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AN EXTENSION OF ROTHE'S METHOD TO NON-CYLINDRICAL DOMAINS

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Abstract. In this paper Rothe's classical method is extended so that it can be used to solve some linear parabolic boundary value problems in non-cylindrical domains. The corresponding existence and uniqueness theorems are proved and some further results and generalizations are discussed and applied.

Keywords: parabolic PDE, numerical method, time-discretization, method of lines, Rothe's method

MSC 2000: 65N40, 35K20

1. Introduction

Let us consider in $\mathbb{R}^{N+1}$ the domain

$$Q = \{(x,t)\colon\ x \in \Omega_t,\ 0 < t < T\}, \tag{1.1}$$

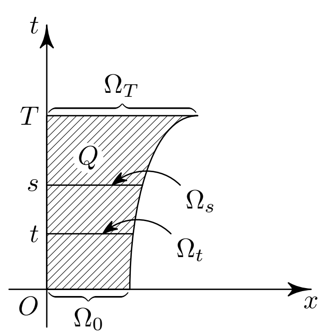

Figure 1. The non-cylindrical domain Q (for $N = 1$).

where $(0,T)$ is a finite interval, $\Omega_t \in C^{k-1,1}(\mathbb{R}^N)$ (here, $C^{k-1,1}(\mathbb{R}^N)$ is the set of all bounded domains in $\mathbb{R}^N$ whose boundary can be locally described by a function from $C^{k-1,1}(\Delta)$, where $\Delta \subset \mathbb{R}^{N-1}$ is a cube; see [4]) and for every $t, s \in (0,T)$, $t < s$,

$$\emptyset \neq \Omega_0 \subset \Omega_t \subset \Omega_s \subset \Omega_T$$

(see Fig. 1). Let us solve the problem

$$\frac{\partial u}{\partial t} + Au = f \quad \text{in } Q, \tag{1.2}$$

$$u(x,0) = 0 \quad \text{in } \Omega_0, \tag{1.3}$$

$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = \ldots = \frac{\partial^{k-1} u}{\partial \nu^{k-1}}(x,t) = 0, \quad 0 < t < T,\ x \in \partial\Omega_t, \tag{1.4}$$

where A is a linear differential operator of order $2k$ in the divergence form:

$$(Au)(x) = \sum_{|i|,|j| \leqslant k} (-1)^{|i|} \partial^i (a_{i,j}(x) \partial^j u)$$

for $x \in \Omega_T$ and ν is the outer normal to $\partial\Omega_t$. The coefficients $a_{i,j}$ of this operator and the function f on the left-hand side of the equation (1.2) depend only on x and are defined a.e. in Ω_T, where i, j are multiindices,

$$i = (i_1, i_2, \ldots, i_N), \quad |i| = i_1 + i_2 + \ldots + i_N,$$

$i_1, i_2, \ldots, i_N$ are nonnegative integers, and

$$\partial^i = \frac{\partial^{|i|}}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_N^{i_N}}.$$

If Q is a cylinder, i.e. $\Omega_t = \Omega_0$ for all $t \in (0,T)$, then there exists a well developed numerical method called Rothe's method, which consists in replacing the time derivative $(\partial u/\partial t)(x,t)$ by the quotient $\big(u(x,t+h) - u(x,t)\big)/h$ and solving a sequence of elliptic problems on the sections Ω_t. This method, which is a modification of the so-called method of lines, was developed by E. Rothe in the thirties and its modern extensions are connected mainly with the names of K. Rektorys [5] and J. Kačur [3]. The history of the method, as well as its modern applications, are described in detail in the paper [1] (see the extensive list of references therein). In [1], the case of a non-cylindrical domain Q is considered up to our knowledge for the first time, and two methods are proposed.

In this paper Rothe's classical method is extended so that it can be used to solve some linear parabolic boundary value problems in non-cylindrical domains. The corresponding existence and uniqueness theorems are proved and some further results and generalizations are discussed and applied. It is organized as follows: In Section 2 we describe our announced extension of Rothe's method in detail. In Section 3 we state the corresponding uniqueness and existence results and also include some necessary preliminaries. The proof of these theorems can be found in Section 4. Finally, in Section 5 we generalize our solution from Section 3 by considering the case with the function on the right-hand side depending also on t. Also some related results are stated and proved.

For simplicity, we assume a homogenous initial condition, and suppose that the coefficients $a_{i,j}$ as well as the right-hand side f depend only on x. In the concluding remarks (see Section 5), we will indicate how the approach can be modified even for coefficients $a_{i,j}(x,t)$, for the right-hand side $f(x,t)$ and for an initial condition $u(x,0)=u_0(x)$, $x\in\Omega_0$.

Remark 1.1. Let us recall that we assume that the domain Q is non-cylindrical with the side boundary nondecreasing with respect to t, but not necessarily increasing.

2. An extension of Rothe's method

Later (see Definition 3.2) we will give the precise meaning in what sense we understand the solution of the problem. First, let us introduce some function spaces.

Let $t\in[0,T]$. Let V_t denote the subset of the Sobolev space $W^{k,2}(\Omega_t)$,

$$V_t=\Big\{v\colon\ v\in W^{k,2}(\Omega_t),\ v=\frac{\partial v}{\partial\nu}=\ldots=\frac{\partial^{k-1}v}{\partial\nu^{k-1}}=0\text{ on }\partial\Omega_t\Big\}=W_0^{k,2}(\Omega_t),$$

and for $(\cdot,\cdot)_t$ the inner product in $L_2(\Omega_t)$, let

$$(\!(u,v)\!)_t=(Au,v)_t=\sum_{|i|,|j|\leqslant k}\int_{\Omega_t}a_{i,j}(x)\partial^i u(x)\partial^j v(x)\,\mathrm{d}x$$

be the bilinear form associated with the operator A.

Assumption 2.1.

(A1) The bilinear form is bounded, i.e. there exists a number $M>0$ such that

$$(\!(u,v)\!)_t\leqslant M\|u\|_{W^{k,2}(\Omega_t)}\|v\|_{W^{k,2}(\Omega_t)}\quad\text{for all }u,v\in W^{k,2}(\Omega_t).$$

(A2) The bilinear form is V_t-elliptic, i.e. there exists a number $m>0$ such that

$$(\!(u,u)\!)_t\geqslant m\|u\|^2_{W^{k,2}(\Omega_t)}\quad\text{for all }u\in V_t.$$

(A3) The bilinear form is V_t-symmetric, i.e.

$$((u,v))_t = ((v,u))_t \quad \text{for all } u, v \in V_t.$$

(A4) The function f belongs to $L_2(\Omega_T)$.

Let us emphasize that the validity of the conditions (A1), (A2) and (A3) is guaranteed if we assume that the coefficients $a_{i,j}$ are bounded,

$$a_{i,j} \in L_\infty(\Omega_T),$$

that A is elliptic and symmetric, i.e. that there exists a $C > 0$ such that

$$\sum_{|i|,|j| \leqslant k} a_{i,j}(x)\xi_i\xi_j \geqslant C \sum_{|i| \leqslant k} \xi_i^2$$

and

$$a_{i,j}(x) = a_{j,i}(x) \quad \text{for all } i, j \ (|i|, |j| \leqslant k)$$

for a.e. $x \in \Omega_T$ and for every $\xi = \{\xi_i \colon \ |i| \leqslant k\} \in \mathbb{R}^m$.

We will prove in the next sections that our Assumption 2.1 ensures the existence of a weak solution of the problem (1.2)–(1.4) in the sense given in Definition 3.2 and that this weak solution is the limit (in a sense given later) of the sequence $\{u_n(x,t)\}$ of functions constructed by Rothe's method.

Rothe's method. Divide the interval $I = [0, T]$ into p subintervals $I_1, I_2, \dots, I_p$ ($I_j = [t_{j-1}, t_j]$, $j = 1, 2, \dots, p$) of length $h = T/p$. According to the initial condition (1.3) we put

$$z_0(x) = 0, \quad x \in \Omega_T,$$

for $t_0 = 0$ and successively for $j = 1, 2, \dots, p$ define functions $z_j(x)$ which are weak solutions of the following elliptic problems:

$$\begin{cases} Az_j + \dfrac{z_j}{h} = f + \dfrac{z_{j-1}}{h} & \text{in } \Omega_{t_j}, \\ z_j = \dfrac{\partial z_j}{\partial \nu} = \dots = \dfrac{\partial^{k-1} z_j}{\partial^{k-1} \nu} = 0 & \text{on } \partial\Omega_{t_j}. \end{cases} \tag{2.1}$$

We obtain these problems if we replace in (1.2) the derivative $\partial u/\partial t$ by the differential quotient

$$\frac{z_j - z_{j-1}}{h}$$

at points $t = t_j$, $j = 1, 2, \ldots, p$. The weak formulation of the problem (2.1) is as follows: To find successively for $j = 1, 2, \ldots, p$ functions

$$z_j \in V_{t_j},$$

where $z_0 = 0$, such that the following integral identities are satisfied:

$$((z_j, v))_{t_j} + \frac{1}{h}(z_j, v)_{t_j} = \Big(f + \frac{z_{j-1}}{h}, v\Big)_{t_j} \quad \text{for all } v \in V_{t_j}.$$

Let us define the form

$$(((u, v)))_{t_j} = ((u, v))_{t_j} + \frac{1}{h}(u, v)_{t_j}.$$

From (A1), (A2) and by the Schwarz inequality it follows that this bilinear form is bounded on $[W^{k,2}(\Omega_{t_j})]^2$ and V_{t_j}-elliptic.

The problem to find a function $z_1 \in V_{t_1}$ which satisfies the integral identity

$$(((z_1, v)))_{t_1} = \Big(f + \frac{z_0}{h}, v\Big)_{t_1} \quad \text{for all } v \in V_{t_1}$$

has exactly one solution (as a consequence of the theory of elliptic boundary value problems; see, e.g. [2]); here $f + z_0/h = (f + z_0/h)|_{\Omega_{t_1}} \in L_2(\Omega_{t_1})$. Further, after redefining again $z_1 \in V_{t_1}$ as

$$z_1(x) = \begin{cases} z_1(x), & x \in \Omega_{t_1}, \\ 0, & x \in \Omega_T \setminus \Omega_{t_1} \end{cases}$$

we get $z_1 \in V_T$. Hence $z_1 \in L_2(\Omega_T)$ and

$$f + \frac{z_1}{h} \in L_2(\Omega_{t_2})$$

(here $f + z_1/h = (f + z_1/h)|_{\Omega_{t_2}}$).

Repeating the above procedure for $j = 2, 3, \ldots, p$ we get functions

$$z_1, z_2, \ldots, z_p \in V_T.$$

Now we construct a function $u_1(x, t)$ called Rothe's function and defined on $\Omega_T \times I$, putting

$$u_1(x, t) = z_{j-1}(x) + \frac{t - t_{j-1}}{h}\big(z_j(x) - z_{j-1}(x)\big)$$

for $t \in I_j$, $j = 1, 2, \ldots, p$, and $x \in \Omega_T$.

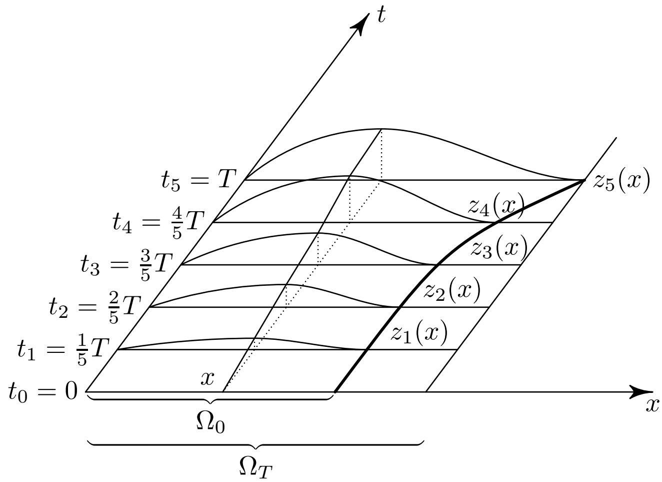

Figure 2. The Rothe function $u_1(x,t)$ (for $p=5$).

For a fixed $x \in \Omega_T$, $u_1(x,t)$ is a piecewise linear function in t on the interval I and for $t = t_j$ it assumes values $z_j(x)$ as shown in Fig. 2. In a similar way let us construct a function $u_2(x,t)$ with the only difference that, instead of dividing the interval $[0,T]$ into p subintervals of length h as above, we divide it into $2p$ subintervals of length $h_2 = T/2p = h/2$. Going on in this way and dividing subsequently the interval $[0,T]$ into $4p, 8p, \ldots, 2^{n-1}p, \ldots$ subintervals, we construct a sequence of functions $u_n(x,t)$ defined on $\Omega_T \times I$ by the relations

$$u_n(x,t) = z^n_{j-1}(x) + \frac{t - t^n_{j-1}}{h_n}\bigl(z^n_j(x) - z^n_{j-1}(x)\bigr) \quad \text{for } t \in I^n_j = [t^n_{j-1}, t^n_j], \tag{2.2}$$

where $t^n_j = jh_n$, $h_n = T/(2^{n-1}p)$ $(j = 1, 2, \ldots, 2^{n-1}p)$.

In this way we get a sequence

$$\{u_n(x,t)\}_{n=1}^{\infty}$$

which is called the Rothe sequence of approximate solutions of the problem (1.2)–(1.4).

Intuitively, we can expect that this sequence will converge to some function $u(x,t)$ which is a solution of the problem (1.2)–(1.4). The next considerations are devoted to the proof of these statements.

3. Some preliminaries and existence and uniqueness results

In this section we start with a brief review about Bochner's space and some auxiliary results which we will use in the next sections.

Denote for brevity $[0,T] = I$. Let H be a Hilbert space whose elements are functions defined on Ω. Let $L_2(I,H)$ be the set of functions $t \mapsto W(t)$ from I into H (more precisely, $W(t) \in H$ for almost all $t \in I$), which are square integrable in the interval I (in the Bochner sense; for details, see [4]). $L_2(I,H)$ is a Hilbert space with the scalar product

$$(W_1, W_2)_{L_2(I,H)} = \int_0^T \big(W_1(t), W_2(t)\big)_H \, \mathrm{d}t.$$

The integral

$$\int_0^t W(\tau)\,\mathrm{d}\tau = w(t)$$

(in the Bochner sense) of a function $W(t) \in L_2(I,H)$ is defined by

$$(w(t), r)_{L_2(I,H)} = \int_0^t (W(\tau), r)_H \,\mathrm{d}\tau$$

for every $r \in H$.

If $W \in L_2(I,H)$, then $w(t)$ can be identified with a function representing a continuous mapping from I into H; thus, we can write

$$w(t) \in \mathrm{AC}(I,H).$$

Moreover, $w(t)$ has a derivative $w'(t)$ almost everywhere in I equal to $W(t)$ in the sense that

$$\lim_{\Delta t \to 0} \Big\| \frac{w(t+\Delta t) - w(t)}{\Delta t} - W(t) \Big\|_H = 0.$$

We denote

$$V^1(0,T;H) = \Big\{ \omega(t)\colon\ V(\omega(t)) = \sup_D \sum_{k=1}^{K} \|\omega(t_k) - \omega(t_{k-1})\|_H < \infty \Big\}$$

where the supremum is taken over all decompositions $D = \{t_0, t_1, \ldots, t_K\}$ of I and

$$L_2(I, V_Q) = \{u \in L_2(I, V_T)\colon\ u(t)|_{\Omega_T \setminus \Omega_t} = 0,\ \text{a.e. in } I\}.$$

The space $L_2(I,V_Q)$ is a subspace of $L_2(I,V_T)$ and also a Hilbert space with the inner product of $L_2(I,V_T)$.

Now we present a simple version of the Gronwall lemma, which we will use in the next section.

Lemma 3.1. *Let A, B and h be given constants and let $\alpha_1, \ldots \alpha_j$ satisfy the conditions*

$$\alpha_1 \leqslant A, \quad \alpha_i \leqslant A + Bh(\alpha_1 + \ldots + \alpha_{i-1}), \; i = 2, \ldots, j.$$

Then

$$\alpha_i \leqslant A\mathrm{e}^{B(i-1)h}, \quad i = 1, 2, \ldots, j.$$

P r o o f. The proof is trivial. □

Definition 3.2. A function $u(t)$ is called a *weak solution* of the problem (1.2)–(1.4) if the following conditions are fulfilled:

1) $u \in L_2(I, V_Q)$,
2) $u \in \mathrm{AC}(I, L_2(\Omega_T))$,
3) $u' \in L_2(I, L_2(\Omega_T))$,
4) $u(0) = 0$,
5) $\int_0^T ((u(t), v(t)))_T \,\mathrm{d}t + \int_0^T \big(u'(t), v(t)\big)_T \,\mathrm{d}t = \int_0^T \big(f(t), v(t)\big)_T \,\mathrm{d}t$ for all $v \in L_2(I, V_Q)$.

First we formulate the uniqueness result.

Theorem 3.3 (Uniqueness)**.** *The weak solution of the problem* (1.2)–(1.4) *is uniquely determined.*

Theorem 3.4 (Existence)**.** *Suppose Assumption* 2.1 *is satisfied. Then there exists a weak solution of the problem* (1.2)–(1.4), *i.e. there exists a function which is a weak (strong) limit of the sequence of Rothe's functions $u_n(t)$ in the space $L_2(I, V_T)$ $\big(L_2(I, L_2(\Omega_T))\big)$.*

4. Proofs

P r o o f of Theorem 3.3. Assume that $\hat{u}(t)$, $\breve{u}(t)$ are weak solutions of the problem (1.2)–(1.4) and denote $u(t) = \hat{u}(t) - \breve{u}(t)$. This function also has the properties 1)–4) of Definition 3.2. From 5) it follows that

$$\int_0^T ((u(t), u(t)))_T \,\mathrm{d}t + \int_0^T (u'(t), u(t))_T \,\mathrm{d}t = 0. \tag{4.1}$$

The second integral is nonnegative, since

$$\begin{aligned}\int_0^T \big(u'(t), u(t)\big)_T \,\mathrm{d}t &= \frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L_2(\Omega_T)}\,\mathrm{d}t \\ &= \frac{1}{2}\|u(T)\|^2_{L_2(\Omega_T)} - \frac{1}{2}\|u(0)\|^2_{L_2(\Omega_T)} \\ &= \frac{1}{2}\|u(T)\|^2_{L_2(\Omega_T)} \geqslant 0.\end{aligned}$$

From this and from (4.1) it follows that

$$\int_0^T ((u(t), u(t)))_T \,\mathrm{d}t \leqslant 0.$$

According to (A2) we have

$$\|u(t)\|_{W^{k,2}(\Omega_T)} = 0 \quad \text{for almost all } t \in I$$

and consequently

$$u(t) = 0 \quad \text{in } I,$$

since u is absolutely continuous in the sense of Bochner. □

Proof of Theorem 3.4. We divide the proof into three parts A, B and C to which we will refer in the next Section 5. In order to prove the convergence of the Rothe sequence, let us first deduce some a priori estimates.

A. Let us consider the integral identity

$$((z_j^n, v))_{t_j^n} + \Big(\frac{z_j^n - z_{j-1}^n}{h_n}, v\Big)_{t_j^n} = (f, v)_{t_j^n} \quad \text{for all } v \in V_{t_j^n}, \tag{4.2}$$

and choose $v = z_j^n$. This is reasonable, since $z_j^n \in V_{t_j^n}$. We get

$$((z_j^n, z_j^n))_{t_j^n} + \Big(\frac{z_j^n - z_{j-1}^n}{h_n}, z_j^n\Big)_{t_j^n} = (f, z_j^n)_{t_j^n}.$$

But since by (A2)

$$((z_j^n, z_j^n))_{t_j^n} \geqslant 0,$$

we have that

$$(z_j^n, z_j^n)_{t_j^n} \leqslant h_n (f, z_j^n)_{t_j^n} + (z_{j-1}^n, z_j^n)_{t_j^n}. \tag{4.3}$$

From (4.3) and by the Schwarz inequality we get

$$\|z_j^n\|^2_{L_2(\Omega_T)} \leqslant h_n \|f\|_{L_2(\Omega_T)} \|z_j^n\|_{L_2(\Omega_T)} + \|z_{j-1}^n\|_{L_2(\Omega_T)} \|z_j^n\|_{L_2(\Omega_T)}$$

and consequently

$$\|z_j^n\|_{L_2(\Omega_T)} \leqslant h_n \|f\|_{L_2(\Omega_T)} + \|z_{j-1}^n\|_{L_2(\Omega_T)}$$

for $j = 1, 2, \ldots, 2^{n-1}p$. Further,

$$\begin{aligned}\|z_j^n\|_{L_2(\Omega_T)} &\leqslant h_n \|f\|_{L_2(\Omega_T)} + \|z_{j-1}^n\|_{L_2(\Omega_T)} \\ &\leqslant 2h_n \|f\|_{L_2(\Omega_T)} + \|z_{j-2}^n\|_{L_2(\Omega_T)} \leqslant \ldots \leqslant jh_n \|f\|_{L_2(\Omega_T)}\end{aligned}$$

and since $jh_n \leqslant 2^{n-1}ph_n = T$, we finally have that

$$\|z_j^n\|_{L_2(\Omega_T)} \leqslant T\,\|f\|_{L_2(\Omega_T)} \tag{4.4}$$

for $j = 1, 2, \ldots, 2^{n-1}p$.

Now choose

$$v = z_j^n - z_{j-1}^n$$

in the integral identity (4.2), which is reasonable, because $z_j^n - z_{j-1}^n \in V_{t_j^n}$. Then we get

$$(\!(z_j^n, z_j^n - z_{j-1}^n)\!)_{t_j^n} + \Big(\frac{z_j^n - z_{j-1}^n}{h_n}, z_j^n - z_{j-1}^n\Big)_{t_j^n} = (f, z_j^n - z_{j-1}^n)_{t_j^n}$$

for $j = 1, 2, \ldots, 2^{n-1}p$. Due to the properties of z_j^n, the integrals in the last identity can be extended to the whole domain Ω_T and we have

$$(\!(z_j^n, z_j^n - z_{j-1}^n)\!)_T + \Big(\frac{z_j^n - z_{j-1}^n}{h_n}, z_j^n - z_{j-1}^n\Big)_T = (f, z_j^n - z_{j-1}^n)_T. \tag{4.5}$$

After adding both sides in (4.5) from $j = 1$ to i, we get

$$\sum_{j=1}^{i} (\!(z_j^n, z_j^n - z_{j-1}^n)\!)_T + \frac{1}{h_n} \sum_{j=1}^{i} (z_j^n - z_{j-1}^n, z_j^n - z_{j-1}^n)_T = \sum_{j=1}^{i} (f, z_j^n - z_{j-1}^n)_T. \tag{4.6}$$

Denote

$$\begin{aligned}S_i^1 &= \sum_{j=1}^{i} (\!(z_j^n, z_j^n - z_{j-1}^n)\!)_T, \\ S_i^2 &= \frac{1}{h_n} \sum_{j=1}^{i} (z_j^n - z_{j-1}^n, z_j^n - z_{j-1}^n)_T, \\ S_i^3 &= \sum_{j=1}^{i} (f, z_j^n - z_{j-1}^n)_T.\end{aligned}$$

Then we can rewrite (4.6) as

$$S_i^1 + S_i^2 = S_i^3. \tag{4.7}$$

Taking into account that (here we use (A3))

$$\begin{aligned} S_i^1 &= \frac{1}{2}\sum_{j=1}^{i}\{2(\!(z_j^n, z_j^n)\!)_T - 2(\!(z_j^n, z_{j-1}^n)\!)_T\} \\ &= \frac{1}{2}\Big\{(\!(z_i^n, z_i^n)\!)_T + \sum_{j=1}^{i}(\!(z_j^n - z_{j-1}^n, z_j^n - z_{j-1}^n)\!)_T\Big\}, \end{aligned}$$

we obtain

$$S_i^1 \geqslant \frac{1}{2}(\!(z_i^n, z_i^n)\!)_T, \tag{4.8}$$

$$S_i^2 = \frac{1}{h_n}\sum_{j=1}^{i}\|z_j^n - z_{j-1}^n\|^2_{L_2(\Omega_T)} \geqslant 0, \tag{4.9}$$

$$S_i^3 = (f, z_i^n)_T.$$

From (4.7) and (4.9) it follows that

$$S_i^1 \leqslant S_i^3, \tag{4.10}$$

$$S_i^2 \leqslant S_i^3. \tag{4.11}$$

Using (4.4), (4.8), (4.10) and the Schwarz inequality, we get

$$\frac{1}{2}(\!(z_i^n, z_i^n)\!)_T \leqslant S_i^1 \leqslant S_i^3 \leqslant \|f\|_{L_2(\Omega_T)}\|z_i^n\|_{L_2(\Omega_T)} \leqslant T\|f\|^2_{L_2(\Omega_T)},$$

i.e.

$$(\!(z_i^n, z_i^n)\!)_T \leqslant 2T\|f\|^2_{L_2(\Omega_T)}.$$

From (A2) it follows that

$$\|z_i^n\|_{W^{k,2}(\Omega_T)} \leqslant \sqrt{\frac{2T}{m}}\|f\|_{L_2(\Omega_T)}. \tag{4.12}$$

Using (4.4), (4.11) and the Schwarz inequality, we get

$$\frac{1}{h_n}\sum_{j=1}^{2^{n-1}p}\|z_j^n - z_{j-1}^n\|^2_{L_2(\Omega_T)} \leqslant T\|f\|^2_{L_2(\Omega_T)}. \tag{4.13}$$

Note that the right-hand sides of the inequalities (4.12) and (4.13) are independent of j and n.

B. Consider now the Rothe sequence

$$\{u_n(t)\}_{n=1}^{\infty}$$

given by (2.2), i.e.

$$u_n(t) = z_{j-1}^n + \frac{t - t_{j-1}^n}{h_n}(z_j^n - z_{j-1}^n) \quad \text{for } t \in I_j^n,\ j = 1, 2, \ldots, 2^{n-1}p.$$

From (4.12) it follows that

$$\begin{aligned}\|u_n(t)\|_{V_T} &= \Big\| z_{j-1}^n + \frac{t - t_{j-1}^n}{h_n}(z_j^n - z_{j-1}^n)\Big\|_{V_T} \\ &\leqslant \Big(1 - \frac{t - t_{j-1}^n}{h_n}\Big)\|z_{j-1}^n\|_{V_T} + \frac{t - t_{j-1}^n}{h_n}\|z_j^n\|_{V_T} \leqslant C\end{aligned}$$

for all $t \in I$ and $n = 1, 2, \ldots$.

Thus, we get

$$\|u_n\|_{L_2(I, V_T)}^2 = \int_0^T \|u_n(t)\|_{V_T}^2 \,\mathrm{d}t \leqslant C^2 T$$

for $n = 1, 2, \ldots$; consequently, the Rothe sequence

$$\{u_n\}_{n=1}^{\infty}$$

is uniformly bounded and has a subsequence $\{u_{n_m}\}_{m=1}^{\infty}$ which converges weakly to a function

$$u \in L_2(I, V_T)$$

since this space is reflexive, i.e. we have

$$u_{n_m} \rightharpoonup u \quad \text{in } L_2(I, V_T). \tag{4.14}$$

We will show that the function u is the desired solution. Denote

$$Z_j^n(x) = \frac{z_j^n(x) - z_{j-1}^n(x)}{h_n}, \quad x \in \Omega_T.$$

Then we can write (2.2) in the form

$$u_n(t) = z_{j-1}^n + Z_j^n(t - t_{j-1}^n) \quad \text{in } I_j^n,\ j = 1, 2, \ldots, 2^{n-1}p.$$

Now we define functions

$$U_n\colon\ t \mapsto L_2(\Omega_T), \quad n = 1, 2, \ldots$$

in the form

$$U_n(t) = \begin{cases} Z_1^n, & t = 0, \\ Z_j^n, & t \in (t_{j-1}^n, t_j^n],\ j = 1, 2, \ldots, 2^{n-1}p. \end{cases}$$

From (4.13) it follows that the sequence

$$\{U_n\}_{n=1}^{\infty}$$

is bounded, i.e.

$$\begin{aligned} \|U_n\|^2_{L_2(I, L_2(\Omega_T))} &= \int_0^T \|U_n(t)\|^2_{L_2(\Omega_T)}\,\mathrm{d}t = \sum_{j=1}^{2^{n-1}p} \int_{t_{j-1}^n}^{t_j^n} \|Z_j^n\|^2_{L_2(\Omega_T)}\,\mathrm{d}t \\ &= \sum_{j=1}^{2^{n-1}p} \Big\| \frac{z_j^n - z_{j-1}^n}{h_n} \Big\|^2_{L_2(\Omega_T)} (t_j^n - t_{j-1}^n) \\ &= \frac{1}{h_n} \sum_{j=1}^{2^{n-1}p} \|z_j^n - z_{j-1}^n\|^2_{L_2(\Omega_T)} \leqslant C. \end{aligned}$$

So we can choose a subsequence

$$\{U_{n_m}\}_{m=1}^{\infty}$$

converging weakly to some function

$$U \in L_2\big(I, L_2(\Omega_T)\big),$$

i.e.

$$U_{n_m} \rightharpoonup U \quad \text{in } L_2\big(I, L_2(\Omega_T)\big). \tag{4.15}$$

Thus the integral

$$\int_0^t U(\tau)\,\mathrm{d}\tau = \omega(t)$$

exists. The functions $U_{n_m}(t)$ and $u_{n_m}(t)$ are connected through the relation

$$\int_0^t U_{n_m}(\tau)\,\mathrm{d}\tau = u_{n_m}(t).$$

Hence from (4.15) and from

$$u_{n_m} \rightharpoonup u \quad \text{in } L_2(I, V_T)$$

it follows that

$$w = u \quad \text{in } L_2\big(I, L_2(\Omega_T)\big).$$

(To obtain the last equality we apply the Lebesgue Dominating Convergence Theorem; for more details see [5], Chapter 11). Then we get that

$$u \in \mathrm{AC}\big(I, L_2(\Omega_T)\big)$$

and

$$u'(t) = U(t) \quad \text{in } L_2(\Omega_T)$$

a.e. in I.

Finally, we obtain

$$u(t) = \int_0^t U(\tau)\,\mathrm{d}\tau$$

and consequently

$$u(0) = 0. \tag{4.16}$$

If we define a sequence $\{\tilde{u}_n(t)\}_{n=1}^{\infty}$ in the form

$$\tilde{u}_n(t) = \begin{cases} z_{j-1}^n, & t \in [t_{j-1}^n, t_j^n),\ j = 1, 2, \ldots, 2^{n-1}p, \\ z_{2^{n-1}p}^n, & t = T \end{cases}$$

then we get that $\{\tilde{u}_n(t)\}_{n=1}^{\infty} \subset L_2(I, V_Q)$. Now we show that this sequence also weakly converges to the function u, i.e.

$$\tilde{u}_n \rightharpoonup u \quad \text{in } L_2(I,\ V_T).$$

By (4.14) and by the properties of weak convergence, we should prove that

$$(v, \tilde{u}_{n_m} - u_{n_m})_{L_2(I,V_T)} \to 0 \quad \text{for all } v \in L_2(I, V_T),$$

i.e.

$$\lim_{m\to\infty} \int_0^T \big(v(t), \tilde{u}_{n_m}(t) - u_{n_m}(t)\big)_{V_T}\,\mathrm{d}t = 0 \quad \text{for all } v \in L_2(I, V_T),$$

as

$$\tilde{u}_{n_m} - u = (\tilde{u}_{n_m} - u_{n_m}) + (u_{n_m} - u).$$

Denote by M the set of all abstract functions $v \in L_2(I, V_T)$ which are equal to a certain function from V_T on some interval $[\alpha, \beta] \subset I$ and are equal to zero outside this interval. Suppose that

$$\alpha = \tilde{\alpha} h_n, \quad \beta = \tilde{\beta} h_n$$

for n large enough, where $\tilde{\alpha}$, $\tilde{\beta}$ are nonnegative integers and

$$0 \leqslant \tilde{\alpha} < \tilde{\beta} \leqslant 2^{n-1} p.$$

Now denote by N the set of all linear combinations of functions of M, i.e. the set of all simple functions. One can show that the set N is dense in $L_2(I, V_T)$. Since every function of N is a linear combination of functions of M, it is enough to prove that

$$\lim_{n_m \to \infty} \int_0^T \big(v(t), \tilde{u}_{n_m}(t) - u_{n_m}(t)\big)_{V_T} \, \mathrm{d}t = 0 \quad \text{for all } v \in M. \tag{4.17}$$

Choose an arbitrary function v from M which is equal to some function $v \in V_T$ in the interval $[\alpha, \beta]$ and is equal to zero outside this interval. Integrating (4.17) we can assume that n_m is large enough and such that

$$\alpha = \tilde{\alpha} h_{n_m}, \quad \beta = \tilde{\beta} h_{n_m}$$

holds, where

$$0 \leqslant \tilde{\alpha} < \tilde{\beta} \leqslant 2^{n_m - 1} p$$

with $\tilde{\alpha}$, $\tilde{\beta}$ nonnegative integers. Then the integral in (4.19) is estimated by h_{n_m} i.e.

$$\begin{aligned}
\left| \int_\alpha^\beta \big(v(t), \tilde{u}_{n_m}(t) - u_{n_m}(t)\big)_{V_T} \, \mathrm{d}t \right| &= \left| \int_{\tilde{\alpha} h_{n_m}}^{\tilde{\beta} h_{n_m}} \big(v(t), \tilde{u}_{n_m}(t) - u_{n_m}(t)\big)_{V_T} \, \mathrm{d}t \right| \\
&= \left| \sum_{j=\tilde{\alpha}}^{\tilde{\beta}} \int_{t_{j-1}^{n_m}}^{t_j^{n_m}} (v, z_j^{n_m} - z_{j-1}^{n_m})_{V_T} \, \mathrm{d}t \right| \\
&= |(v, z_\alpha^{n_m} - z_\beta^{n_m})_{V_T}| h_{n_m} \\
&\leqslant \|v\|_{V_T} (\|z_\alpha^{n_m}\|_{V_T} + \|z_\beta^{n_m}\|_{V_T}) h_{n_m} \leqslant C \|v\|_{V_T} h_{n_m}
\end{aligned}$$

and we get

$$\left| \int_\alpha^\beta \big(v(t), \tilde{u}_{n_m}(t) - u_{n_m}(t)\big)_{V_T} \, \mathrm{d}t \right| \leqslant C h_{n_m}.$$

Hence, it follows that if

$$\alpha = \tilde{\alpha} h_{n_m}, \quad \beta = \tilde{\beta} h_{n_m}$$

then our statement holds, since otherwise we approximate the numbers α and β by $\tilde{\alpha} h_{n_m}$ and $\tilde{\beta} h_{n_m}$, respectively (here $\tilde{\alpha}$ and $\tilde{\beta}$ are nonnegative integers). This approximation leads to an additional term on the right-hand side of the last inequality which converges to zero when $n_m \to \infty$.

From the above considerations it follows that

$$u \in L_2(I, V_Q),$$

since the sequence $\{\tilde{u}_n\}_{n=1}^{\infty} \subset L_2(I, V_Q)$ and the set $L_2(I, V_Q)$ is a convex closed set in $L_2(I, V_T)$ (here, we apply Theorem 25.2 in [2], i.e. the fact that every convex closed set in reflexive Banach space is weakly closed).

So, we have proved that the function u satisfies the conditions 1)–4) of Definition 3.2. Moreover, this function also satisfies the integral identity 5).

C. Consider the integral identity (4.2), written for n_m:

$$((z_j^{n_m}, v))_{t_j^{n_m}} + \Big(\frac{z_j^{n_m} - z_{j-1}^{n_m}}{h_{n_m}}, v\Big)_{t_j^{n_m}} = (f, v)_{t_j^{n_m}} \quad \text{for all } v \in V_{t_j^{n_m}}.$$

Let $v \in L_2(I, V_Q)$. We can write

$$((\tilde{u}_{n_m}(t), v(t)))_T + \big(U_{n_m}(t), v(t)\big)_T = \big(f(t), v(t)\big)_T$$

for almost all $t \in [0, T]$, where $\tilde{u}_{n_m}(t)$, $U_{n_m}(t)$ are defined as above and $f(t) \equiv f$ in $[0, T]$. After integrating over the interval I we get

$$\int_0^T ((\tilde{u}_{n_m}(t), v(t)))_T \,\mathrm{d}t + \int_0^T \big(U_{n_m}(t), v(t)\big)_T \,\mathrm{d}t = \int_0^T \big(f(t), v(t)\big)_T \,\mathrm{d}t.$$

All these integrals exist, since $v \in L_2(I, V_Q)$ (consequently, $v \in L_2(I, L_2(\Omega_T))$), $\tilde{u}_{n_m} \in L_2(I, V_Q)$, $U_{n_m} \in L_2(I, L_2(\Omega_T))$, $f \in L_2(I, L_2(\Omega_T))$.

From (4.14) and (4.15) we have

$$\int_0^T ((\tilde{u}_{n_m}(t), v(t)))_T \,\mathrm{d}t \to \int_0^T ((u(t), v(t)))_T \,\mathrm{d}t,$$
$$\int_0^T \big(U_{n_m}(t), v(t)\big)_T \,\mathrm{d}t \to \int_0^T \big(u'(t), v(t)\big)_T \,\mathrm{d}t$$

as $m \to \infty$. Altogether, for a fixed function $v \in L_2(I, V_Q)$ and for $m \to \infty$ we get

$$\int_0^T ((u(t), v(t)))_T \,\mathrm{d}t + \int_0^T \big(u'(t), v(t)\big)_T \,\mathrm{d}t = \int_0^T \big(f(t), v(t)\big)_T \,\mathrm{d}t.$$

However, the function $v(t)$ was arbitrary as an element of $L_2(I, V_Q)$, and consequently the last equality holds for every function $v \in L_2(I, V_Q)$.

From the uniqueness of the solution it follows that not only the subsequence $\{u_{n_m}\}_{m=1}^{\infty}$, but the whole sequence $\{u_n\}_{n=1}^{\infty}$ converges weakly in $L_2(I, V_T)$ to the function u. $\square$

5. Further results and concluding remarks

The proofs in Section 4 show that our results may be generalized and supplemented in a number of ways. Here we just begin by considering the following generalization of the problem described by (1.2)–(1.4):

$$\frac{\partial u}{\partial t} + A(x,t)u = f(x,t) \quad \text{in } Q, \tag{5.1}$$

$$u(x,0) = u_0 \quad \text{in } \Omega_0, \tag{5.2}$$

$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = \ldots = \frac{\partial^{k-1} u}{\partial \nu^{k-1}}(x,t) = 0, \quad 0 < t < T,\ x \in \partial\Omega_t, \tag{5.3}$$

where A is a linear differential operator of order $2k$ in the divergence form:

$$(Au)(x,t) = \sum_{|i|,|j| \leqslant k} (-1)^{|i|} \partial^i \big(a_{i,j}(x,t) \partial^j u\big)$$

for $(x,t) \in Q$ and ν the outer normal to $\partial\Omega_t$. The coefficients $a_{i,j}$ of this operator and the right-hand side f are now functions defined a.e. in Q and u_0 is defined in Ω_0.

Denote by $(\cdot,\cdot)_\tau$ the inner product in $L_2(\Omega_\tau)$, let

$$(\!(u,v)\!)_{(t,\tau)} = \big(A(t)u, v\big)_\tau = \sum_{|i|,|j| \leqslant k} \int_{\Omega_\tau} a_{i,j}(x,t) \partial^i u(x) \partial^j v(x) \,\mathrm{d}x$$

be the bilinear form associated with the operator A where we assume that for $t < \tau$ the coefficients of A are extended as

$$a_{i,j}(x,t) = \begin{cases} a_{i,j}(x,t), & x \in \Omega_t, \\ 0, & x \in \Omega_\tau \setminus \Omega_t. \end{cases}$$

The existence of a solution will be ensured by the following assumptions.

Assumption 5.1. It is assumed that there exist constants $M, m > 0$ such that for every $t \in (0,T)$ we have

(A1) boundedness:

$$(\!(u,v)\!)_{(t,t)} \leqslant M \|u\|_{W^{k,2}(\Omega_t)} \|v\|_{W^{k,2}(\Omega_t)} \quad \text{for all } u, v \in W^{k,2}(\Omega_t);$$

(A2) V_t-ellipticity:

$$(\!(u,u)\!)_{(t,t)} \geqslant m\|u\|^2_{W^{k,2}(\Omega_t)} \quad \text{for all } u \in V_t;$$

(A3) V_t-symmetry:

$$(\!(u,v)\!)_{(t,t)} = (\!(v,u)\!)_{(t,t)} \quad \text{for all } u,v \in V_t;$$

(A4) the Lipschitz condition on the coefficients of the operator A:
the inequality

$$\frac{|(\!(u,v)\!)_{(t+h,t)} - (\!(u,v)\!)_{(t,t)}|}{h} \leqslant M\|u\|_{W^{k,2}(\Omega_t)}\|v\|_{W^{k,2}(\Omega_t)} \quad \text{for all } u,v \in V_t$$

holds for all $h \in (0, T-t)$;

(A5) there exists a function $F \in V^1\big(0,T; L_2(\Omega_T)\big) \cap C\big(I; L_2(\Omega_T)\big)$ such that

$$F(x,t) = f(x,t) \quad \text{for all } (x,t) \in Q$$

and we extend our function f to the set $\Omega_T \times [0,T]$ as

$$f(x,t) = \begin{cases} f(x,t), & (x,t) \in Q, \\ 0, & \Omega_T \times [0,T] \setminus Q; \end{cases}$$

(A6) the initial condition u_0 belongs to $W^{k,2}_0(\Omega_0)$.

The assumptions (A1)–(A4) are satisfied if we assume that the coefficients of the differential operator satisfy the following conditions: there exist numbers $C, c > 0$ such that for all $t \in (0,T)$

$$\|a_{i,j}(\cdot,t)\|_{L_\infty(\Omega_t)} \leqslant C,$$
$$\sum_{|i|,|j|\leqslant k} a_{i,j}(x,t)\xi_i\xi_j \geqslant c\sum_{|i|\leqslant k} \xi_i^2 \quad \text{for all } \xi = \{\xi_i\colon\ |i| \leqslant k\} \in \mathbb{R}^m,$$
$$a_{i,j}(x,t) = a_{j,i}(x,t) \quad \text{for all } i,j\ (|i|,|j| \leqslant k)$$

for a.e. $x \in \Omega_t$, and

$$\|a_{i,j}(\cdot,t+h) - a_{i,j}(\cdot,t)\|_{L_\infty(\Omega_t)} \leqslant Mh.$$

The notion of a solution of the problem introduced above will be given now.

Definition 5.2. A function u is called a *weak solution* of the problem (5.1)–(5.3) if it satisfies the following conditions:

1) $u \in L_2(I, V_Q)$,
2) $u \in \mathrm{AC}\big(I, L_2(\Omega_T)\big)$,
3) $u' \in L_2\big(I, L_2(\Omega_T)\big)$,
4) $u(0) = u_0$,
5) $\int_0^T ((u(t), v(t)))_{(t,T)}\, \mathrm{d}t + \int_0^T \big(u'(t), v(t)\big)_T \,\mathrm{d}t = \int_0^T \big(f(t), v(t)\big)_T \,\mathrm{d}t$ for all $v \in L_2(I, V_Q)$.

We shall prove the existence of the weak solution and show that this weak solution is the limit of the sequence $\{u_n(x,t)\}$ of functions constructed by Rothe's method. Moreover, we should point out that we will deal with the *homogeneous* initial condition, where the right-hand side in (5.2) is zero. The step to *nonhomogeneous* initial condition can be made via a "translation" $u + u_0$ (see e.g. [5], Section 13).

Theorem 5.3 (Existence and Uniqueness)**.** *Under Assumption* 5.1*, the problem*

$$\frac{\partial u}{\partial t} + A(x,t)u = f(x,t) \quad in\ Q,$$
$$u(x,0) = 0 \quad in\ \Omega_0,$$
$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = \ldots = \frac{\partial^{k-1} u}{\partial \nu^{k-1}}(x,t) = 0, \quad x \in \partial\Omega_t,\ 0 < t < T$$

has exactly one weak solution, i.e. exactly one function which is a weak limit of the sequence of Rothe's functions $u_n(t)$ in the space $L_2(I, V_T)$.

Proof. (Uniqueness) The uniqueness of the solution is proved analogously as in Theorem 3.3.

(Existence) We follow the lines of the proof of Theorem 3.4 and concentrate on inequalities which are essential for this extended case.

A$'$. Let us consider the integral identity

$$((z_j^n, v))_{(t_j^n, t_j^n)} + \Big(\frac{z_j^n - z_{j-1}^n}{h_n}, v\Big)_{t_j^n} = (f_j^n, v)_{t_j^n} \quad \text{for all } v \in V_{t_j^n}, \tag{5.4}$$

where $f_j^n = f(\cdot, t_j^n)$, and choose $v = z_j^n$. This makes sense, since $z_j^n \in V_{t_j^n}$. We get

$$((z_j^n, z_j^n))_{(t_j^n, t_j^n)} + \Big(\frac{z_j^n - z_{j-1}^n}{h_n}, z_j^n\Big)_{t_j^n} = (f_j^n, z_j^n)_{t_j^n}.$$

However, since by (A2)

$$((z_j^n, z_j^n))_{(t_j^n, t_j^n)} \geqslant 0,$$

we have that

$$(z_j^n, z_j^n)_{t_j^n} \leqslant h_n (f_j^n, z_j^n)_{t_j^n} + (z_{j-1}^n, z_j^n)_{t_j^n}.$$

By the Schwarz inequality we get

$$\|z_j^n\|^2_{L_2(\Omega_{t_j^n})} \leqslant h_n \|f_j^n\|_{L_2(\Omega_{t_j^n})} \|z_j^n\|_{L_2(\Omega_{t_j^n})} + \|z_{j-1}^n\|_{L_2(\Omega_{t_j^n})} \|z_j^n\|_{L_2(\Omega_{t_j^n})}$$

and consequently

$$\|z_j^n\|_{L_2(\Omega_{t_j^n})} \leqslant h_n \|f_j^n\|_{L_2(\Omega_{t_j^n})} + \|z_{j-1}^n\|_{L_2(\Omega_{t_j^n})}$$

for $j = 1, 2, \ldots, 2^{n-1}p$. Further,

$$\begin{aligned} \|z_j^n\|_{L_2(\Omega_{t_j^n})} &\leqslant h_n \|f_j^n\|_{L_2(\Omega_{t_j^n})} + \|z_{j-1}^n\|_{L_2(\Omega_{t_j^n})} \\ &\leqslant h_n (\|f_j^n\|_{L_2(\Omega_{t_j^n})} + \|f_{j-1}^n\|_{L_2(\Omega_{t_j^n})}) + \|z_{j-2}^n\|_{L_2(\Omega_{t_j^n})} \\ &\leqslant \ldots \leqslant h_n \sum_{k=1}^{j} \|f_k^n\|_{L_2(\Omega_{t_j^n})} \leqslant j h_n V(f) \end{aligned}$$

and since $jh_n \leqslant 2^{n-1} p h_n = T$, we finally have that

$$\|z_j^n\|_{L_2(\Omega_T)} \leqslant T V(f) \tag{5.5}$$

for $j = 1, 2, \ldots, 2^{n-1}p$, where $V(f) = V^1(f) + \sup\limits_{(0,T)} \|f(t)\|_{L_2(\Omega_t)}$ and $V^1(f)$ is a total variation of f in the sense of Bochner.

Let us choose the function

$$v = z_j^n - z_{j-1}^n$$

in the integral identity (5.4), which makes sense, because $z_j^n - z_{j-1}^n \in V_{t_j^n}$. Then we get

$$(\!(z_j^n, z_j^n - z_{j-1}^n)\!)_{(t_j^n, t_j^n)} + \Big(\frac{z_j^n - z_{j-1}^n}{h_n}, z_j^n - z_{j-1}^n\Big)_{t_j^n} = (f_j^n, z_j^n - z_{j-1}^n)_{t_j^n}$$

for $j = 1, 2, \ldots, 2^{n-1}p$. After summing both sides from $j = 1$ to i, we get

$$\begin{aligned} \sum_{j=1}^{i} (\!(z_j^n, z_j^n - z_{j-1}^n)\!)_{(t_j^n, t_j^n)} &+ \frac{1}{h_n} \sum_{j=1}^{i} (z_j^n - z_{j-1}^n, z_j^n - z_{j-1}^n)_{t_j^n} \\ &= \sum_{j=1}^{i} (f_j^n, z_j^n - z_{j-1}^n)_{t_j^n}. \end{aligned} \tag{5.6}$$

If we denote

$$S_i^1 = \sum_{j=1}^{i} ((z_j^n, z_j^n - z_{j-1}^n))_{(t_j^n, t_j^n)}, \qquad S_i^2 = \frac{1}{h_n} \sum_{j=1}^{i} (z_j^n - z_{j-1}^n, z_j^n - z_{j-1}^n)_{t_j^n},$$

$$S_i^3 = \sum_{j=1}^{i} (f_j^n, z_j^n - z_{j-1}^n)_{t_j^n},$$

then we can rewrite (5.6) as

$$S_i^1 + S_i^2 = S_i^3. \tag{5.7}$$

Taking into account that (here we use (A4))

$$\begin{aligned}
S_i^1 &= \frac{1}{2} \sum_{j=1}^{i} \bigl\{ 2((z_j^n, z_j^n))_{(t_j^n, t_j^n)} - 2((z_j^n, z_{j-1}^n))_{(t_j^n, t_j^n)} \bigr\} \\
&= \frac{1}{2} \biggl\{ ((z_i^n, z_i^n))_{(t_i^n, t_i^n)} + \sum_{j=1}^{i} ((z_j^n - z_{j-1}^n, z_j^n - z_{j-1}^n))_{(t_j^n, t_j^n)} \\
&\quad + \sum_{j=1}^{i} [((z_{j-1}^n, z_{j-1}^n))_{(t_{j-1}^n, t_{j-1}^n)} - ((z_{j-1}^n, z_{j-1}^n))_{(t_j^n, t_j^n)}] \biggr\},
\end{aligned}$$

by (A2) and (A3) we get

$$S_i^1 \geqslant \frac{m}{2} \|z_i^n\|^2_{W^{k,2}(\Omega_{t_i^n})} - \frac{h_n M}{2} \sum_{j=1}^{i} \|z_{j-1}^n\|^2_{W^{k,2}(\Omega_{t_j^n})}.$$

So, we obtain

$$S_i^1 \geqslant \frac{m}{2} \|z_i^n\|^2_{W^{k,2}(\Omega_{t_i^n})} - \frac{h_n M}{2} \sum_{j=1}^{i} \|z_{j-1}^n\|^2_{W^{k,2}(\Omega_{t_j^n})}, \tag{5.8}$$

$$S_i^2 = \frac{1}{h_n} \sum_{j=1}^{i} \|z_j^n - z_{j-1}^n\|^2_{L_2(\Omega_{t_j^n})}, \tag{5.9}$$

$$\begin{aligned}
S_i^3 &= \bigl(f(t_i^n), z_i^n\bigr)_{t_i^n} + \sum_{j=1}^{i} \bigl(f(t_{j-1}^n) - f(t_j^n), z_{j-1}^n\bigr)_{t_j^n} \\
&\leqslant \|f_i^n\|_{L^2(\Omega_{t_i^n})} \|z_i^n\|_{L^2(\Omega_{t_i^n})} + \sum_{j=1}^{i} \|f_j^n - f_{j-1}^n\|_{L^2(\Omega_{t_j^n})} \|z_{j-1}^n\|_{L^2(\Omega_{t_j^n})} \\
&\leqslant TV(f)^2 + TV(f) \sum_{j=1}^{i} \|f_j^n - f_{j-1}^n\|_{L^2(\Omega_{t_j^n})} \leqslant 2TV(f)^2,
\end{aligned}$$

hence

$$S_i^3 \leqslant 2TV(f)^2. \tag{5.10}$$

From (5.7)–(5.10) it follows that

$$\frac{m}{2}\|z_i^n\|^2_{W^{k,2}(\Omega_{t_i^n})} - \frac{h_n M}{2}\sum_{j=1}^{i}\|z_{j-1}^n\|^2_{W^{k,2}(\Omega_{t_j^n})} \leqslant S_i^1 \leqslant S_i^3 \leqslant 2TV(f)^2,$$

$$\|z_i^n\|^2_{W^{k,2}(\Omega_{t_i^n})} \leqslant \frac{4T}{m}V(f)^2 + \frac{h_n M}{m}\sum_{j=1}^{i}\|z_{j-1}^n\|^2_{W^{k,2}(\Omega_{t_j^n})}.$$

Here we apply Lemma 3.1:

$$\|z_i^n\|^2_{W^{k,2}(\Omega_{t_i^n})} \leqslant \frac{4T}{m}V(f)^2 \mathrm{e}^{(ih_n M)/m} \leqslant \frac{4T}{m}V(f)^2\mathrm{e}^{TM/m}$$

and so we get

$$\|z_i^n\|_{W^{k,2}(\Omega_T)} \leqslant 2\sqrt{\frac{T}{m}}V(f)\mathrm{e}^{TM/2m} \tag{5.11}$$

for all $i = 1, 2, \ldots, 2^{n-1}p$. According to (5.8)

$$\begin{aligned}
S_i^2 = S_i^3 - S_i^1 &\leqslant 2TV(f)^2 - \sum_{j=1}^{i}\big[((z_{j-1}^n, z_{j-1}^n))_{(t_{j-1}^n, t_{j-1}^n)} - ((z_{j-1}^n, z_{j-1}^n))_{(t_j^n, t_j^n)}\big] \\
&\leqslant 2TV(f)^2 + 2Mh_n\sum_{j=1}^{i}\|z_{j-1}^n\|^2_{W^{k,2}(\Omega_T)} \\
&\leqslant 2TV(f)^2 + 2MC_1^2 ih_n \\
&\leqslant 2TV(f)^2 + 2MC_1^2 T = C_2.
\end{aligned}$$

This implies that

$$\frac{1}{h_n}\sum_{j=1}^{2^{n-1}p}\|z_j^n - z_{j-1}^n\|^2_{L_2(\Omega_T)} \leqslant C_2. \tag{5.12}$$

Note that the right-hand sides of the inequalities (5.11) and (5.12) are independent of i and n.

B$'$. Now, we can almost literally repeat our considerations from part B of the proof of Theorem 3.4. Then we get $u \in L_2(I, V_Q)$, $u \in \mathrm{AC}\big(I, L_2(\Omega_T)\big)$ and $u(0) = 0$.

C′. Now, we show that the function $u(t)$ satisfies the integral identity 4) of Definition 5.2. Consider the integral identity (5.4), written for n_m:

$$(\!(z_j^{n_m}, v)\!)_{(t_j^{n_m}, t_j^{n_m})} + \Big(\frac{z_j^{n_m} - z_{j-1}^{n_m}}{h_{n_m}}, v\Big)_{t_j^{n_m}} = (f_j^{n_m}, v)_{t_j^{n_m}} \quad \text{for all } v \in V_{t_j^{n_m}}.$$

Let $v \in L_2(I, V_Q)$. We can write

$$(\!(\tilde{u}_{n_m}(t), v(t))\!)_{(T_{n_m}(t),T)} + \big(U_{n_m}(t), v(t)\big)_T = \big(f_{n_m}(t), v(t)\big)_T$$

for almost all $t \in I$, where $\tilde{u}_{n_m}(t)$, $U_{n_m}(t)$ are defined in the proof of Theorem 3.4 and

$$T_{n_m}(t) = \begin{cases} t_1^{n_m}, & t = 0, \\ t_j^{n_m}, & t \in \tilde{I_j^n} = (t_{j-1}^n, t_j^n],\ j = 1, \dots, 2^{n-1}p, \end{cases}$$

$$f_{n_m}(t) = \begin{cases} f_1^{n_m}, & t = 0, \\ f_j^{n_m}, & t \in \tilde{I_j^n} = (t_{j-1}^n, t_j^n],\ j = 1, \dots, 2^{n-1}p. \end{cases}$$

After integrating over the interval I we get

$$\int_0^T (\!(\tilde{u}_{n_m}(t), v(t))\!)_{(T_{n_m}(t),T)}\,\mathrm{d}t + \int_0^T \big(U_{n_m}(t), v(t)\big)_T\,\mathrm{d}t = \int_0^T \big(f_{n_m}(t), v(t)\big)_T\,\mathrm{d}t.$$

All these integrals exist and

$$\int_0^T \big(U_{n_m}(t), v(t)\big)_T\,\mathrm{d}t \to \int_0^T \big(u'(t), v(t)\big)_T\,\mathrm{d}t$$

and

$$\int_0^T (\!(\tilde{u}_{n_m}(t), v(t))\!)_{(T_{n_m}(t),T)}\,\mathrm{d}t \to \int_0^T (\!(u(t), v(t))\!)_{(t,T)}\,\mathrm{d}t$$

as $m \to \infty$. By (A4) and the definition of the sequence $\{f_{n_m}\}_{m=1}^{\infty}$ it follows that

$$\int_0^T \big(f_{n_m}(t), v(t)\big)_T\,\mathrm{d}t \to \int_0^T \big(f(t), v(t)\big)_T\,\mathrm{d}t \quad \text{as } n_m \to \infty,$$

since applying Hölder inequality we get

$$\begin{aligned}
&\Big|\int_0^T \big(f_{n_m}(t) - f(t), v(t)\big)_T\,\mathrm{d}t\Big| \\
&\qquad \leqslant \int_0^T \|f_{n_m}(t) - f(t)\|_{L_2(\Omega_T)} \|v(t)\|_{L_2(\Omega_T)}\,\mathrm{d}t \\
&\qquad \leqslant \max_I \|f_{n_m}(t) - f(t)\|_{L_2(\Omega_T)} \int_0^T \|v(t)\|_{L_2(\Omega_T)}\,\mathrm{d}t \\
&\qquad = C(v) \max_I \|f_{n_m}(t) - f(t)\|_{L_2(\Omega_T)}
\end{aligned}$$

and the claim follows from the uniform convergence of $\{f_{n_m}(t)\}_{m=1}^{\infty}$.

Altogether, for the fixed function $v \in L_2(I, V_Q)$ and for $m \to \infty$ we get

$$\int_0^T ((u(t), v(t)))_{(t,T)} \, \mathrm{d}t + \int_0^T \big(u'(t), v(t)\big)_T \, \mathrm{d}t = \int_0^T \big(f(t), v(t)\big)_T \, \mathrm{d}t.$$

But the function $v(t)$ was arbitrary as an element of $L_2(I, V_Q)$, and consequently the last equality holds for all functions $v \in L_2(I, V_Q)$.

From the uniqueness of the solution it follows that not only the subsequence $\{u_{n_m}\}_{m=1}^{\infty}$, but the whole sequence $\{u_n\}_{n=1}^{\infty}$ converges weakly in $L_2(I, V_T)$ to the function u.

The proof is complete. □

Proposition 5.4 (Continuous dependence of the solution on f)**.** *The weak solution of the problem* (5.1)–(5.3) *satisfies*

$$\|u\|_{L_2(I, L_2(\Omega_T))} \leqslant T^{3/2} V(f).$$

P r o o f. The assertion follows from the inequality (5.5), i.e.

$$\begin{aligned} \|u_n(t)\|_{L_2(\Omega_T)} &= \Big\| z_{j-1}^n + \frac{t - t_{j-1}^n}{h_n}(z_j^n - z_{j-1}^n) \Big\|_{L_2(\Omega_T)} \\ &\leqslant \Big(1 - \frac{t - t_{j-1}^n}{h_n}\Big) \|z_{j-1}^n\|_{L_2(\Omega_T)} \\ &\quad + \frac{t - t_{j-1}^n}{h_n} \|z_j^n\|_{L_2(\Omega_T)} \leqslant T V(f) \end{aligned}$$

where $t \in [t_{j-1}, t_j]$, $j = 1, 2, \ldots, 2^{n-1}p$, and hence

$$\|u_n\|_{L_2(I, L_2(\Omega_T))}^2 = \int_0^T \|u_n(t)\|_{L_2(\Omega_T)}^2 \, \mathrm{d}t \leqslant T^3 V(f)^2.$$

□

Corollary 5.5. *Let $u(t)$ and $\tilde{u}(t)$ be weak solutions of the problem* (5.1)–(5.3) *with right-hand sides f and $\tilde{f}$, respectively. Then*

$$\|u(t) - \tilde{u}(t)\|_{L_2(I, L_2(\Omega_T))} \leqslant T^{3/2} V(f - \tilde{f}).$$

P r o o f. Obviously, the function $u(t) - \tilde{u}(t)$ is the weak solution of the problem (5.1)–(5.3) with the right-hand side $f - \tilde{f}$. Hence, the last inequality follows immediately from Proposition 5.4. □

References

[1] *J. Dasht, J. Engström, A. Kufner, and L.-E. Persson*: Rothe's method for parabolic equations on non-cylindrical domains. Advances in Algebra and Analysis *1* (2006), 1–22.

[2] *S. Fučík, A. Kufner*: Nonlinear Differential Equations. Elsevier, Amsterdam-Oxford-New York, 1980. zbl

[3] *J. Kačur*: Method of Rothe in Evolution Equations. Teubner, Leipzig, 1985.

[4] *A. Kufner, O. John, and S. Fučík*: Function Spaces. Academia, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1977. zbl

[5] *K. Rektorys*: The Method of Discretization in Time and Partial Differential Equations. D. Reidel, Dordrecht-Boston-London, 1982. zbl

[6] *K. Rektorys*: On application of direct variational methods to the solution of parabolic boundary value problems of arbitrary order in space variables. Czech. Math. J. *21* (1971), 318–339. zbl

[7] *K. Rektorys*: Variational Methods in Mathematics, Science and Engineering. D. Reidel, Dordrecht-Boston-London, 1980. zbl

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