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UNILATERAL DYNAMIC CONTACT OF VON KÁRMÁN PLATES WITH SINGULAR MEMORY*

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Dedicated to Jiří V. Outrata on the occasion of his 60th birthday.

Abstract. The solvability of the contact problem is proved provided the plate is simply supported. The singular memory material is assumed. This makes it possible to get a priori estimates important for the strong convergence of gradients of velocities of solutions to the penalized problem.

Keywords: von Kármán plate, unilateral dynamic contact, singular memory, existence of solutions

MSC 2000: 35L85, 74D10, 74K20

1. Introduction and notation

Contact problems occur frequently in practical situations involving strings, beams, membranes, plates and shells. An optimal control problem for a string in contact has been solved in [8]. Especially, dynamic contact problems play a very important role in the present investigations. Since the existence of solutions for purely elastic materials is difficult to prove even in the simplest cases, some kind of physically admissible viscosity helps a lot to solvability of such problems. Considering moderately large deflections, we investigate the nonlinear von Kármán model for the material with a singular memory. Dynamic problems for viscoelastic von Kármán system with the emphasis on decay rates of solutions were treated in [10] where the viscosity does not appear in the equation for the Airy stress function and no contact is considered. The solvability of quasistatic contact problems for such model was solved in [2] and [3] while the dynamic contact problem for short memory material has been studied in [1].

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The aim of the present paper is to prove the solvability of the dynamic contact problem for von Kármán plates made of a material with a singular memory. We do it with the help of penalization of the contact condition. The interpolation technique and the compact imbedding theorem play a crucial role in the transition to the original Signorini contact problem.

We consider a bounded convex polygonal or $C^{3,1}$ domain $\Omega \subset \mathbb{R}^2$ (cf. Remark 2.5) with boundary Γ and a bounded time interval $I \equiv (0,T)$. The unit outer normal vector is denoted by $\boldsymbol{n} = (n_1, n_2)$, $\boldsymbol{\tau} = (-n_2, n_1)$ is the unit tangent vector. We denote by $\boldsymbol{w} \equiv (w_1, w_2, w_3)$ the in-plane and perpendicular displacement. The further notation is as follows:

$$\frac{\partial}{\partial s} \equiv \partial_s, \quad \frac{\partial^2}{\partial s \partial r} \equiv \partial_{sr}, \quad \partial_i = \partial_{x_i}, \quad i = 1, \ldots, N,$$
$$\dot{v} = \frac{\partial v}{\partial t}, \quad \ddot{v} = \frac{\partial^2 v}{\partial t^2}, \qquad Q = I \times \Omega, \quad S = I \times \Gamma.$$

The strain tensor is defined as $\varepsilon_{ij}(\boldsymbol{w}) = \frac{1}{2}(\partial_i w_j + \partial_j w_i + \partial_i w_3 \partial_j w_3) - x_3 \partial_{ij} w_3$, $i,j = 1,2$, $\varepsilon_{i3} \equiv 0$, $i = 1,2,3$.

We denote by δ_{ij} the Kronecker symbol and employ the Einstein summation convention. The constitutional law has the form

$$\begin{aligned}\sigma_{ij}(\boldsymbol{w}) = {} & \frac{1}{1-\nu^2}\mathfrak{E}\big((1-\nu)\varepsilon_{ij}(\boldsymbol{w}) + \nu\delta_{ij}\varepsilon_{kk}(\boldsymbol{w})\big) \\ & + \frac{E}{1-\nu^2}\big((1-\nu)\varepsilon_{ij}(\boldsymbol{w}) + \nu\delta_{ij}\varepsilon_{kk}(\boldsymbol{w})\big).\end{aligned} \tag{1}$$

Here the Young modulus of elasticity E is a positive constant,

$$\mathfrak{E}\colon\ v \mapsto \int_0^t K(t-s)\big(v(t,\cdot) - v(s,\cdot)\big)\,\mathrm{d}s. \tag{2}$$

The kernel K of the singular memory term is assumed to be integrable over $\mathbb{R}_+$ and to have the form

$$\begin{aligned}& K : t \mapsto t^{-2\alpha} q(t) + r(t), \quad t \in \mathbb{R}_+ \equiv (0, +\infty) \text{ with } \alpha \in \Big(0, \frac{1}{2}\Big), \\ & K : t \mapsto 0, \quad t \leqslant 0.\end{aligned} \tag{3}$$

Both q and r belong to $C^1(\mathbb{R}_+)$; they are non-negative and non-increasing functions. Moreover, $q(t) > 0$ for t in a right neighbourhood of the origin.

We set

$$a = \frac{h^2}{12}, \quad b = \frac{h^2}{12\varrho(1-\nu^2)},$$

where h is the the plate thickness and ϱ is the density of the material. We denote

$$(4)\qquad [u,v] \equiv \partial_{11}u\partial_{22}v + \partial_{22}u\partial_{11}v - 2\partial_{12}u\partial_{12}v.$$

We use the following notation of the function spaces: by $W_p^k(M)$ with $k \geqslant 0$ and $p \in [1,\infty]$ the Sobolev (for a noninteger k the Sobolev-Slobodetskii) spaces are denoted provided they are defined on a domain or an appropriate manifold M. By $\mathring{W}_p^k(M)$ we denote the spaces with zero traces on ∂M. If $p=2$ we use the notation $H^k(M)$, $\mathring{H}^k(M)$. For the anisotropic spaces $W_p^k(M)$, $k=(k_1,k_2) \in \mathbb{R}_+^2$, k_1 is related to the time while k_2 to the space variables (with the obvious consequences for $p=2$) provided M is a time-space domain. The duals to $\mathring{H}^k(M)$ are denoted by $H^{-k}(M)$.

2. Formulation of the problem for a simply supported plate

First we introduce the classical formulation of the problem to be solved. Applying the approach of [4] and the constitutive law (1) we arrive at the following system for the deflection u and the Airy stress function v:

$$(5)\qquad \left.\begin{aligned} \ddot{u} - a\Delta\ddot{u} + b(\mathfrak{E}\Delta^2 u + E\Delta^2 u) - [u,v] = f + g,\\ u \geqslant 0,\ \ g \geqslant 0,\ \ ug = 0,\\ \Delta^2 v + \mathfrak{E}[u,u] + E[u,u] = 0 \end{aligned}\right\} \text{ on } Q$$

with the boundary condition

$$(6)\qquad u = u_0, \quad \mathcal{M}(u) = 0, \quad v = \partial_n v = 0 \text{ on } S,$$

and the initial condition

$$(7)\qquad u(0,\cdot) = u_0, \quad \dot{u}(0,\cdot) = u_1 \text{ on } \Omega.$$

Here $\mathcal{M}(u) = b[\mathfrak{E}m(u) + Em(u)]$, where $m(u) = \Delta u + (1-\nu)(2n_1n_2\partial_{12}u - n_1^2\partial_{22}u - n_2^2\partial_{11}u)$.

For $u, y \in L_2(I; H^2(\Omega))$ we define the following bilinear form

$$(8)\qquad A\colon\ (u,y) \mapsto b\big(\partial_{kk}u\partial_{kk}y + \nu(\partial_{11}u\partial_{22}y + \partial_{22}u\partial_{11}y) + 2(1-\nu)\partial_{12}u\partial_{12}y\big)$$

almost everywhere on Q and introduce a cone $\mathcal{C}$ as

$$(9)\qquad \mathcal{C} := \big\{y \in u_0 + (L_2(I;V) \cap H^1(Q));\ y \geqslant 0\big\},$$

where

$$V = H^2(\Omega) \cap \mathring{H}^1(\Omega). \tag{10}$$

Then the variational formulation of the problem (5)–(7) has the following form: Look for $\{u, v\} \in \mathcal{C} \times L_2(I; \mathring{H}^2(\Omega))$ such that $\dot{u} \in L_2(I; H^1(\Omega))$ and

$$\begin{aligned} &\int_Q \big(A(\mathfrak{E}u + Eu, y_1 - u) - a\nabla\dot{u}\cdot\nabla(\dot{y}_1 - \dot{u}) - \dot{u}(\dot{y}_1 - \dot{u}) - [u,v](y_1 - u)\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_\Omega \big(a\nabla\dot{u}\cdot\nabla(y_1 - u) + \dot{u}(y_1 - u)\big)(T,\cdot)\,\mathrm{d}x \\ &\quad \geqslant \int_\Omega \big(a\nabla u_1\cdot\nabla(y_1(0,\cdot) - u_0) + u_1\big(y_1(0,\cdot) - u_0\big)\big)\,\mathrm{d}x + \int_Q f(y_1 - u)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{11}$$

$$\int_\Omega \big(\Delta v \Delta y_2 + (\mathfrak{E}[u,u] + E[u,u])y_2\big)\,\mathrm{d}x = 0 \quad \forall\,(y_1, y_2) \in \mathcal{C} \times \mathring{H}^2(\Omega). \tag{12}$$

We define the bilinear operator $\Phi\colon H^2(\Omega)^2 \to \mathring{H}^2(\Omega)$ by means of the variational equation

$$\int_\Omega \Delta\Phi(u,v)\Delta\varphi\,\mathrm{d}x = \int_\Omega [u,v]\varphi\,\mathrm{d}x, \quad \varphi \in \mathring{H}^2(\Omega). \tag{13}$$

The equation (13) has a unique solution, because $[u,v] \in L_1(\Omega) \hookrightarrow H^2(\Omega)^*$. The well-defined operator Φ is evidently compact and symmetric. The domain Ω fulfils the assumptions enabling us to apply Lemma 1 from [9] due to which $\Phi\colon H^2(\Omega)^2 \to W_p^2(\Omega)$, for any $p \in (2,\infty)$, and

$$\|\Phi(u,v)\|_{W_p^2(\Omega)} \leqslant c\|u\|_{H^2(\Omega)}\|v\|_{W_p^1(\Omega)} \quad \forall\,(u,v) \in H^2(\Omega)^2. \tag{14}$$

With its help we reformulate the system (11), (12) into the following variational inequality:

Problem $\mathcal{P}$. We look for $u \in \mathcal{C}$ such that $\dot{u} \in L_2(I; H^1(\Omega))$ and the inequality

$$\begin{aligned} &\int_Q \big(A(\mathfrak{E}u + Eu, y - u) - a\nabla\dot{u}\cdot\nabla(\dot{y} - \dot{u}) - \dot{u}(\dot{y} - \dot{u})\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_Q [u, \mathfrak{E}\Phi(u,u) + E\Phi(u,u)](y - u)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_\Omega \big(a\nabla\dot{u}\cdot\nabla(y - u) + \dot{u}(y - u)\big)(T,\cdot)\,\mathrm{d}x \\ &\quad \geqslant \int_\Omega \big(a\nabla u_1\cdot(\nabla y(0,\cdot) - \nabla u_0) + u_1(y(0,\cdot) - u_0)\big)\,\mathrm{d}x \\ &\qquad + \int_Q f(y - u)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{15}$$

holds for any $y \in \mathcal{C}$.

In the sequel we assume

$$u_0 \in H^2(\Omega), \quad u_0 \geqslant U \text{ on } \Omega, \quad u_1 \in H^1(\Omega), \quad f \in L_2(I; H^{-1}\Omega)) \tag{16}$$

for a certain positive constant U. Furthermore, a certain "smallness" of the memory will be needed:

$$\int_0^{+\infty} K(s)\,\mathrm{d}s < E. \tag{17}$$

To be able to solve this problem we penalize it first.

For the limit procedures required in the proof of the existence results the following theorems and corollaries from [5, Chapter 2] will be crucial:

Theorem 2.1 (Embedding theorem). *Let $M \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary. Let $p, q \in (1, \infty)$, $\gamma \in [0,1]$ and $\alpha \in (\gamma, 1]$ be numbers such that the inequality*

$$\frac{1}{\alpha}\Big(\frac{N}{p} - \frac{N}{q} + \gamma\Big) \leqslant 1 \tag{18}$$

holds. Then the Sobolev-Slobodetskii space $W_p^\alpha(M)$ is continuously embedded into $W_q^\gamma(M)$. If the inequality (18) is strict, then the embedding is compact for any real $q \geqslant 1$. For $q = \infty$ this is true under the convention $1/q = 0$.

Corollary 2.2. *Let M and I be as above. Let p_i, q_i belong to $(1, +\infty)$, α_i belong to $(0,1]$ and γ_i to $[0, \alpha_i)$, $i = 1, 2$. Assume that (18) holds with $i = 1$ and N replaced by 1 and that it simultaneously holds for $i = 2$. Then $W_{p_1}^{\alpha_1}(I; W_{p_2}^{\alpha_2}(M))$ can be imbedded into $W_{q_1}^{\gamma_1}(I; W_{q_2}^{\gamma_2}(M))$. If both inequalities are strict, the imbedding is compact. The last assertion still holds if q_i is infinite, provided we use the convention $1/q_i = 0$, $i = 1, 2$.*

Theorem 2.3 (Interpolation theorem). *Let M be as above, let k_1, k_2 belong to $[0,1]$, let p_1, p_2 belong to $(1, +\infty)$ and Θ_λ to $[0,1]$. Then there exists a constant c such that for all $u \in W_{p_1}^{k_1}(M) \cap W_{p_2}^{k_2}(M)$ the following estimate holds:*

$$\|u\|_{W_p^k(M)} \leqslant c\|u\|_{W_{p_1}^{k_1}(M)}^{\Theta_\lambda} \|u\|_{W_{p_2}^{k_2}(M)}^{1-\Theta_\lambda}$$

with $k = \Theta_\lambda k_1 + (1 - \Theta_\lambda)k_2$ and $1/p = \Theta_\lambda/p_1 + (1 - \Theta_\lambda)/p_2$. The assertion remains true if $k_1 = k_2 = 0$ and p_1, p_2 belong to $[1, +\infty]$.

Corollary 2.4 (Generalization). *Let M, k_1, k_2, p_1, p_2 be as above. Let I be a bounded interval in $\mathbb{R}$, let κ_1, κ_2 belong to $[0,1]$, let q_1, q_2 belong to $(1,+\infty)$ and Θ_λ to $[0,1]$. Then there exists a constant c such that for all $u \in W^{\kappa_1}_{q_1}(I; W^{k_1}_{p_1}(M)) \cap W^{\kappa_2}_{q_2}(I; W^{k_2}_{p_2}(M))$ it holds*

$$\|u\|_{W^{\kappa}_{q}(I;W^{k}_{p}(M))} \leqslant c\|u\|^{\Theta_\lambda}_{W^{\kappa_1}_{q_1}(I;W^{k_1}_{p_1}(M))}\|u\|^{1-\Theta_\lambda}_{W^{\kappa_2}_{q_2}(I;W^{k_2}_{p_2}(M))},$$

where $k = \Theta_\lambda k_1 + (1-\Theta_\lambda)k_2$, $\kappa = \Theta_\lambda \kappa_1 + (1-\Theta_\lambda)\kappa_2$, $1/q = \Theta_\lambda/q_1 + (1-\Theta_\lambda)/q_2$ and $1/p = \Theta_\lambda/p_1 + (1-\Theta_\lambda)/p_2$. If $\kappa_1 = \kappa_2 = 0$ and q_1, q_2 belong to $[1,+\infty]$, the assertion still holds.

Remark 2.5. In order to apply Lemma 1 from [8] containing the estimate (14) we need the regularity $v \in H^3(\Omega)$ for a weak solution of the Dirichlet problem

$$\Delta^2 v = g \ \text{ on } \ \Omega, \quad v = \partial_n v = 0 \ \text{ on } \ \Gamma, \quad g \in H^{-1}(\Omega).$$

The regularity result for a $C^{3,1}$ domain Ω is due to Theorem 2.2, Chapter 4 from [11]. In the case of a convex polygonal domain we apply Theorem 2.1 from [12]. It is probable that the requirement can be weakened.

3. Penalized problem

We penalize the unilateral contact condition in the standard way replacing the second row in (5) by the condition $g = -u^-/\eta$. Thus we arrive at the penalized

Problem $\mathcal{P}_\eta$. Look for $u \in L_2(I;V) + u_0$ such that $\ddot{u} \in L_2(Q)$, the equation

$$(19)\qquad \int_Q \big(\ddot{u}z - a\ddot{u}\Delta z + A(\mathfrak{E}u, z) + EA(u,z) + [u, \mathfrak{E}\Phi(u,u) + E\Phi(u,u)]z - \eta^{-1}u^- z\big)\,\mathrm{d}x\,\mathrm{d}t = \int_Q fz\,\mathrm{d}x\,\mathrm{d}t$$

holds for any $z \in L_2(I;V)$ and the conditions (7) remain valid. In fact, a solution of $\mathcal{P}_\eta$ should be denoted by u_η, but in this section we drop the index η for the sake of simplicity.

We shall verify the existence and the uniqueness of a solution to the penalized problem.

Theorem 3.1. *Let $f \in L_2(I; H^{-1}(\Omega))$, $u_0 \in H^2(\Omega)$, $u_1 \in H^1(\Omega)$, $i = 0, 1$. Then there exists a unique solution u of the problem $\mathcal{P}_\eta$.*

P r o o f. (i) *Existence.* Let us denote by $\{w_i \in V;\ i \in \mathbb{N}\}$ an orthonormal basis of $\mathring{H}^1(\Omega)$ with respect to the inner product

$$(\cdot,\cdot)_a\colon\ (v, w) \mapsto \int_\Omega (a\nabla v \cdot \nabla w + vw)\,\mathrm{d}x$$

fulfilling the eigenvalue problem

$$\int_\Omega A(w_i, v)\,\mathrm{d}x = \int_\Omega \lambda_i(a\nabla w_i \cdot \nabla v + w_i v)\,\mathrm{d}x \quad \forall\, v \in V.$$

We construct the Galerkin approximation u_m of a solution in the form

$$u_m(t) = \sum_{i=1}^m \alpha_i(t) w_i + u_0, \quad \alpha_i(t) \in \mathbb{R},\ \ i = 1, \dots, m,\ \ m \in \mathbb{N}$$

given by the solution of the approximate problem

$$\begin{aligned}(20)\qquad &\int_\Omega \big(a\nabla \ddot{u}_m(t) \cdot \nabla w_i + \ddot{u}_m(t) w_i + A(\mathfrak{E}u_m(t) + Eu_m(t), w_i)\\ &\qquad + [u_m(t), w_i](\mathfrak{E}\Phi(u_m, u_m)(t) + E\Phi(u_m, u_m)(t)) - \eta^{-1} u_m(t)^- w_i\big)\,\mathrm{d}x\\ &= \int_\Omega f(t) w_i\,\mathrm{d}x, \quad i = 1, \dots, m,\\ (21)\qquad &u_m(0) = u_{0m}, \quad \dot{u}_m(0) = u_{1m}, \quad u_{im} \to u_i \text{ in } H^{2-i}(\Omega), \quad i = 0, 1.\end{aligned}$$

The matrix $\mathbb{A} = (a_{ij})$, $a_{ij} = \int_\Omega (a\nabla w_i \cdot \nabla w_j + w_i w_j)\,\mathrm{d}x$ is positive definite. The system (20) can then be expressed in the form

$$\ddot{\alpha}_i = F_i(t, \alpha_1, \dots, \alpha_m), \quad i = 1, \dots, m.$$

Its right-hand side satisfies the conditions for the local existence of a solution fulfilling the initial conditions corresponding to the functions u_{0m}, u_{1m}. Hence there exists a Galerkin approximation $u_m(t)$ defined on some interval $[0, t_m]$, $0 < t_m \leqslant T$. To derive the *a priori* estimates for solutions of (20), (21), we multiply the equation (20) by $\dot{\alpha}_i(t)$, sum over i and integrate on the interval $[0, s]$, $s \leqslant t_m$.

Taking into account the property

$$(22)\qquad \int_\Omega [u, v] y\,\mathrm{d}x = \int_\Omega [u, y] v\,\mathrm{d}x$$

if at least one element of $\{u, v, y\}$ belongs to $\overset{\circ}{H}{}^2(\Omega)$, cf. [4], and using the standard integration by parts and the properties of the kernel function K we get

$$
\begin{aligned}
(23)\quad &\int_{Q_s}\Big(\frac{1}{2}\partial_t\Big(a|\nabla\dot u_m|^2+|\dot u_m|^2+EA(u_m,u_m)+\frac{E}{2}(\Delta\Phi(u_m,u_m))^2+\eta^{-1}(u_m^-)^2\Big)\\
&\qquad+\frac{K(s-t)}{2}\Big(A\big(u_m(s)-u_m(t),u_m(s)-u_m(t)\big)\\
&\qquad\qquad+\frac{1}{2}\big(\Delta(\Phi(u_m,u_m)(s)-\Phi(u_m,u_m)(t))\big)^2\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\frac{1}{2}\int_{Q_s}\int_0^t K_t'(t-r)A\big(u_m(t)-u_m(r),u_m(t)-u_m(r)\big)\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad-\frac{1}{4}\int_{Q_s}\int_0^t K_t'(t-r)\big(\Delta\Phi(u_m,u_m)(t)-\Delta\Phi(u_m,u_m)(r)\big)^2\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}x\\
&\quad=\int_{Q_s} f\dot u_m\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

By virtue of (3) and what follows, the identity (23) leads to the *a priori* estimates independent of the penalty parameter η, of $m\in\mathbb{N}$ as well as of $t_m\in I$:

$$
\begin{aligned}
(24)\qquad &\|u_m\|^2_{H^\alpha(I;H^2(\Omega))}+\|\dot u_m\|^2_{L_\infty(I;H^1(\Omega))}+\|u_m\|^2_{L_\infty(I;H^2(\Omega))}\\
&\qquad+\|\Phi(u_m,u_m)\|^2_{H^\alpha(I;H^2(\Omega))}\leqslant c\equiv c(f,u_0,u_1),
\end{aligned}
$$

where for a Banach space X

$$\|v\|^2_{H^\alpha(I;X)}\equiv\int_I\|v\|^2_X\,\mathrm{d}t+\int_I\int_I\frac{\|v(t)-v(s)\|^2_X}{|t-s|^{1+2\alpha}}\,\mathrm{d}s\,\mathrm{d}t.$$

Moreover, the estimate (14) implies

$$(25)\qquad \|\Phi(u_m,u_m)\|^2_{L_2(I;W^2_p(\Omega))}\leqslant c_p\equiv c_p(f,u_0,u_1)\quad\forall p>2$$

and naturally the solution u_m exists on the whole interval $[0,T]$.

The estimate (25) further implies

$$
\begin{aligned}
(26)\qquad &[u_m,\Phi(u_m,u_m)]\in L_2(I;L_r(\Omega)),\quad r=\frac{2p}{p+2},\\
&\|[u_m,\Phi(u_m,u_m)]\|_{L_2(I;L_r(\Omega))}\leqslant c_r\equiv c_r(f,u_0,u_1).
\end{aligned}
$$

Moreover, using (24) we arrive at the important *dual* estimate

$$(27)\qquad \|\ddot u_m\|^2_{L_2(Q)}\leqslant c_\eta,\quad m\in\mathbb{N}.$$

Indeed, we have just proved that the sequence of remainders $a\Delta\ddot{u}_m - \ddot{u}_m$ is bounded in $L_2(I;V^*)$ and via integration by parts we get

$$\|\ddot{u}_m\|_{L_2(Q)} = \sup_{\|f\|_{L_2(Q)}\leqslant 1} (\ddot{u}_m, f)_Q \leqslant c \sup_{\|v\|_{L_2(I;V)}\leqslant 1} (\ddot{u}_m, v - a\Delta v)_Q \leqslant k, \tag{28}$$

where we employ also the properties of the Green operator for the elliptic problem $v - a\Delta v = f$ with homogeneous Dirichlet boundary value condition and the right-hand side in $L_2(\Omega)$ for Ω of the class C^2 or convex polygonal as well as the fact that $\eta > 0$ is fixed.

We proceed with the convergence of the Galerkin approximation. Applying the estimates (24)–(27) we obtain for any $p \in [1,\infty]$, a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$), and a function u the following convergences

$$\begin{aligned}
&\dot{u}_m \rightharpoonup^* \dot{u} && \text{in } L_\infty(I;H^1(\Omega)),\\
&u_m \rightharpoonup u && \text{in } H^\alpha(I;H^2(\Omega)),\\
&\ddot{u}_m \rightharpoonup \ddot{u} && \text{in } L_2(Q),\\
&\dot{u}_m \to \dot{u} && \text{in } L_q(I;W^1_{2+\theta}(\Omega)) \text{ for any } 2 \leqslant q \in \mathbb{R}\\
& && \text{and a small } \theta \equiv \theta(\alpha) > 0,\\
&u_m \to u && \text{in } L_{2+\theta}(I;W^1_q(\Omega)) \cap C^0(I;W^1_{2+\theta}(\Omega))\\
& && \text{for any real } q \geqslant 2 \text{ and a small } \theta \equiv \theta(\alpha) > 0,\\
&\Phi(u_m,u_m) \rightharpoonup \Phi(u,u) && \text{in } L_2(I;W^2_p(\Omega)) \cap H^\alpha(I;H^2(\Omega)).
\end{aligned} \tag{29}$$

To get the fourth convergence we first interpolate the second and third one via Corollary 2.4 with $\Theta_\lambda = 2/(2-\alpha)+\theta$, $0<\theta$ arbitrarily small and we apply Corollary 2.2 to the result. Then we interpolate this result once again with the first convergence, where we replace ∞ by an arbitrarily large real $\tilde{p}$. The first part of the fifth convergence follows from the second one and the second part from the fourth one via Corollary 2.2. The last convergence is a consequence of (14) and the second and the fifth convergence. Indeed, taking a functional $F \in L_2(I;W^2_p(\Omega))^*$ (with the norm denoted by $\|\cdot\|_*$) we obtain

$$\langle F, \Phi(u_m,u_m) - \Phi(u,u)\rangle = \langle F, \Phi(u_m - u, u)\rangle + \langle F, \Phi(u_m, u_m - u)\rangle.$$

We have

$$|\langle F, \Phi(v,u)\rangle| \leqslant \|F\|_* \|u\|_{C^0(I;W^1_p(\Omega))} \|v\|_{L_2(I;H^2(\Omega))} \quad \forall\, v \in L_2(I;H^2(\Omega))$$

with $p = 2+\theta$, θ from (29). The second convergence in (29) then implies

$$\langle F, \Phi(u_m - u, u)\rangle \to 0.$$

Further, we have the estimate

$$|\langle F, \Phi(u_m, u_m - u)\rangle| \leqslant \|F\|_* \|u_m\|_{L_2(I;H^2(\Omega))} \|u_m - u\|_{C^0(I;W^1_p(\Omega))}.$$

The boundedness of the sequence $\{u_m\}$ in $L_2(I; H^2(\Omega))$ and the strong convergence $u_m \to u$ in $C^0(I; W^1_p(\Omega))$ for $p = 2 + \theta$ then imply the convergence

$$\langle F, \Phi(u_m, u_m - u)\rangle \to 0$$

and the last convergence in (29) follows.

Let $\mu \in \mathbb{N}$ and $z_\mu = \sum_{i=1}^{m} \varphi_i(t) w_i$, $\varphi_i \in \mathcal{D}(0,T)$, $i = 1, \ldots, \mu$. We have for arbitrary $m \in \mathbb{N}$ and $t \in I$ the relation

$$\begin{aligned}\int_\Omega &\big(a\nabla \ddot{u}_m(t) \cdot \nabla z_\mu(t) + \ddot{u}_m(t) z_\mu(t) + A((\mathfrak{E}u_m + E u_m)(t), z_\mu(t)) \\ &\quad + [u_m(t), z_\mu(t)](\mathfrak{E}\Phi(u_m, u_m)(t) + E\Phi(u_m, u_m)(t)) - \eta^{-1} u_m(t)^- z_\mu(t)\big)\,\mathrm{d}x \\ &= \int_\Omega f(t) z_\mu(t)\,\mathrm{d}x.\end{aligned}$$

The convergence process (29) and the property (22) imply that the function u fulfils

$$\begin{aligned}\int_\Omega &\big(a\nabla \ddot{u} \cdot \nabla z_\mu + \ddot{u} z_\mu + A(\mathfrak{E}u, z_\mu) + EA(u, z_\mu) \\ &\quad + [u, \mathfrak{E}\Phi(u,u) + E\Phi(u,u)] z_\mu - \eta^{-1} u^- z_\mu\big)\,\mathrm{d}x\,\mathrm{d}t = \int_Q f z\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The functions $\{z_\mu\}$ form a dense subset of the set $L_2(I; H^2(\Omega))$, hence the function u fulfils the identity (19). The initial conditions (7) follow due to (21) and the proof of the existence of a solution is complete.

(ii) *Uniqueness.* Let u, $\hat{u}$ be two solutions of Problem $\mathcal{P}_\eta$ and let $w = u - \hat{u}$. We have for arbitrary $s \in I$ the relations

$$\begin{aligned}\int_{Q_s} &\big(a\nabla \ddot{w} \cdot \nabla z + \ddot{w} z + A(\mathfrak{E}w, z) + EA(w, z) \\ &+ ([u, \mathfrak{E}\Phi(u,u) + E\Phi(u,u)] - [\hat{u}, \mathfrak{E}\Phi(\hat{u},\hat{u}) + E\Phi(\hat{u},\hat{u})])z \\ &- \eta^{-1}(u^- - \hat{u}^-)z\big)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \forall\, z \in L_2(I; H^2(\Omega)), \\ &w(0,\cdot) = \dot{w}(0,\cdot) = 0 \quad \text{on } \Omega.\end{aligned}$$

After setting $z = \dot{w}$ we get

$$\begin{aligned}&\frac{1}{2}\Big(a\|\nabla \dot{w}\|^2_{L_2(\Omega)} + \|\dot{w}\|^2_{L_2(\Omega)} + E\int_\Omega A(w,w)\,\mathrm{d}x\Big)(s) + \int_{Q_s} A(\mathfrak{E}w, \dot{w})\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_s} \big([\hat{u}, \mathfrak{E}\Phi(\hat{u},\hat{u}) + E\Phi(\hat{u},\hat{u})] - [u, \mathfrak{E}\Phi(u,u) + E\Phi(u,u)] + \eta^{-1}(u^- - \hat{u}^-)\big)\dot{w}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using the estimate (26) with u and $\hat{u}$ instead of u_m and the imbedding $L_q(\Omega) \hookrightarrow H^1(\Omega)$, $q = 2p/(p-2)$ we obtain the inequality

$$\|\dot{w}\|^2_{H^1(\Omega)}(s) \leqslant \frac{c}{\eta} \int_0^s \|\dot{w}\|^2_{H^1(\Omega)}(t)\,\mathrm{d}t \quad \text{for every } s \in I.$$

The Gronwall lemma implies

$$\|\dot{w}\|_{H^1(\Omega)}(s) = 0 \quad \text{for every } s \in I$$

and the uniqueness of the solution follows due to the zero initial conditions for $w \equiv u - \hat{u}$. □

Let us remark that the solution u satisfies again the η-independent *a priori* estimates (24).

4. Existence of solutions to the original problem

Our task now is to perform the limit process $\eta \searrow 0$. To get it, we need a new η-independent *dual* estimate for the solutions u_η of the problems $\mathcal{P}_\eta$. For this we need an η-independent estimate of the penalty term. To get it, we put $z = u_\eta - u_0$ into (19), where we rewrite u by u_η. From the validity of (24) and the strict positivity of u_0 we easily derive the penalty estimate

$$\|\eta^{-1} u_\eta^-\|_{L_1(Q)} \leqslant c \tag{30}$$

which is η independent and we recall the obvious imbedding

$$L_1(Q) \hookrightarrow L_1(I; H^{-1-\theta}(\Omega)) \quad \forall\, \theta \in \Bigl(0, \frac{1}{2}\Bigr).$$

As earlier this gives the estimate of $\ddot{u}_\eta - \Delta\ddot{u}_\eta$, but here in $L_1(I; V^*))$ only. However, an appropriate analogue of the estimate (28) leads to the dual estimate

$$\|\ddot{u}_\eta\|_{L_1(I; L_2(\Omega))} \leqslant c \tag{31}$$

with c independent of η. Hence $\dot{u}_\eta$ is bounded in $W^{1-\theta}_{1+\theta'}(I; L_2(\Omega))$ for any $\theta \in (0,1)$ and for $\theta' \equiv \theta'(\theta) \searrow 0$ if $\theta \searrow 0$. Interpolating this space with the space $L_p(I; H^1(\Omega))$ for some real $p > 2$ big enough, we get that

$$\|\dot{u}_\eta\|_{H^{1/2-\theta}(I; H^{1/2}(\Omega))} \leqslant C \quad \text{with } 0 < \theta \text{ arbitrarily small.} \tag{32}$$

Interpolating the result in (32) with the fact that u_η is bounded in $H^\alpha(I, H^2(\Omega))$, we get that u_η is again bounded in some $H^{1+\theta_1}(I; H^{1+\theta_2}(\Omega))$ for some $\theta_1, \theta_2 > 0$ dependent on α, i.e. $\dot u_\eta$ is bounded in $H^{\theta_1}(I; H^{1+\theta_2}(\Omega))$. This space is compactly imbedded in $L_2(I, W_s^1(\Omega))$ for some $s > 2$.

Hence there exist sequences $\eta_k \searrow 0$ and $u_{\eta_k} \equiv u_k$ and a function u such that the following convergences

$$u_k \rightharpoonup u \quad \text{in } H^\alpha(I; H^2(\Omega)), \tag{33}$$

$$\dot u_k \to \dot u \quad \text{in } L_2(I; W_s^1(\Omega)), \tag{34}$$

$$u_k \to u \quad \text{in } C^0(I; W_s^1(\Omega)), \tag{35}$$

$$\Phi(u_k, u_k) \rightharpoonup \Phi(u, u) \quad \text{in } L_2(I; W_p^2(\Omega)) \tag{36}$$

are valid. These convergences are sufficient to prove that u is a solution of Problem $\mathcal{P}$. Indeed, we perform the integration by parts in the terms in (19) where $\ddot u_k$ occurs. Then for these terms the strong convergences (34) and (35) are sufficient for the limit process. Moreover, we add $A(-Eu - \mathfrak{E}u, z)$ to both sides of (19), setting $z = u_k - u$. The estimate for Φ, the non-negativeness of the kernel K and the condition (17) yield

$$u_k \to u \in L_2(I; H^2(\Omega)). \tag{37}$$

For the limit process in the terms with Φ we employ the obvious strong convergence $u_k \to u$ in $L_\infty(Q)$ and the weak convergence $[u_k, \Phi(u_k, u_k)] \rightharpoonup [u, \Phi(u, u)]$ in $(L_\infty(Q))^*$ which follows from (36) and (37). Hence the following theorem holds.

Theorem 4.1. *Let the relation* (3) *for some* $\alpha \in (0, 1/2)$ *and the assumptions* (16) *and* (17) *be satisfied. Then there exists a solution to Problem* $\mathcal{P}$.

R e m a r k 4.2. The boundary conditions for the simply supported plate played a key role in deriving the dual estimates (27) and (31). The dynamic contact problem for a viscoelastic clamped plate with a short memory has been solved in [1] but the same problem for the clamped plate with a long memory remains unsolved.

References

[1] *I. Bock, J. Jarušek*: Unilateral dynamic contact of viscoelastic von Kármán plates. Adv. Math. Sci. Appl. *16* (2006), 175–187. zbl

[2] *I. Bock, J. Lovíšek*: On unilaterally supported viscoelastic von Kármán plates with a long memory. Math. Comput. Simul. *61* (2003), 399–407. zbl

[3] *I. Bock, J. Lovíšek*: On a contact problem for a viscoelastic von Kármán plate and its semidiscretization. Appl. Math. *50* (2005), 203–217. zbl

[4] *P. G. Ciarlet, P. Rabier*: Les équations de von Kármán. Springer-Verlag, Berlin, 1980. zbl

[5] *C. Eck, J. Jarušek, and M. Krbec*: Unilateral contact problems. Variational Methods and Existence Theorems. Pure and Applied Mathematics No. 270. Chapman & Hall/CRC, Boca Raton-London-New York-Singapore, 2005. zbl

[6] *J. Jarušek*: Solvability of unilateral hyperbolic problems involving viscoelasticity via penalization. Proc. of "Conference EQUAM", Varenna 1992 (R. Salvi, ed.). SAACM *3* (1993), 129–140.

[7] *J. Jarušek*: Solvability of the variational inequality for a drum with a memory vibrating in the presence of an obstacle. Boll. Unione Mat. Ital. VII. Ser., A *8* (1994), 113–122. zbl

[8] *J. Jarušek, J. V. Outrata*: On sharp optimality conditions in control of contact problems with strings. Nonlinear Anal. *67* (2007), 1117–1128.

[9] *H. Koch, A. Stahel*: Global existence of classical solutions to the dynamic von Kármán equations. Math. Methods Appl. Sci. *16* (1993), 581–586. zbl

[10] *J. E. Muñoz Rivera, G. Perla Menzala*: Decay rates of solutions to a von Kármán system for viscoelastic plates with memory. Q. Appl. Math. *57* (1999), 181–200. zbl

[11] *J. Nečas*: Les méthodes directes en théorie des équations elliptiques. Masson/Academia, Paris/Praha, 1967.

[12] *A. Oukit, R. Pierre*: Mixed finite element for the linear plate problem: the Hermann-Miyoshi model revisited. Numer. Math. *74* (1996), 453–477. zbl

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