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APPROXIMATE SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS*

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Abstract. The methods of arbitrarily high orders of accuracy for the solution of an abstract ordinary differential equation are studied. The right-hand side of the differential equation under investigation contains an unbounded operator which is an infinitesimal generator of a strongly continuous semigroup of operators. Necessary and sufficient conditions are found for a rational function to approximate the given semigroup with high accuracy.

Keywords: abstract differential equations, semigroups of operators, rational approximations, A-stability

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1. Preliminaries

In this paper we will deal with approximations of the solution to an abstract differential equation of the form

$$u'(t) = Au(t), \quad t \in (0, T) \tag{1.1}$$

with the initial condition

$$u(0) = \eta. \tag{1.2}$$

The values of the unknown function $u(t)$ lie in a (complex) Banach space $\mathcal{X}$ and A is a closed linear operator with domain which is dense in $\mathcal{X}$.

For the reader's convenience we recall some special results from the functional analysis in this section, namely, some theorems concerning abstract functions of

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a real variable, the operator calculus for unbounded operators, and the theory of semigroups of operators. We will not prove them, the details may be found in many textbooks on functional analysis, e.g., in [1], [3], [4] and [6].

Theorem 1.1. *Let* $\varphi\colon (0, h_0) \to \mathcal{X}$, *let* $\varphi'(h)$ *exist for any* $h \in (0, h_0)$, *and let, moreover,* $\lim\limits_{h\to 0} \varphi'(h)$ *exist. Then* $\lim\limits_{h\to 0} \varphi(h)$ *also exists, and the function* $\varphi(h)$ *defined by its limit as* h *tends to zero has the right derivative at* $h = 0$ *equal to* $\lim\limits_{h\to 0} \varphi'(h)$.

Theorem 1.2. *Let* $\varphi(h)$ *have the* n*th derivative at* $h = 0$. *Then*

$$\lim_{h\to 0} \frac{\varphi(h) - \varphi(0) - \varphi'(0)h^n/1! - \ldots - \varphi^{(n)}(0)h^n/n!}{h^n} = 0.$$

Theorem 1.3. *Let* $\varphi(h)$ *have the* $(n+1)$*st derivative in* $[0, h_0]$, *and let* $\|\varphi^{(n+1)}(h)\| \leqslant M$ *for* $h \in [0, h_0]$. *Then*

$$\left\|\varphi(h) - \left[\varphi(0) + \varphi'(0)\frac{h}{1!} + \ldots + \varphi^{(n)}(0)\frac{h^n}{n!}\right]\right\| \leqslant M\frac{h^{n+1}}{(n+1)!}$$

for all $h \in [0, h_0]$.

Further, let us recall some concepts of the operator calculus for unbounded operators. By the operator calculus we mean the technique which enables us to define functions of operators. Thus, let $\mathcal{X}$ be a Banach space and A a closed operator mapping the subspace $\mathcal{D} \subseteq \mathcal{X}$ into $\mathcal{X}$, and let $\mathcal{D}$ be dense in $\mathcal{X}$. The set $r(A)$ of such complex λ that the operator $R(\lambda, A) = (\lambda I - A)^{-1}$ exists, is bounded and defined on the whole space $\mathcal{X}$, is called the *resolvent set*. The operator $R(\lambda, A)$ is called the resolvent of A and the complement of the set $r(A)$ is called the spectrum of the operator A and is denoted by $\sigma(A)$.

The symbol $\mathcal{F}(A)$ will denote the family of all functions f of a complex variable which are regular in a neighbourhood of the spectrum and at infinity. This neighbourhood need not be connected and may depend on the individual f. Let V be an open set containing the spectrum $\sigma(A)$ of A, let the boundary Γ of V consist of a finite number of rectifiable Jordan curves, and let f be regular on $V \cup \Gamma$. Further, let Γ be positive orientated with respect to V. Then the function $f(A)$ of the operator A is defined by

$$f(A) = f(\infty)I + \frac{1}{2\pi \mathrm{i}}\int_\Gamma f(\lambda)R(\lambda, A)\,\mathrm{d}\lambda. \tag{1.3}$$

This formula defines a *bounded* operator the properties of which are summarized in the following theorem.

Theorem 1.4. *Let $f, g \in \mathcal{F}(A)$. Then*

(a) $(f+g)(A) = f(A) + g(A)$;

(b) $(fg)(A) = f(A)g(A)$;

(c) $\sigma(f(A)) = f(\sigma(A) \cup \infty)$;

(d) *if $f \in \mathcal{F}(A)$, $g \in \mathcal{F}(f(A))$ and $F(\xi) = g(f(\xi))$, then $F \in \mathcal{F}(A)$ and $F(A) = g(f(A))$;*

(e) *if U is any bounded operator which commutes with A, then it commutes also with any $f(A)$.*

Further, we want to define the powers of an (unbounded) operator A. The formula (1.3) cannot be used here since any nonconstant polynomial has a pole at infinity. Consequently, we succeed to define a polynomial of A only on a proper subset of $\mathcal{X}$. It is clear that it is sufficient to define a power of A and we will do it in the following natural recurrent way:

$$A^0 = I, \quad A^1 = A,$$
$$\mathcal{D}(A^n) = \{x\colon\ x \in \mathcal{D}(A^{n-1}) \ \text{and} \ A^{n-1}x \in \mathcal{D}(A)\},$$
$$A^n = A(A^{n-1}).$$

The basic properties of a polynomial of A and its relation to $f(A)$ are given in the following theorems.

Theorem 1.5. *Let A be a closed operator with a nonempty resolvent set and let P be a polynomial of degree n. Then $P(A)$ is also a closed operator. If, moreover, the domain of A is dense in $\mathcal{X}$, then the domain of $P(A)$ is also dense in $\mathcal{X}$.*

Theorem 1.6. *Let P be a polynomial of degree n and let $f \in \mathcal{F}(A)$ have a root of order m, $0 \leqslant m \leqslant \infty$ at infinity. Then*

(a) *if $x \in \mathcal{D}(A^n)$, then $f(A)x \in \mathcal{D}(A^{n+m})$ ($n+m = \infty$ if $m = \infty$) and $P(A)f(A)x = f(A)P(A)x$;*

(b) *if $0 \leqslant n \leqslant m$ and $g(\lambda) = P(\lambda)f(\lambda)$, then $g \in \mathcal{F}(A)$ and $g(A) = P(A)f(A)$.*

Theorem 1.7. *Let P be a polynomial. Then $P(\sigma(A)) = \sigma(P(A))$.*

Finally, let us present some results concerning the theory of semigroups of operators. The set $\{U(t);\ 0 \leqslant t < \infty\}$ of bounded linear operators in $\mathcal{X}$ is said to form a *strongly continuous semigroup of operators* if

(i) $U(s+t) = U(s)U(t)$, $s, t \geqslant 0$,

(ii) $U(0) = I$,

(iii) the function $U(\cdot)x$ is continuous (in the topology of the space $\mathcal{X}$) on $[0,\infty)$ for any $x \in \mathcal{X}$.

Define, for any $h > 0$, an operator A_h by

$$A_h = \frac{U(h)x - x}{h}, \quad x \in \mathcal{X},$$

and suppose that $\mathcal{D}(A)$ is the set of all $x \in \mathcal{X}$ for which $\lim\limits_{h\to 0} A_h x$ exists. Then the operator A with the domain $\mathcal{D}(A)$ defined by

$$Ax = \lim_{h\to 0} A_h x, \quad x \in \mathcal{X}$$

will be called the *infinitesimal generator* of the semigroup of operators $U(t)$. The following theorems on infinitesimal generators are of basic importance for us.

Theorem 1.8. *The domain $\mathcal{D}(A)$ of A is dense in $\mathcal{X}$ and A is a closed operator on $\mathcal{D}(A)$. Further, if $\eta \in \mathcal{D}(A)$, then $U(t)\eta \in \mathcal{D}(A)$ for $0 \leqslant t < \infty$ and $\mathrm{d}U(t)\eta/\mathrm{d}t = AU(t)\eta = U(t)A\eta$.*

Theorem 1.9. *The limit $\omega_0 = \lim\limits_{t\to\infty} \log \|U(t)\|/t$ exists, and $\omega_0 < \infty$. Any λ with $\operatorname{Re}\lambda > \omega_0$ belongs to the resolvent set of A, and we have*

$$R(\lambda, A)\eta = \int_0^\infty \exp(-\lambda t)U(t)\eta \,\mathrm{d}t, \quad \eta \in \mathcal{X}, \ \operatorname{Re}\lambda > \omega_0.$$

Note that the existence of ω_0 implies that, for any $\omega > \omega_0$, there exists a constant M_ω such that

$$\|U(t)\| \leqslant M_\omega \exp(\omega t) \tag{1.4}$$

for any $t \geqslant 0$.

Theorem 1.10. *For a closed operator with domain which is dense in $\mathcal{X}$, to be the infinitesimal generator of a strongly continuous semigroup of operators it is necessary and sufficient that there exist real constants M and ω such that*

$$\|R^n(\lambda, A)\| \leqslant \frac{M}{(\operatorname{Re}\lambda - \omega)^n}, \quad n = 1, 2, \dots \tag{1.5}$$

for any λ such that $\operatorname{Re}\lambda > \omega$.

2. Auxiliary lemmas

In what follows we suppose that A is the infinitesimal generator of a strongly continuous semigroup of operators $U(t)$.

Lemma 2.1. *Let $F(z)$ be a rational function which has poles in the (open) right-hand halfplane and is regular at infinity. Then*

$$\lim_{h\to 0} F^{(i)}(hA)x = F^{(i)}(0)x$$

for any $x \in \mathcal{X}$ and any $i = 0, 1, \ldots$

Proof. Let $\tilde{V}$ be a bounded domain containing the poles of $F(z)$. Obviously, it can be chosen in such a way that its closure lies in the open right-hand halfplane and that its boundary Γ is a Jordan curve. Further, let V be the complement of $\tilde{V}$. Then the function $F^{(i)}(z)$ is regular in V and at infinity for any $i = 0, 1, \ldots$ The spectrum of the operator hA lies, for sufficiently small h, in V and we can write

$$F^{(i)}(hA) = F^{(i)}(\infty)I + \frac{1}{2\pi\mathrm{i}} \int_\Gamma F^{(i)}(\lambda) R(\lambda, hA)\,\mathrm{d}\lambda \tag{2.1}$$

for $h \leqslant h_0$. The operator A is an infinitesimal generator of a semigroup $U(t)$. Thus, Theorem 1.9 implies

$$R(\lambda, hA)x = \frac{1}{h} R\Big(\frac{\lambda}{h}, hA\Big)x = \frac{1}{h}\int_0^\infty \exp\Big(-\frac{\lambda}{h}t\Big)U(t)x\,\mathrm{d}t = \int_0^\infty \exp(-\lambda t)U(ht)x\,\mathrm{d}t$$

for any $x \in \mathcal{X}$ and for any λ with $\operatorname{Re}\lambda > h\omega_0$. Consequently,

$$\lim_{h\to 0} R(\lambda, hA)x = \lim_{h\to 0}\int_0^\infty \exp(-\lambda t)U(ht)x\,\mathrm{d}t \tag{2.2}$$

for any $x \in \mathcal{X}$ and for any λ with $\operatorname{Re}\lambda > h\omega_0$ and the more so for $\operatorname{Re}\lambda > h_0\omega_0$. Since $\|U(ht)\| \leqslant M_\omega \exp(\omega ht)$ for any $\omega > \omega_0$ (see (1.4)) we conclude that

$$\|\exp(-\lambda t)U(ht)x\| \leqslant M_\omega \exp(-(\lambda - \omega h)t)\|x\| \leqslant M_\omega \exp(-(\lambda - \omega h_0)t)\|x\|.$$

Thus, the function $\exp(-\lambda U(ht))x$ has an integrable majorant independent of h for any λ with $\operatorname{Re}\lambda > h_0\omega$. Consequently, the limit sign in (2.2) can be interchanged with the integral sign so that we obtain

$$\lim_{h\to 0} R(\lambda, h)x = \frac{1}{\lambda}x$$

for any $x \in \mathcal{X}$ and for any $\operatorname{Re}\lambda > h_0\omega$. Using the assumption that A is the infinitesimal generator of $U(t)$ again we find that

$$\|R(\lambda, hA)\| \leqslant \frac{M}{\operatorname{Re}\lambda - \omega h_0}$$

for $\operatorname{Re}\lambda > \omega h_0$ and $h \leqslant h_0$. Hence we can pass in (2.1) to the limit under the integration sign, which yields

$$\begin{aligned}
\lim_{h\to 0} F^{(i)}(hA)x &= F^{(i)}(\infty)x + \frac{1}{2\pi\mathrm{i}}\int_\Gamma F^{(i)}(\lambda)\lim_{h\to 0} R(\lambda, hA)x\,\mathrm{d}\lambda \\
&= F^{(i)}(\infty)x + \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{F^{(i)}(\lambda)}{\lambda}x\,\mathrm{d}\lambda \\
&= F^{(i)}(\infty)x + F^{(i)}(0)x - F^{(i)}(\infty)x.
\end{aligned}$$

The lemma is proved. $\square$

Lemma 2.2. *Let $0 < \lambda_F < \varrho_F$ and let*

$$\begin{aligned}
\Omega &= \{z\colon\ \operatorname{Re} z > \lambda_F,\ |z| < \varrho_F\}, \\
\Omega_h &= \{z\colon\ hz \in \Omega\}, \\
\Omega_{h_0,h_1} &= \bigcup_{h_0\leqslant h\leqslant h_1} \Omega_h,
\end{aligned}$$

where $0 < h_0 < h_1$ are given. Further, denote by Γ_{h_0,h_1} the boundary of Ω_{h_0,h_1} and let

$$\begin{aligned}
K_h &= \Big\{z,\ \frac{1}{h}z \in \Gamma_{h_0,h_1}\Big\}, \\
K &= \bigcup_{h_0\leqslant h\leqslant h_1} K_h.
\end{aligned}$$

Then $K \cap \Omega = \emptyset$ and K is compact.

P r o o f. It is obvious that

$$\begin{aligned}
\Omega_{h_0,h_1} = \Big\{z;\ \frac{\lambda_F}{h_1} < \operatorname{Re} z < \frac{\lambda_F}{h_0},\ |\operatorname{Im} z| < \frac{(\varrho_F^2 - \lambda_F^2)^{1/2}}{\lambda_F}\operatorname{Re} z\Big\} \\
\cup \Big\{z;\ \operatorname{Re} z \geqslant \frac{\lambda_F}{h_0},\ |z| < \frac{\varrho_F}{h_0}\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_{h_0,h_1} = \Big\{z;\ \operatorname{Re} z = \frac{\lambda_F}{h_1},\ |z| \leqslant \frac{\varrho_F}{h_1}\Big\} \cup \Big\{z;\ \operatorname{Re} z \geqslant \frac{\lambda_F}{h_0},\ |z| = \frac{\varrho_F}{h_0}\Big\} \\
\cup \Big\{z;\ \frac{\lambda_F}{h_1} < \operatorname{Re} z < \frac{\lambda_F}{h_0},\ |\operatorname{Im} z| = \frac{(\varrho_F^2 - \lambda_F^2)^{1/2}}{\lambda_F}\operatorname{Re} z\Big\} \\
\cup \Big\{z;\ \operatorname{Re} z = \frac{\lambda_F}{h_0},\ \frac{(\varrho_F^2 - \lambda_F^2)^{1/2}}{h_0} \leqslant |\operatorname{Im} z| \leqslant \frac{\varrho_F}{h_0}\Big\}.
\end{aligned}$$

This implies immediately that

$$K_h = \Big\{z;\ \operatorname{Re} z = h\frac{\lambda_F}{h_1},\ |z| \leqslant h\frac{\varrho_F}{h_1}\Big\} \cup \Big\{z;\ \operatorname{Re} z \geqslant h\frac{\lambda_F}{h_0},\ |z| = h\frac{\varrho_F}{h_0}\Big\}$$
$$\cup \Big\{z;\ h\frac{\lambda_F}{h_1} < \operatorname{Re} z < h\frac{\lambda_F}{h_0},\ |\operatorname{Im} z| = \frac{(\varrho_F^2 - \lambda_F^2)^{1/2}}{\lambda_F}\operatorname{Re} z\Big\}$$

and, consequently,

$$K = \Big\{z;\ \frac{h_0}{h_1}\lambda_F \leqslant \operatorname{Re} z \leqslant \lambda_F,\ |\operatorname{Im} z| \leqslant \frac{(\varrho_F^2 - \lambda_F^2)^{1/2}}{\lambda_F}\operatorname{Re} z\Big\}$$
$$\cup \Big\{z;\ \varrho_F \leqslant |z| \leqslant \frac{h_1}{h_0}\varrho_F,\ |\operatorname{Im} z| \leqslant \frac{(\varrho_F^2 - \lambda_F^2)^{1/2}}{\lambda_F}\operatorname{Re} z\Big\}$$
$$\cup \Big\{z;\ \varrho_F \leqslant |z| \leqslant \frac{h_1}{h_0}(\lambda_F + \varrho_F), |\operatorname{Im} z| = \frac{(\varrho_F^2 - \lambda_F^2)^{1/2}}{\lambda_F}\operatorname{Re} z\Big\}.$$

The assertion of the lemma now immediately follows. □

Lemma 2.3. *Let the assumptions of Lemma* 2.1 *be satisfied. Then the function* $F(hA)\colon (0,h_0) \to B(\mathcal{X})$[1] *of the variable* h *has derivatives of arbitrary orders for any sufficiently small positive* h *and we have*

$$\frac{\mathrm{d}^i}{\mathrm{d}h^i}F(hA) = A^i F^{(i)}(hA), \quad i = 0,1,\ldots \tag{2.3}$$

P r o o f. First, let us note that the operator on the right-hand side of (2.3) is a bounded operator for any i and for any sufficiently small h. Indeed, the function $F^{(i)}(hz)$ is regular at infinity for any $h > 0$ and for $i = 1,2,\ldots$, and its poles lie in the halfplane $\operatorname{Re} z \geqslant b(h)$ with $\lim\limits_{h\to 0} b(h) = \infty$. Moreover, it has a root of order at least $i+1$ at infinity. Consequently, $z^i F^{(i)}(hz) \in \mathcal{F}(A)$ for sufficiently small h as follows from Theorem 1.6. Note that the function $z^i F^{(i)}(hz)$ has a root of order at least 1 at infinity.

Let us prove our assertion by induction. For $i = 1$ it is, obviously, true. So, let us suppose that it is true for some i and let us prove it for $i := i+1$. It follows from the properties of F that there exist constants ϱ_F and λ_F such that the poles of $F^{(i)}(z)$ and $F^{(i+1)}(z)$ lie in Ω from Lemma 2.2. But this yields that the poles of $z^i F^{(i)}(hz)$ and $z^{i+1}F^{(i+1)}(hz)$ lie in Ω_{h_0,h_1} of the same lemma for any $0 < h_0 \leqslant h \leqslant h_1$.

[1] $B(\mathcal{X})$ is the space of linear bounded operators in $\mathcal{X}$.

Moreover, let us choose h_1 so small that the spectrum of A lies in the complement of Ω_{h_0,h_1}. This is, obviously, possible. Thus, for any h, $h+\varepsilon \in [h_0,h_1]$, we have

$$\begin{aligned}(2.4)\qquad &\frac{A^iF^{(i)}((h+\varepsilon)A)-A^iF^{(i)}(hA)}{\varepsilon}\\ &\quad=\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_{h_0,h_1}}\frac{\lambda^iF^{(i)}((h+\varepsilon)\lambda)-\lambda^iF^{(i)}(h\lambda)}{\varepsilon}R(\lambda,A)\,\mathrm{d}\lambda.\end{aligned}$$

Further, we have for $\lambda\in\Gamma_{h_0,h_1}$

$$\begin{aligned}(2.5)\qquad &\Bigl|\frac{F^{(i)}((h+\varepsilon)\lambda)-F^{(i)}(\lambda)}{\varepsilon}-\lambda F^{(i+1)}(\lambda)\Bigr|\\ &\quad=\frac{1}{2}|\varepsilon||\lambda^2F^{(i+1)}((h+\theta\varepsilon)\lambda)|\\ &\quad\leqslant\frac{1}{2}|\varepsilon|\max_{\lambda\in\Gamma_{h_0,h_1}}|\lambda|^2\max_{\substack{h_0\leqslant h_1\\ \lambda\in\Gamma_{h_0,h_1}}}|F^{(i+2)}(h\lambda)|\\ &\quad=\frac{1}{2}|\varepsilon|\max_{\lambda\in\Gamma_{h_0,h_1}}|\lambda|^2\max_{z\in K}|F^{(i+2)}(z)|.\end{aligned}$$

The set K is disjoint with Ω and compact, hence, the continuous function $F^{(i+2)}(z)$ is bounded on K. Consequently, (2.5) implies that

$$\lim_{\varepsilon\to 0}\frac{F^{(i)}((h+\varepsilon)\lambda)-F^{(i)}(h\lambda)}{\varepsilon}=\lambda F^{(i+1)}(h\lambda)$$

uniformly with respect to $\lambda\in\Gamma_{h_0,h_1}$. Thus, we can pass in (2.4) to the limit under the integration sign obtaining finally

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{F^{(i)}((h+\varepsilon)A)-F^{(i)}(hA)}{\varepsilon}\\ &=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{h_0,h_1}}\lambda^{i+1}F^{(i+1)}(h\lambda)R(\lambda,A)\,\mathrm{d}\lambda=A^{i+1}F^{(i+1)}(hA).\end{aligned}$$

This identity proves the lemma. □

Lemma 2.4. *Let F satisfy the assumptions of Lemma* 2.1. *Then the function $\varphi(h)=F(hA)x$ with $x\in\mathcal{D}(A^m)$ has m derivatives in $[0,h_0]$ and*

$$(2.6)\qquad \frac{\mathrm{d}^i}{\mathrm{d}h^i}\varphi(h)=F^{(i)}(hA)A^ix\quad \textit{for } i=0,\dots,m \textit{ and } h>0$$

and

$$(2.7)\qquad \frac{\mathrm{d}^i}{\mathrm{d}h^i}\varphi(h)=F^{(i)}(0)A^ix\quad \textit{for } i=0,\dots,m \textit{ and } h=0.$$

P r o o f. Formula (2.6) follows directly from Lemma 2.3; formula (2.7) follows from Lemmas 2.3, 2.1 and from Theorem 1.1. □

Lemma 2.5. *Let F satisfy the assumptions of Lemma* 2.1. *Then there exists a constant M such that*

$$\|F^{(i)}(hA)\| \leqslant M, \quad i = 0, 1, \ldots$$

for any $h \in [0, h_0]$.

Proof. The assertion of the lemma follows immediately from the uniform boundedness principle. In fact, the function $F^{(i)}(hA)x$ is continuous on $[h_0, h_1]$ for any $x \in \mathcal{X}$ as follows from Lemma 2.3 and, thus, it is bounded. □

3. Main results

Let us investigate the problem (1.1), (1.2) and let A be the infinitesimal generator of a continuous semigroup of operators $U(t)$, $t \in [0, T]$. The function $U(t)\eta$ is continuous and differentiable in $[0, T]$ for any $\eta \in \mathcal{D}(A)$ and satisfies (1.1) for any $t \in [0, T]$ (see Theorem 1.8). Thus, this function represents the *classical* solution of the problem (1.1)–(1.2). However, $U(t)\eta$ has sense and is continuous for *any* $\eta \in \mathcal{X}$. Thus, this function may be referred to as the *generalized* solution even though it may not be differentiable.

Let F be a rational function with poles in the right-hand halfplane and let it be regular at infinity. Further, let the coefficients of the polynomials in the numerator and denominator of F be real and let F approximate the exponential with order p, i.e., let

$$\exp(z) = F(z) + O(z^{p+1}) \quad \text{for } z \to 0, \tag{3.1}$$

where p is a positive integer. Divide the interval $[0, T]$ into N subintervals $[t_j, t_{j+1}]$ of length $h = T/N$ by the mesh points $0 = t_0 < t_1 < \ldots < t_N = T$. The semigroup $U(t)$ is the generalization of the exponential function of the operator tA. Therefore, it seems natural to approximate the solution of (1.1)–(1.2) at the points t_j, $j = 0, 1, \ldots, N$ by the sequence $\{u_j\} \subset \mathcal{X}$ defined recurrently by

$$u_{j+1} = F(hA)u_j, \quad j = 0, \ldots, N-1, \ldots, \ u_0 = \eta. \tag{3.2}$$

In the sequel we will investigate the conditions under which it will be really so. We begin with determining what we will mean by convergence.

Definition 3.1. The method given by (3.2) is said to be *convergent* on the class of problems (1.1)–(1.2) if

$$\lim_{\substack{h \to 0 \\ jh \to t}} u_j = U(t)\eta$$

for any $\eta \in \mathcal{X}$ and any $t \in [0, T]$.

Theorem 3.1. *Let a rational function F be given having its poles in the right-hand halfplane and being regular at infinity. Further, let F define the method* (3.2) *which is convergent for the class of problems* (1.1)–(1.2). *Then there exists a constant $M = M(t)$ such that*

$$\|F^j(hA)\| \leqslant M \tag{3.3}$$

for any sufficiently small h and for any j satisfying $0 \leqslant jh \leqslant t$.

P r o o f. Obviously, $u_j = F^j(hA)$. Thus, we have

$$\lim_{\substack{h\to 0\\ jh\to t}} F^j(hA)\eta = U(t)\eta.$$

Let us show now that the set $\{F^j(hA)\eta;\ 0 < h \leqslant h_0,\ jh \leqslant t\}$ is bounded for any fixed $\eta \in \mathcal{X}$. We will prove it by contradiction. Thus, let us suppose that there exist $\eta \in \mathcal{X}$, $t \in [0,T]$ and sequences j_k and h_k such that $j_k h_k \leqslant t$ and

$$\|F^{j_k}(h_k A)\eta\| \to \infty. \tag{3.4}$$

Taking into account the inequality $j_k h_k \leqslant t$ we may assume that $j_k h_k \to t_0$. But then the sequence $F^{j_k}(h_k A)\eta$ is convergent and, consequently, bounded. This fact contradicts (3.4). Hence, we have $\|F^j(hA)\| \leqslant M(\eta)$ and the assertion of Theorem 3.1 follows from the uniform boundedness principle. □

Theorem 3.2. *Let a rational function F be given with poles in the right-hand halfplane and regular at infinity. Further, let* (3.1) *hold with some $p \geqslant 1$. Finally, let* (3.3) *be satisfied. Then the method given by recurrences* (3.2) *is convergent.*

P r o o f. For the exact solution $u(t) = U(t)\eta$ we have

$$\begin{aligned} u(t_{j+1}) = U(t_{j+1})\eta &= U(h)U(t_j)\eta \\ &= F(hA)U(t_j)\eta - [F(hA) - U(h)]U(t_j)\eta \\ &= F(hA)u(t_j)\eta - U^j(h)[F(hA) - U(h)]\eta \end{aligned}$$

since the operators $U(t^j)$ and $F(hA)$ commute (see Theorem 1.4). Thus, the error $e_j = u_j - u(t_j)$ of the approximate solution satisfies

$$e_{j+1} = F(hA)e_j + U^j(h)[F(hA) - U(h)]\eta, \quad e_0 = 0,$$

or

$$e_j = \sum_{s=0}^{j-1} F^{j-1-s}(hA)U^s(h)[F(hA) - U(h)]\eta.$$

From this identity, from (3.3) and from (1.4) we obtain

$$\|e_j\| \leqslant M(T)M_\omega \exp(\omega T) j \|[F(hA) - U(h)]\eta\| \tag{3.5}$$

with some $T > t$.

Let us suppose now that $\eta \in \mathcal{D}(A^{p+1})$. Then Theorem 1.8 and Lemma 1.4 imply that the function $[F(hA) - U(h)]\eta$ has $p+1$ derivatives and that

$$\frac{\mathrm{d}^i}{\mathrm{d}h^i}[F(hA) - U(h)]\eta\Big|_{h=0} = [F^{(i)}(0) - 1]A^i\eta, \quad i = 0, \ldots, p \tag{3.6}$$

and

$$\frac{\mathrm{d}^{p+1}}{\mathrm{d}h^{p+1}}[F(hA) - U(h)]\eta = [F^{(p+1)}(hA) - U(h)]A^i\eta.$$

By Lemma 2.5 it follows immediately that

$$\left\|\frac{\mathrm{d}^{p+1}}{\mathrm{d}h^{p+1}}[F(hA) - U(h)]\eta\right\| \leqslant M_{p+1}\|A^{p+1}\eta\|$$

for any sufficiently small h. Using now Theorem 1.3 and observing that (3.1) implies $F^{(i)}(0) = 1$ for $i = 0, \ldots, p$ we obtain from (3.6)

$$\|[F(hA) - U(h)]\eta\| \leqslant M_{p+1}\frac{h^{p+1}}{(p+1)!}\|A^{p+1}\eta\|.$$

If we substitute this estimate into (3.5) we have

$$\|e_j\| \leqslant \tilde{M}\|A^{p+1}\eta\|h^p,$$

where $\tilde{M}$ is a constant depending only on T. Hence, the method is convergent for any $\eta \in \mathcal{D}(A^{p+1})$ (and this convergence is, obviously, uniform on any finite interval of t's).

To conclude the proof it is now sufficient to take into account that $\mathcal{D}(A^{p+1})$ is dense in $\mathcal{X}$ and to remember the well-known theorem stating that the sequence $\{T_h\}$ of bounded linear operators in a Banach space $\mathcal{X}$ is convergent for any $\eta \in \mathcal{X}$ if and only if

(i) the sequence of norms $\{\|T_h\|\}$ is bounded, and
(ii) $\{T_h\eta\}$ is convergent on a dense subset of $\mathcal{X}$. □

Note that we have proved in fact more than is stated in Theorem 3.2. We have proved that the method of the order p leads in the case of sufficiently smooth data to the convergence of order h^p.

4. Concluding remarks

In this section, let us briefly check condition (3.4) which controls the convergence of the method (3.2).

Let us begin with a trivial example when the given method is the Rothe method. In that case, the function F in (3.2) has the form

$$F(hA) = (I - hA)^{-1} = \frac{1}{h} R\Big(\frac{1}{h}, A\Big).$$

Using now (1.5) (A is the infinitesimal generator of a continuous semigroup of operators), we obtain the estimate

$$\|R^j(hA)\| \leqslant \frac{M}{(1 - h\omega)^{t/h}}$$

and this inequality holds for any sufficiently small h and for any j satisfying $jh \leqslant t$. But the right-hand term of the last inequality is bounded since $\lim_{h\to 0}(1 - h\omega)^{-t/h} = \exp(t\omega)$. Consequently, any power of $F(hA)$ is bounded and the Rothe method is convergent.

To be able to make more general conclusions, suppose that $\|F(x)\| < 1$ for negative x,[2] and that the space $\mathcal{X}$ is a Hilbert space and A is selfadjoint. In that situation, the spectrum of A is real, $F(hA)$ is also selfadjoint (the values of F are real for real z's) and, hence the norm of $F^j(hA)$ is equal to the spectral radius

$$\|F^j(hA)\| = \varrho(F^j(hA)) \leqslant \varrho^j(F(hA)).$$

From Theorem 1.4 we have

$$\varrho(F(hA)) \leqslant \sup_{\lambda \leqslant \omega} |F(h\lambda)|. \tag{4.1}$$

Now observe that the assumption $|F(h\lambda)| < 1$ for $\lambda < 0$ implies that

$$\sup_{\lambda \leqslant \omega} |F(hA)| \leqslant 1 + Mh,$$

where M is a constant independent of h. The reason is that F is regular at the point $z = 0$. Using this result in (4.1) we obtain immediately that

$$\|F^j(hA)\| \leqslant (1 + Mh)^j,$$

which proves (3.3).

[2] In the language of ordinary differential equations, this property is known as $A(0)$-*stability*.

The general situation is substantially more complicated. If we assume that the resolvent of A is compact we hope that we succeed to prove, at least, that the method converges in the case of the A-stable method.

Let us remark that an effective way leading to methods with rational F in (3.2) may be the Runge-Kutta implicit methods (see, e.g., [2]) or the overimplicit methods introduced in [5].

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