_{A\vee C}^0),\\
&B\,\&\,C \Rightarrow H(w_{B\&C}^0 \ominus w_B^0 \ominus w_C^0 \oplus w_{B\vee C}^0),\\
&A\,\&\,B\,\&\,C \Rightarrow H(w_\&^0 \ominus w_{A\&B}^0 \ominus w_{A\&C}^0 \ominus w_{B\&C}^0 \oplus w_A^0 \oplus w_B^0 \oplus w_C^0 \ominus w_\vee^0),\\
&A \vee B \Rightarrow H(w_{A\vee B}^0 \ominus w_\vee^0),\\
&A \vee C \Rightarrow H(w_{A\vee C}^0 \ominus w_\vee^0),\\
&B \vee C \Rightarrow H(w_{B\vee C}^0 \ominus w_\vee^0),\\
&A \vee B \vee C \Rightarrow H(w_\vee^0 \ominus w_0).
\end{aligned}$$

Similarly we can compute Möbius weights for a knowledge base with four or more questions.

In general, for a knowledge base with one hypothesis H and n questions/symptoms $A, B, C, \ldots, N$, we can compute Möbius weights as:

$$w_0 = w_0^0,$$

$$w_\vee = w_\vee^0 \ominus w_0,$$

$$w_{A\vee B\vee C\vee\cdots\vee K} = \bigoplus_{i=0}^{n-k}\left((-1)^i \bigoplus_{|d|=k+i,\ A\vee B\vee\cdots\vee K\subset d} w_d^0\right)$$

$$w_A = \bigoplus_{i=0}^{n-1}\left((-1)^i \bigoplus_{|d|=i+1,\ A\subset d} w_d^0\right)$$

$$w_{A\&B\&C\&\cdots\&K} = \bigoplus_{i=0}^{k-1}\left((-1)^i \bigoplus_{|c|=k-i,\ c\subset A\&B\&\cdots\&K} w_c^0\right) \oplus (-1)^k w_{A\vee B\vee C\vee\cdots\vee K}^0,$$

where $w_\vee$ is an abbreviation for a weight $w_{A\vee B\vee C\vee\cdots\vee N}$ of the rule with the maximal possible disjunction in the antecedent, $a \subset b$ means b implies a, $|c|$ is a length (number of conjuncts) of the conjunction c, conjunction $c = A\&B\&C\&\cdots\&K$ has k elements i.e. $|c| = k$.

If we compare the formula for computing $w_{A\&B\&C\&\cdots\&K}$ with a similar one which is used in knowledge bases without a disjunction we can mention a significant similarity. From the comparison of these formulas we obtain the following one.

$$w_{A\&B\&C\&\cdots\&K} = (-1)^k (w_{A\vee B\vee C\vee\cdots\vee K}^0 \ominus w_0)$$

Now, we have general formulas to compute Möbius weights of rules from any weakly sound low *ecd* knowledge base with one goal, where all possible elementary conjunctions or elementary disjunctions of questions are used as antecedents. (All possible conditional weights are already explicitly included in the source knowledge base. Negated questions are handled separately from the original ones like the new ones.)

We can also use the formulas for deciding which types of new rules will be added into the transformed knowledge base and which ones will not, see the section called Simplifications. For this we need to know how to compute an estimation of implicit weights of possible rules which are not included in the source knowledge base.

Estimations of implicit weights of "rules" which are not included in a source knowledge base

As it was suggested in the previous subsection, we can use the formulas developed there to specify which types of rules are added into a knowledge base during Möbius transformation and which ones are not.

The formulas need all w^0's, all conditional weights which are definable on a set of questions and goals given by a source knowledge base. But, a lot of them are not given in a usual source knowledge base, i.e. not all rules with syntactically possible antecedents are included in the knowledge base. So, we have to estimate these values.

A rule $Ant \Rightarrow H$, literals of which are relevant to H, is not included in a source knowledge base Θ either if Ant is not possible or almost impossible in real situations or if *an expert thinks that Ant expresses nothing new for the hypothesis*, i.e. everything that expresses Ant has already been expressed by applicable rules which are already included in Θ. Thus, we have to compute an expected value w^0_{Ant} from the contributions of other rules, which are applicable, provided that just Ant holds, i.e. from rules of which the antecedents A are implied by Ant ($A \subset Ant$).

To distinguish *explicit conditional weights* w^0_{Ant} of rules of a source knowledge base from computed estimations of those which are not given (resp. which are given implicitly through other rules), we shall denote *estimated implicit weights* as w^x_{Ant}.

It looks like the expected implicit weight w^x_{Ant} of a rule $Ant \Rightarrow H$ should be something like a combination of Möbius weights of all the rules applicable, provided that just Ant is true. But unfortunately, a generation of w^x_{Ant} is more complicated, in general.

It is quite simple in the case of disjunctive rules. In the case of conjunctive rules, where w^0_A of all conjuncts of the antecedent are explicitly given or $w^0_{A\&Conj}$ is given for a $Conj$ ($A \subset A\&Conj \subset Ant$), the value w^x_{Ant} should be the same as in a knowledge base with only conjunctive rules. Complications start with conjunctive rules, where w^0_A of conjuncts of antecedents are not given and where a value should be transfered from disjunctive rules into conjunctive ones.

The further text is concerning founded *ecd* knowledge bases, thus we can omit a transfer of weights from disjunctions into conjunctions from our consideration. We get the following formulas for estimation of implicit weights:

$$w_0^x = w_0,$$

$$w_{Disj}^x = \bigoplus_{D \subset Disj} w_D,$$

$$w_{Conj}^x = \bigoplus_{C \subset Conj} w'_C,$$

where D is a disjunctive or empty antecedent from the source knowledge base Θ_0, C is a conjunctive or empty antecedent from Θ_0, and w'_C is a Möbius weight of the rule $C \Rightarrow H$ from the knowledge base Θ'_0, which is Θ_0 without disjunctive rules.

We can close the section by the formulation of the existence theorem.

Theorem 3. If Θ_0 is a weakly sound low *ecd* knowledge base, then Möbius transform of the knowledge base Θ_0 exists.

Idea of the proof. We can perform Möbius transformation separately for every hypothesis. The rest follows from previous text provided that Θ_0 is a founded *ecd* knowledge base. In the question of general *ecd* knowledge bases see [3].

4. SIMPLIFICATIONS

According to the previous section, we know how to compute Möbius weights, i.e. we can perform Möbius transformation for any founded *ecd* knowledge base.

Now, we are going to specify which rules are not necessary to add within the process of Möbius transformation. We consider the rules $Ant \Rightarrow H$ which are not included into the source knowledge base, i.e. w_{Ant}^0 is not given there. Whether a rule is to be really added or not, depends on its Möbius weight w_{Ant}. By an added rule we mean such a rule that $w_{Ant} \neq 0$, while if $w_{Ant} = 0$ we say that the rule is not added.

We can eliminate some types of rules to be added by a symbolic computation of their Möbius weight. But usually, we cannot assert that some type of rules will be added, because the actual value of its weight w_{Ant} depends on actual values of conditional weights from the source knowledge base.

Now, we shall formulate some lemmata to describe which types of rules are to be or are not to be added in the knowledge base.

Lemma 4. There are no disjunctive rules added to a knowledge base during Möbius transformation.

Lemma 5. If Θ is a founded *ecd* knowledge base, then there are no rules $A_1 \& A_2 \& \cdots \& A_k \Rightarrow H$ added into the knowledge base within the process of Möbius transformation, where $A_1 \vee A_2 \vee \cdots \vee A_k \vee B_1 \vee \cdots \vee B_l$ is not an antecedent of some rule from the source knowledge base.

Lemma 6. If Θ is a founded *ecd* knowledge base, then conjunctive translations of all maximal disjunctive rules are added [5] into the knowledge base within the process of Möbius transformation (if they are not already included in the source knowledge base Θ).

Let $A_1 \vee A_2 \vee \cdots \vee A_k \Rightarrow H$ be a maximal disjunctive rule and $A_1 \& A_2 \& \cdots \& A_k \Rightarrow H$ its conjunctive translation added according to the lemma. Usually, there are also all rules added with a subconjunction of $A_1 \& A_2 \& \cdots \& A_k$ in the antecedent. But, there are also counter-examples on a symbolic level, see [3].

In both previous lemmata, the assumption of foundness of the *ecd* knowledge base is necessary. For general *ecd* knowledge bases which are not founded, there are counter-examples against both of the lemmata included in [3].

Summarizing the above lemmata we see, that antecedents of rules added by Möbius transformation into an *ecd* knowledge base are all conjunctive translations of antecedents of maximal disjunctive rules and subconjunctions of these translations. (But, not necessarily all subconjunctions of these conjunctive translations have to form an antecedent of an added rule). Formally, we have the following:

Theorem 7. Let Θ be a weakly sound low founded *ecd* knowledge base. During a process of Möbius transformation of Θ, rules are added into the knowledge base (if they are not already included in Θ and) if and only if they are in one of the two following types:

- All rules $A_1 \& A_2 \& \cdots \& A_n \Rightarrow H$, where $A_1 \vee A_2 \vee \cdots \vee A_n \Rightarrow H$, is a maximal disjunctive rule from Θ.
- Rules $A_1 \& A_2 \& \cdots \& A_n \Rightarrow H$, where $A_1 \vee A_2 \vee \cdots \vee A_n \vee Disj \Rightarrow H$, is a rule from Θ and $Disj$ is any disjunction. (In general, all such rules are added, but counter-examples exist.)

For proofs and assertions on general *ecd* knowledge bases see [3].

5. ALGORITHM OF MÖBIUS TRANSFORMATION

From the Theorem 3 we have an existence of Möbius transform for weakly sound low founded *ecd* knowledge bases. Using Theorem 7, we can formulate the following algorithm of Möbius transformation of a knowledge base Θ.

(∗) Go ahead through all hypothesis H:
and perform items (0)–(4).

(0) Construct a set Rel of questions relevant to H.
Put $w_0 = w_0^0$.
Create an empty set of maximal disjunctions $MaxD$.

[5] Notice, here we have an exception and we can claim that rules are added: $w_{A_1 \& A_2 \& \cdots \& A_k} = (-1)^{k-1}(w_{A_1 \vee A_2 \vee \cdots \vee A_k} \ominus w_0)$, and we expect $w_{A_1 \vee A_2 \vee \cdots \vee A_k} \neq w_0$ for maximal disjunctive rules not to be redundant.

(1) Go ahead through all disjunctions D in Θ relevant to H:
Put Sum equal to the $\oplus$-sum of Möbius weights of all rules $D \vee D' \Rightarrow H$.
IF there is no such rule, THEN insert D into $MaxD$ and put $w_D = w_D^0 \ominus w_0$,
ELSE put $w_D = w_D^0 \ominus Sum$.
If $|D| = 1$, then sign D in Rel.

(2) Go through all unsigned questions Q from Rel:
IF there is no rule $Q \vee D \Rightarrow H$, THEN put $w_Q = 0$,
ELSE give warning "Assumption does not hold for hypothesis H." and STOP.

(3) Go through all maximal disjunctions MD from $MaxD$:
for MD and every subdisjunction SMD of MD -
create all new rules $Ant \Rightarrow H$ which are already not included in Θ, where Ant is a conjunctive translation of MD or SMD.

(4) Go ahead through all conjunctions $|C| > 1$ in Θ relevant to H:
Put Sum equal to the $\oplus$-sum of Möbius weights ($w_{C'}$) of all rules $C' \Rightarrow H$, where $C' \subset C$ (C implies C').
If w_C^0 is not given ($C \Rightarrow H$ is an added rule), then put w_C^0 equal to the $\oplus$-sum of Möbius weights ($w_{C'}$) of all rules $C' \Rightarrow H$, where C' is a subconjunction of C.
Keep w_C^0 and put $w_C = w_C^0 \ominus Sum$.
(During the construction of Möbius transform it is not necessary to distinguish between w_C^0 and w_C^x, they can be represented by the same variable denoted by w_C^0.)

(*) Save all rules with weights $w_{H_i,Ant_{ij}} \neq 0$ – Möbius transform of Θ.
STOP.

It is possible to show that this algorithm ends and produces Möbius transform of any weakly sound low founded *ecd* knowledge base Θ.

6. CONCLUSION

Generalized Möbius transformation is a theoretical tool for the construction of more correct generalizations of expert systems both of MYCIN-like and fuzzy expert systems based on a composition of fuzzy relations.

Möbius transformation has been generalized to *ecd* knowledge bases, i.e. knowledge bases whose rules have antecedents either in the form of an elementary conjunction (as before) or in the form of an elementary disjunction (new ones) of questions.

The principal difference between original and generalized Möbius transformation consists in a complicated transfer of weights of rules with disjunctive antecedents D_i to weights of other rules with conjunctive ones C_i, where C_i implies D_i.

Original Möbius transformation is only the transformation of weights. While within the generalized one, moreover, some new rules are often added into the knowledge base.

An *estimation of implicit (expected) weights* for these added rules was shown for a class of *ecd* knowledge bases. The *existence theorem* was proved for this class of knowledge bases. Finally, an *algorithm* of the construction of this generalized Möbius transform of knowledge base is described.

A challenge for the future is an admission of rules with more complicated antecedents or a consideration of knowledge bases with several different conjunctions and/or disjunctions.

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REFERENCES

[1] K.-P. Adlassnig: Fuzzy set theory in medical diagnosis. IEEE Trans. Systems Man Cybernet *16* (1986), 2, 260–265.

[2] M. Daniel: A generalization of Möbius transformation. In: 4th Workshop on Uncertainty Processing (R. Jiroušek and G. D. Kleiter, eds.), Prague 1997, pp. 31–42.

[3] M. Daniel: A Generalization of Möbius Transformation for Knowledge Bases which Include Rules with Disjunction in Antecedent. Technical Report No. V-706, ICS AS CR, Prague 1997.

[4] M. Daniel, P. Hájek and P. H. Nguyen: CADIAG-2 and MYCIN-like systems. Artificial Intelligence in Medicine *9* (1997), 241–259.

[5] P. Hájek: Combining functions in consulting systems and dependence of premisses. In: Artificial Intelligence and Information – Control Systems of Robots (I. Plander, ed.), North-Holland, Amsterdam, 1984, pp. 163–166.

[6] P. Hájek, T. Havránek and R. Jiroušek: Uncertain Information Processing in Expert Systems. CRC Press, Boca Raton 1992.

[7] P. Hájek and J. J. Valdés: An analysis of MYCIN-like expert systems. Mathware & Soft Computing *1* (1994), 45–68.

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