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A PRIORI ESTIMATES AND SOLVABILITY OF A NON-RESONANT GENERALIZED MULTI-POINT BOUNDARY VALUE PROBLEM OF MIXED DIRICHLET-NEUMANN-DIRICHLET TYPE INVOLVING A p-LAPLACIAN TYPE OPERATOR

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Abstract. This paper is devoted to the problem of existence of a solution for a non-resonant, non-linear generalized multi-point boundary value problem on the interval $[0,1]$. The existence of a solution is obtained using topological degree and some a priori estimates for functions satisfying the boundary conditions specified in the problem.

Keywords: generalized multi-point boundary value problems, p-Laplace type operator, non-resonance, a priori estimates, topological degree

MSC 2000: 34B10, 34B15, 34L30

1. Introduction

Let φ be an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, $f\colon [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ a function satisfying Carathéodory conditions and $e\colon [0,1]\to\mathbb{R}$ be a function in $L^1[0,1]$. Let $\xi_i,\tau_j\in(0,1)$, $a_i,b_j\in\mathbb{R}$, $i=1,2,\ldots,m-2$, $j=1,2,\ldots,n-2$, $0<\xi_1<\xi_2<\ldots<\xi_{m-2}<1$, $0<\tau_1<\tau_2<\ldots<\tau_{n-2}<1$ be given. We study the problem of existence of solutions for the generalized multi-point boundary value problem

$$(1)\qquad \begin{gathered}(\varphi(x'))'=f(t,x,x')+e(t),\quad \text{a.e. on } [0,1],\\ x(0)=\sum_{i=1}^{m-2}a_ix'(\xi_i),\quad x(1)=\sum_{j=1}^{n-2}b_jx(\tau_j)\end{gathered}$$

in the non-resonance case. We say that this problem is non-resonant if the associated problem

$$(2) \qquad (\varphi(x'))' = 0, \quad \text{a.e. on } [0,1], \\ x(0) = \sum_{i=1}^{m-2} a_i x'(\xi_i), \quad x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j)$$

has the trivial solution as its only solution. This is the case if the "non-resonance condition"

$$(3) \qquad \left(\sum_{i=1}^{m-2} a_i\right)\left(1 - \sum_{j=1}^{n-2} b_j\right) \neq \sum_{j=1}^{n-2} b_j \tau_j - 1$$

holds. This problem was studied by Gupta, Ntouyas, and Tsamatos in [20] when the homeomorphism φ from $\mathbb{R}$ onto $\mathbb{R}$ is the identity homeomorphism, i.e., for second order ordinary differential equations when $a_i, b_j \in \mathbb{R}$ have the same sign for all $i = 1, 2, \ldots, m-2$, $j = 1, 2, \ldots, n-2$. The study of multi-point boundary value problems for nonlinear second order ordinary differential equations was initiated by Il'in and Moiseev in [23], [24] who were motivated by the works of Bitsadze and Samarskiĭ on nonlocal linear elliptic boundary value problems, [2], [3], [4], and it has been the subject of many papers, see for example [5], [6], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [22], [25], [30] and [31]. More recently multipoint boundary value problems involving a p-Laplacian type operator or the more general operator $-(\varphi(x'))'$ have been studied in [1], [7], [8], [9], [10], [26], to mention just a few.

We present in Section 2 some a priori estimates for functions x that satisfy the boundary conditions in (1). Our a priori estimates utilize the non-resonance condition for the boundary value problem (1). In Section 3 we present an existence theorem for the boundary value problem (1) using the degree theory.

2. A priori estimates

We shall assume the following throughout the rest of the paper:

(a) For any constant $K > 0$,

$$(4) \qquad \alpha(K) := \limsup_{z \to \infty} \frac{\varphi(Kz)}{\varphi(z)} < \infty.$$

(b) For any σ, $0 \leqslant \sigma < 1$,

$$(5) \qquad \tilde{\alpha}(\sigma) := \limsup_{z \to \infty} \frac{\varphi(\sigma z)}{\varphi(z)} < 1.$$

(c) $\xi_i, \tau_j \in (0,1)$, $a_i, b_j \in \mathbb{R}$, $i = 1,2,\ldots,m-2$, $j = 1,2,\ldots,n-2$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \ldots < \tau_{n-2} < 1$ are such that the non-resonance condition (3) holds.

We observe that when the non-resonance condition (3) holds then at least one of $1 - \sum\limits_{j=1}^{n-2} b_j$, $\sum\limits_{j=1}^{n-2} b_j\tau_j - 1$ is non-zero. For $a \in \mathbb{R}$, let us denote $a^+ = \max(a,0)$, $a^- = \max(-a,0)$ so that $a = a^+ - a^-$ and $|a| = a^+ + a^-$. Furthermore, let us define (6)

$$\sigma_1 \equiv \begin{cases} \min\left\{ \dfrac{\sum\limits_{j=1}^{n-2} b_j^+}{1 + \sum\limits_{j=1}^{n-2} b_j^-}, \dfrac{1 + \sum\limits_{j=1}^{n-2} b_j^-}{\sum\limits_{j=1}^{n-2} b_j^+} \right\} \in [0,1), & \text{if } 1 - \sum\limits_{j=1}^{n-2} b_j \neq 0 \text{ and } \sum\limits_{j=1}^{n-2} b_j^+ \neq 0, \\ 0, & \text{if } 1 - \sum\limits_{j=1}^{n-2} b_j \neq 0 \text{ and } \sum\limits_{j=1}^{n-2} b_j^+ = 0, \\ 1, & \text{if } 1 - \sum\limits_{j=1}^{n-2} b_j = 0. \end{cases}$$

Notice that $\sigma_1 \in [0,1]$.

Theorem 1. *Let assumption* (c) *hold. Also let a function x be such that x, x' are absolutely continuous on* $[0,1]$ *and* $x(0) = \sum\limits_{i=1}^{m-2} a_i x'(\xi_i)$, $x(1) = \sum\limits_{j=1}^{n-2} b_j x(\tau_j)$. *Then there exists an $M \in (0,\infty)$ such that*

$$\|x\|_\infty \leqslant M\|x'\|_\infty. \tag{7}$$

P r o o f. We see from $x(t) = x(0) + \int_0^t x'(s)\,\mathrm{d}s$ and the assumption that $x(0) = \sum\limits_{i=1}^{m-2} a_i x'(\xi_i)$ that

$$|x(t)| \leqslant \left(\sum_{i=1}^{m-2} |a_i| + 1\right)\|x'\|_\infty \quad \text{for } t \in [0,1]. \tag{8}$$

Accordingly we get

$$\|x\|_\infty \leqslant \left(\sum_{i=1}^{m-2} |a_i| + 1\right)\|x'\|_\infty. \tag{9}$$

Now, when $1 - \sum\limits_{j=1}^{n-2} b_j = 0$, estimate (7) holds with $M = 1 + \sum\limits_{i=1}^{m-2} |a_i|$. Next, let us assume in the following that $1 - \sum\limits_{j=1}^{n-2} b_j \neq 0$.

Now, we see from $x(1)-x(\tau_j)=\int_{\tau_j}^{1} x'(s)\,\mathrm{d}s$, for $j=1,2,\dots,n-2$ and from the assumption $x(1)=\sum\limits_{j=1}^{n-2} b_j x(\tau_j)$ that $\Big(\sum\limits_{j=1}^{n-2} b_j-1\Big)x(1)=\sum\limits_{j=1}^{n-2} b_j\int_{\tau_j}^{1} x'(s)\,\mathrm{d}s$. It follows that

$$|x(1)|\leqslant \frac{\sum\limits_{j=1}^{n-2}|b_j(1-\tau_j)|}{\Big|1-\sum\limits_{j=1}^{n-2} b_j\Big|}\|x'\|_\infty. \tag{10}$$

Next, we use the equations $x(t)-x(\tau_j)=\int_{\tau_j}^{t} x'(s)\,\mathrm{d}s$ for $t\in[0,1]$, $j=1,2,\dots,n-2$, and the assumption $x(1)=\sum\limits_{j=1}^{n-2} b_j x(\tau_j)$ to get

$$x(t)=\frac{1}{\sum\limits_{j=1}^{n-2} b_j}\left(x(1)+\sum_{j=1}^{n-2} b_j\int_{\tau_j}^{t} x'(s)\,\mathrm{d}s\right)\quad \text{for } t\in[0,1]. \tag{11}$$

It follows from (10), (11) and with $\mu_j=\max(\tau_j,1-\tau_j)$ for $j=1,2,\dots,n-2$ that

$$\|x\|_\infty\leqslant \frac{1}{\Big|\sum\limits_{j=1}^{n-2} b_j\Big|}\left(\frac{\sum\limits_{j=1}^{n-2}|b_j(1-\tau_j)|}{\Big|1-\sum\limits_{j=1}^{n-2} b_j\Big|}+\sum_{j=1}^{n-2}\mu_j|b_j|\right)\|x'\|_\infty. \tag{12}$$

Similarly, starting from the equation $x(t)=x(1)-\int_t^1 x'(s)\,\mathrm{d}s$, we obtain the estimate

$$\|x\|_\infty\leqslant \left(\frac{\sum\limits_{j=1}^{n-2}|b_j(1-\tau_j)|}{\Big|1-\sum\limits_{j=1}^{n-2} b_j\Big|}+1\right)\|x'\|_\infty. \tag{13}$$

Next, since $x(1)=\sum\limits_{j=1}^{n-2} b_j x(\tau_j)$, we see that

$$x(1)+\sum_{j=1}^{n-2} b_j^- x(\tau_j)=\sum_{j=1}^{n-2} b_j^+ x(\tau_j).$$

It follows, by the intermediate value theorem, that there must exist χ_1, χ_2 in $[0,1]$ such that

$$\left(1+\sum_{j=1}^{n-2} b_j^-\right)x(\chi_1)=\left(\sum_{i=1}^{m-2} b_j^+\right)x(\chi_2). \tag{14}$$

If now one of $x(\chi_1)$, $x(\chi_2)$ is zero, we see using one of the two equations

$$x(t) = x(\chi_k) + \int_{\chi_k}^{t} x'(s)\,\mathrm{d}s, \quad k = 1, 2, \ t \in [0, 1] \tag{15}$$

that

$$\|x\|_\infty \leqslant \|x'\|_\infty. \tag{16}$$

If both $x(\chi_1)$, $x(\chi_2)$ are non-zero it is easy to see from (14) that $x(\chi_1) \neq x(\chi_2)$, since we have assumed that $1 - \sum_{j=1}^{n-2} b_j \neq 0$, so that $1 + \sum_{j=1}^{n-2} b_j^- \neq \sum_{j=1}^{n-2} b_j^+$. It then follows easily from (14) and (15) that

$$\|x\|_\infty \leqslant \frac{1}{1-\sigma_1}\|x'\|_\infty, \tag{17}$$

where σ_1 is defined in (6).

Estimate (7) is now immediate from (9), (12), (13), (16), (17) with

$$M = \min\left\{ \frac{1}{\left|\sum_{j=1}^{n-2} b_j\right|} \left(\sum_{j=1}^{n-2} |b_j|\mu_j + \frac{\sum_{j=1}^{n-2} |b_j(1-\tau_j)|}{\left|1 - \sum_{j=1}^{n-2} b_j\right|} \right), \right.$$
$$\left. 1 + \frac{\sum_{j=1}^{n-2} |b_j(1-\tau_j)|}{\left|1 - \sum_{j=1}^{n-2} b_j\right|}, \frac{1}{1-\sigma_1}, 1 + \sum_{i=1}^{m-2} |a_i| \right\}$$

when $1 - \sum_{j=1}^{n-2} b_j \neq 0$. This completes the proof of the theorem. □

Lemma 2. *Let us set*

$$A = \left(1 - \sum_{j=1}^{n-2} b_j\right)^+ + \sum_{j=1}^{n-2} [b_j(1-\tau_j)]^+ + \sum_{i=1}^{m-2}\left[a_i\left(1 - \sum_{j=1}^{n-2} b_j\right)\right]^+ \tag{18}$$

and

$$B = \left(1 - \sum_{j=1}^{n-2} b_j\right)^- + \sum_{j=1}^{n-2} [b_j(1-\tau_j)]^- + \sum_{i=1}^{m-2}\left[a_i\left(1 - \sum_{j=1}^{n-2} b_j\right)\right]^-. \tag{19}$$

Then

$$A \neq B$$

provided the non-resonance condition (3) *holds.*

P r o o f. We note that

$$\begin{aligned} A - B &= \Bigl(1 - \sum_{j=1}^{n-2} b_j\Bigr) + \sum_{j=1}^{n-2} b_j(1-\tau_j) + \sum_{i=1}^{m-2} a_i\Bigl(1 - \sum_{j=1}^{n-2} b_j\Bigr) \\ &= 1 - \sum_{j=1}^{n-2} b_j + \sum_{j=1}^{n-2} b_j - \sum_{j=1}^{n-2} b_j\tau_j + \sum_{i=1}^{m-2} a_i\Bigl(1 - \sum_{j=1}^{n-2} b_j\Bigr) \\ &= 1 - \sum_{j=1}^{n-2} b_j\tau_j + \Bigl(\sum_{i=1}^{m-2} a_i\Bigr)\Bigl(1 - \sum_{j=1}^{n-2} b_j\Bigr) \neq 0 \end{aligned}$$

in view of the non-resonance assumption (3). Hence $A \neq B$. This completes the proof of the lemma. □

Let us define σ^* by

$$\sigma^* = \min\Bigl\{\frac{A}{B}, \frac{B}{A}\Bigr\} \in [0,1), \tag{20}$$

where A, B are defined in Lemma 2. Accordingly, we see that $\sigma^* \in [0,1)$. Furthermore, in view of (5) we have $\tilde\alpha(\sigma^*) < 1$.

Let $\varepsilon > 0$ be such that $\tilde\alpha(\sigma^*) + \varepsilon < 1$ and let the constant C_ε be such that

$$\varphi(\sigma^* z) \leqslant \bigl(\tilde\alpha(\sigma^*) + \varepsilon\bigr)\varphi(z) + C_\varepsilon \quad \text{for every } \ z \in \mathbb{R}. \tag{21}$$

Theorem 3. *Let assumption* (c) *hold. Also let the function x be such that x, x' is absolutely continuous on* $[0,1]$ *with* $(\varphi(x'))' \in L^1(0,1)$ *and* $x(0) = \sum\limits_{i=1}^{m-2} a_i x'(\xi_i)$, $x(1) = \sum\limits_{j=1}^{n-2} b_j x(\tau_j)$. *Then*

$$\|\varphi(x')\|_\infty \leqslant \frac{1}{1 - \tilde\alpha(\sigma^*) - \varepsilon}\|(\varphi(x'))'\|_{L^1(0,1)} + \frac{C_\varepsilon}{1 - \tilde\alpha(\sigma^*) - \varepsilon}, \tag{22}$$

where ε and C_ε are as in (21).

P r o o f. For $j = 1, 2, \ldots, n-2$ we see using the mean value theorem that there exist λ_j in $[0,1]$ such that

$$x(1) - x(\tau_j) = (1-\tau_j)x'(\lambda_j),$$

and we see using $x(1)=\sum\limits_{j=1}^{n-2} b_j x(\tau_j)$ that

$$\left(\sum_{j=1}^{n-2} b_j-1\right)x(1)=\sum_{j=1}^{n-2} b_j(1-\tau_j)x'(\lambda_j). \tag{23}$$

Also, we see that there exists a $\lambda\in[0,1]$ such that

$$x(1)-x(0)=x'(\lambda). \tag{24}$$

Now, we see from equations (23), (24) that

$$\begin{aligned}\left(\sum_{j=1}^{n-2} b_j-1\right)x'(\lambda)&=\left(\sum_{j=1}^{n-2} b_j-1\right)\bigl(x(1)-x(0)\bigr)\\&=\sum_{j=1}^{n-2} b_j(1-\tau_j)x'(\lambda_j)-\left(\sum_{j=1}^{n-2} b_j-1\right)\left(\sum_{i=1}^{m-2} a_i x'(\xi_i)\right)\\&=\sum_{j=1}^{n-2} b_j(1-\tau_j)x'(\lambda_j)+\left(\sum_{i=1}^{m-2} a_i\left(1-\sum_{j=1}^{n-2} b_j\right)x'(\xi_i)\right).\end{aligned}$$

It follows that

$$\left(1-\sum_{j=1}^{n-2} b_j\right)x'(\lambda)+\sum_{j=1}^{n-2} b_j(1-\tau_j)x'(\lambda_j)+\sum_{i=1}^{m-2} a_i\left(1-\sum_{j=1}^{n-2} b_j\right)x'(\xi_i)=0.$$

Similar to the proof of Theorem 1 (see (14)), we use (18), (19) and the intermediate value theorem to see that there are υ_1, υ_2 in $[0,1]$ such that

$$Ax'(\upsilon_1)-Bx'(\upsilon_2)=0. \tag{25}$$

Suppose now that one of $x'(\upsilon_1)$, $x'(\upsilon_2)$ is zero. We then see from one of the equation

$$\varphi(x'(t))=\varphi(x'(\upsilon_k))+\int_{\upsilon_k}^{t}(\varphi(x'))'(s)\,\mathrm{d}s,\quad k=1,2,\ \ t\in[0,1] \tag{26}$$

that

$$\|\varphi(x')\|_\infty\leqslant\|(\varphi(x'))'\|_{L^1(0,1)}. \tag{27}$$

Let us, next, suppose that both $x'(\upsilon_1)$, $x'(\upsilon_2)$ are non-zero. Since now $A\neq B$, in view of Lemma 2 we see from equation (25) that

$$x'(\upsilon_1)\neq x'(\upsilon_2).$$

We now use the equations

$$\varphi(x'(t)) = \varphi(x'(\upsilon_1)) + \int_{\upsilon_k}^{t} (\varphi(x'))'(s)\,\mathrm{d}s = \varphi\Big(\frac{B}{A}x'(\upsilon_2)\Big) + \int_{\upsilon_k}^{t} (\varphi(x'))'(s)\,\mathrm{d}s,$$
$$\varphi(x'(t)) = \varphi(x'(\upsilon_2)) + \int_{\upsilon_k}^{t} (\varphi(x'))'(s)\,\mathrm{d}s = \varphi\Big(\frac{A}{B}x'(\upsilon_1)\Big) + \int_{\upsilon_k}^{t} (\varphi(x'))'(s)\,\mathrm{d}s,$$

along with the definition of σ^* given in (20), (21) and the estimate (27) to obtain the estimate (22). This completes the proof of the theorem. $\square$

3. Existence Theorem

Let φ be an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, $f\colon [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ a function satisfying the Carathéodory conditions and $e\colon [0,1]\to\mathbb{R}$ a function in $L^1[0,1]$. Let assumption (c) hold.

Theorem 4. *Let $f\colon [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function satisfying the Carathéodory conditions such that there exist non-negative functions $d_1(t)$, $d_2(t)$ and $r(t)$ in $L^1(0,1)$ such that*

$$|f(t,u,v)| \leqslant d_1(t)\varphi(|u|) + d_2(t)\varphi(|v|) + r(t)$$

for a.e. $t\in[0,1]$ and all u, $v\in\mathbb{R}$. Suppose, further, that

$$\alpha(M)\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)} < 1 - \tilde{\alpha}(\sigma^*) \tag{28}$$

where M is defined in Theorem 1, $\alpha(M)$ is defined in (4), σ^* and $\tilde{\alpha}(\sigma^*)$ are defined in (20), (21). Then, for every given function $e(t)\in L^1[0,1]$, the boundary value problem (1) has at least one solution $x\in C^1[0,1]$.

Proof. We consider the family of boundary value problems

$$\begin{gathered}(\varphi(x'))' = \lambda f(t,x,x') + \lambda e(t), \quad \text{a.e. on } [0,1],\ \lambda\in[0,1],\\ x(0) = \sum_{i=1}^{m-2} a_i x'(\xi_i), \quad x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j).\end{gathered} \tag{29}$$

Also, we define an operator $\Psi\colon C^1[0,1]\times[0,1]\to C^1[0,1]$ by setting for $(x,\lambda)\in C^1[0,1]\times[0,1]$

$$\begin{aligned}\Psi(x,\lambda)(t) = x(0) + \int_0^t \varphi^{-1}\Big(\varphi(x'(0)) + \lambda\int_0^s \big(f(\tau,x(\tau),x'(\tau)) + e(\tau)\big)\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\ + \Big(x(0) - \sum_{i=1}^{m-2} a_i x'(\xi_i)\Big) + t\Big(x(1) - \sum_{j=1}^{n-2} b_j x(\tau_j)\Big).\end{aligned} \tag{30}$$

Let us suppose that $x \in C^1[0,1]$ is a solution to the operator equation, for some $\lambda \in [0,1]$,

$$x = \Psi(x, \lambda). \tag{31}$$

Solving equation (31) at $t = 0$ we see that x satisfies the boundary condition

$$x(0) = \sum_{i=1}^{m-2} a_i x'(\xi_i).$$

Next, we differentiate equation (31) with respect to t to get

$$\begin{aligned} \varphi(x'(t)) = \varphi(x'(0)) + \lambda \int_0^t \bigl(f(\tau, x(\tau), x'(\tau)) + e(\tau)\bigr)\,\mathrm{d}\tau \\ + x(1) - \sum_{j=1}^{n-2} b_j x(\tau_j). \end{aligned} \tag{32}$$

Solving now equation (32) at $t = 0$ we see that x satisfies the boundary condition

$$x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j),$$

and differentiating equation (32) with respect to t we get

$$\bigl(\varphi(x'(t))\bigr)' = \lambda f\bigl(t, x(t), x'(t)\bigr) + \lambda e(t) \quad \text{for a.e. } t \in [0,1] \text{ and each } \lambda \in [0,1].$$

Thus we see that if $x \in C^1[0,1]$ is a solution to the operator equation $x = \Psi(x, \lambda)$ for some $\lambda \in [0,1]$ then x is a solution to the boundary value problems (29) for the same $\lambda \in [0,1]$. Conversely, it is easy to see that if $x \in C^1[0,1]$ is a solution to the boundary value problems (29) for some $\lambda \in [0,1]$ then $x \in C^1[0,1]$ is a solution to the operator equation $x = \Psi(x, \lambda)$ for the same $\lambda \in [0,1]$.

Next, it is easy to show, following standard arguments, that $\Psi\colon C^1[0,1] \times [0,1] \to C^1[0,1]$ is a completely continuous operator.

We shall next show that there is a constant $R > 0$, independent of $\lambda \in [0,1]$, such that if $x \in C^1[0,1]$ is a solution to (31), equivalently to the boundary value problem (29), for some $\lambda \in [0,1]$, then $\|x\|_{C^1[0,1]} < R$.

We note first that if $x \in C^1[0,1]$ satisfies

$$x = \Psi(x, 0), \tag{33}$$

then $x(t) = 0$ for all $t \in [0,1]$. Indeed, from the definition of Ψ or from the boundary value problem (29) it follows that $x(t) = x(0) + x'(0)t$. It then follows from the two boundary conditions in (29) and the non-resonance assumption (3) that $x(0) = x'(0) = 0$, which implies that $x(t) = 0$ for all $t \in [0,1]$.

We shall assume now that $\lambda \in (0,1]$. We shall also assume that σ^*, as defined in (20), is positive, since the proof for the case $\sigma^* = 0$ is simpler. Let us choose $\varepsilon > 0$ such that $\tilde{\alpha}(\sigma^*) + \varepsilon < 1$ and

$$\big(\alpha(M) + \varepsilon\big)\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)} < 1 - \tilde{\alpha}(\sigma^*) - \varepsilon, \tag{34}$$

which is possible to do in view of our assumption (28). Here M is defined in Theorem 1 and $\alpha(M)$ is defined in (4) so that for the $\varepsilon > 0$ chosen above there exists a constant $C_\varepsilon^1 > 0$ such that

$$\varphi(Mz) \leqslant \big(\alpha(M) + \varepsilon\big)\varphi(z) + C_\varepsilon^1 \quad \text{for every } z \in \mathbb{R}. \tag{35}$$

Also, from Theorem 3 we see that for the chosen $\varepsilon > 0$ there is a constant $C_\varepsilon^2 > 0$ such that

$$\varphi(\|x'\|_\infty) \leqslant \frac{1}{1 - \tilde{\alpha}(\sigma^*) - \varepsilon}\|(\varphi(x'))'\|_{L^1(0,1)} + C_\varepsilon^2. \tag{36}$$

We now see from the equation in (29), using our assumptions on the function f, Theorem 1, and estimates (35), (36) that

$$\begin{aligned}
\|(\varphi(x'))'\|_{L^1(0,1)} &\leqslant \varphi(\|x\|_\infty)\|d_1\|_{L^1(0,1)} + \varphi(\|x'\|_\infty)\|d_2\|_{L^1(0,1)} \\
&\quad + \|r\|_{L^1(0,1)} + \|e\|_{L^1(0,1)} \\
&\leqslant \varphi(M\|x'\|_\infty)\|d_1\|_{L^1(0,1)} + \varphi\|x'\|_\infty)\|d_2\|_{L^1(0,1)} \\
&\quad + \|r\|_{L^1(0,1)} + \|e\|_{L^1(0,1)} \\
&\leqslant \big((\alpha(M) + \varepsilon)\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}\big)\varphi(\|x'\|_\infty) \\
&\quad + \|r\|_{L^1(0,1)} + \|e\|_{L^1(0,1)} + C_\varepsilon^1\|d_1\|_{L^1(0,1)} \\
&\leqslant \frac{\big(\alpha(M) + \varepsilon\big)\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}}{1 - \tilde{\alpha}(\sigma^*) - \varepsilon}\|(\varphi(x'))'\|_{L^1(0,1)} + C_\varepsilon,
\end{aligned}$$

where $C_\varepsilon = \|r\|_{L^1(0,1)} + \|e\|_{L^1(0,1)} + C_\varepsilon^1\|d_1\|_{L^1(0,1)} + C_\varepsilon^2[(\alpha(M) + \varepsilon)\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}]$. It follows from (34) that there exists a constant R_0, independent of $\lambda \in [0,1]$, such that if $x \in C^1[0,1]$ is a solution to the boundary value problem (29) for some $\lambda \in [0,1]$ then

$$\|(\varphi(x'))'\|_{L^1(0,1)} \leqslant R_0.$$

This combined with (36) and (7) gives that there exists a constant $R > 0$ such that

$$\|x\|_{C^1[0,1]} < R.$$

This then implies that $\deg_{\mathrm{LS}}\big(I - \Psi(\cdot,\lambda), B(0,R), 0\big)$ is well-defined for all $\lambda \in [0,1]$, where $B(0,R)$ is the ball with center 0 and radius R in $C^1[0,1]$. Here I denotes the identity mapping from $C^1[0,1]$ onto $C^1[0,1]$ and $\deg_{\mathrm{LS}}$ denotes the Leray-Schauder degree.

Let X denote the two-dimensional subspace of $C^1[0,1]$ given by

$$X = \{\alpha + \beta t \ \text{ for } \ \alpha, \beta \in \mathbb{R}\}. \tag{37}$$

Let us define the isomorphism $i\colon\ \mathbb{R}^2 \to X$ by

$$i\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = i_{\binom{\alpha}{\beta}} \in X \quad \text{for} \quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{R}^2, \tag{38}$$

where

$$i_{\binom{\alpha}{\beta}}(t) = \alpha + \beta t \quad \text{for } \ t \in [0,1]. \tag{39}$$

Also, we define a 2×2 matrix $\mathbb{A}$ by setting

$$\mathbb{A} = \begin{pmatrix} -1 & \sum\limits_{i=1}^{m-2} a_i \\ -\Big(1 - \sum\limits_{j=1}^{n-2} b_j\Big) & -\Big(1 - \sum\limits_{j=1}^{n-2} b_j\tau_j\Big) \end{pmatrix}. \tag{40}$$

We note that $\det \mathbb{A} = 1 - \sum\limits_{j=1}^{n-2} b_j\tau_j + \Big(\sum\limits_{i=1}^{m-2} a_i\Big)\Big(1 - \sum\limits_{j=1}^{n-2} b_j\Big) \neq 0$ in view of the non-resonance assumption (3).

Next, we define a function $G\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$G\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \mathbb{A}\cdot\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix} -\alpha + \beta\Big(\sum\limits_{i=1}^{m-2} a_i\Big) \\ -\alpha\Big(1 - \sum\limits_{j=1}^{n-2} b_j\Big) - \beta\Big(1 - \sum\limits_{j=1}^{n-2} b_j\tau_j\Big) \end{pmatrix} \tag{41}$$

for $\binom{\alpha}{\beta} \in \mathbb{R}^2$.

We note that for $v(t) = \alpha + \beta t \in X$ we have

$$\big(I - \Psi(\cdot,0)\big)(v) = i_{G\binom{\alpha}{\beta}},$$

and it follows that

$$G = i^{-1} \circ \big((I - \Psi(\cdot, 0))|_X \circ i.$$

Now, we see from the homotopy invariance property of the Leray-Schauder degree that

$$\begin{aligned}
\deg_{\mathrm{LS}}\big(I - \Psi(\cdot, 1), B(0, R), 0\big) &= \deg_{\mathrm{LS}}\big(I - \Psi(\cdot, 0), B(0, R), 0\big) \\
&= \deg_{\mathrm{B}}\big(I - \Psi(\cdot, 0)|_X, X \cap B(0, R), 0\big) \\
&= \deg_{\mathrm{B}}\big(G, \mathbb{B}(0, R), 0\big),
\end{aligned}$$

where $\mathbb{B}(0, R)$ denotes the ball of radius R in $\mathbb{R}^2$ with center at the origin. Finally, we have, using standard results for the Brouwer degree and denoting it by $\deg_{\mathrm{B}}$ (see [27], [28], [29]), that

$$\deg_{\mathrm{B}}\big(G, \mathbb{B}(0, R), 0\big) = \begin{cases} 1, & \text{if } \det \mathbb{A} > 0, \\ -1, & \text{if } \det \mathbb{A} < 0. \end{cases}$$

Accordingly, we see from the non-resonance assumption (3), i.e.,

$$\det \mathbb{A} = \Big(1 - \sum_{i=1}^{m-2} a_i\Big)\Big(1 - \sum_{j=1}^{n-2} b_j \tau_j\Big) + \Big(\sum_{i=1}^{m-2} a_i \xi_i\Big)\Big(1 - \sum_{j=1}^{n-2} b_j\Big) \neq 0,$$

that $\deg_{\mathrm{LS}}\big(I - \Psi(\cdot, 1), B(0, R), 0\big) \neq 0$ and there is $x \in B(0, R) \subset C^1[0, 1]$ that satisfies

$$x = \Psi(x, 1);$$

equivalently, x is a solution to the boundary value (1). This completes the proof of the theorem. □

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